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Short Note

# Rationally realizing sets, five years later

A subset $X$ of the Euclidean $n$-space $E^n$ can be rationally realizable in space $E^m$ if there exists a subset Y of the rational $m$-space $Q^m$ which is congruent to $X$.

We showed in [5] that a set $X$ in the space $E^n$ can be *rationally* realizable in $E^m$, for some $m$, if, and only if, the following holds:

(*) for every $x, y \in X$, $\mathrm{dist}^2(x, y)$ is a rational number.

The proof we gave used the famous Gram-Schmidt Orthogonalization algorithm (without normalizing). In fact we showed that if a set $X$ satisfies the condition (*) and if $d$ denotes the dimension of the affine hull of $X$, then $X$ is a subset of the set:
$Q\sqrt{k_1} \times Q\sqrt{k_2} \times \ldots \times Q\sqrt{k_d}$, in which $k_1$, $k_2$, ..., $k_d$ are positive integers; i.e.,
$(Q\sqrt{k_1}) \times (Q\sqrt{k_2}) \times \ldots \times (Q\sqrt{k_d}) = \{(q_1\sqrt{k_1}, q_2\sqrt{k_2}, \ldots, q_d\sqrt{k_d}) \,|\, q_i \in Q, k_i \in Q^+\}$.

It follows that for such sets $X$, all the $d$-dimensional volumes of simplices, formed by vertices in $X$, are of the form $q\sqrt{(k_1 k_2 \ldots k_d)}$ for some rational number $q$.

This property implies that $X$ can be rationally embbeded in the space $Q^{4d}$, because each $Q\sqrt{k_i}$ can be embedded in $Q^4$, using Lagrange's Four Squares Theorem. For example, $Q\sqrt{7}$ can be embedded in $Q^4$ and it cannot be embedded in any other $Q^d$, for $d < 4$.

Since I gave my talk at the Segre Conference in 2004, I was trying to reduce the upper bound $4d$. It is clear from the process that if $X$ satisfies the condition (*), then so does the set RatAff($X$), defined by $\mathrm{RatAff}(X) = \{\Sigma\lambda_i x_i \,|\, x_i \in X, \lambda_i \in Q, \Sigma\lambda_i = 1\}$.

In addition, If $X$ is rationally realizable in $E^m$, then so does RatAff($X$). It is clear from the Gram-Schmidt process that if RatAff($X$) contains two points at a rational distance,

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they can be chosen to be the first two points in the process, thus guaranteeing that $\sqrt{k_1} = 1$, hence $Q\sqrt{k_1} = Q$. In general, if there is a Diophantine solution to the equation $\Sigma k_j x_j^2 = y^2$, then RatAff($X$) contains two points at an integral $y$ distance, thus allowing us to (repeatedly) reduce the upper bound.

I have recently learned that my results have been already established some years ago. Maehara [1, 2, 3] showed that if a set $X$ satisfies condition (*), and its affine hull has dimension $d$, then it can be rationally embedded in some rational space $Q^m$, where $m$ is less or equal to $3d + 1$. Kumada [4] showed that such a set $X$ can be embedded already in $Q^{d+3}$; he gave examples where the least upper bound can be either $Q^d$, $Q^{d+1}$, $Q^{d+2}$ or $Q^{d+3}$.

## REFERENCES


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