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Beniamino Segre and Italian Geometry

Conferenza tenuta nel convegno in onore del Prof. B. Segre

ABSTRACT. — A personal view of Beniamino Segre's research activity in algebraic geometry.

1. - GEOMETRY IN ROME IN THE EARLY FIFTIES

Upon Francesco Severi's decision, the choice of new junior fellows (Discepoli Ricercatori) of the Istituto Nazionale di Alta Matematica (INdAM) for the academic year 1951-52 would be made, through formal interviews with candidates coming from all over Italy.

The interviews were made at the beginning of November 1951 by a committee chaired by Severi, with Luigi Fantappié, Mauro Picone, Antonio Signorini and Giulio Krall as members.

Beniamino Segre was not in the committee because in the same days he was engaged as a judge in a national competition for the selection of a full professor of geometry in the University of Turin. The choice fell on Aldo Andreotti, who, within a few weeks, moved to Turin. Almost in the same days, I came to Rome as Discepolo Ricercatore at INdAM, and I met for the first time Beniamino Segre who, on Severi's advice, became my research advisor.

Beniamino Segre was very busy at that time (as, in fact, at all times of his life). Besides discharging several academic duties outside the University of Rome, he was teaching an undergraduate course in the Department of Mathematics of the University and an advanced course in the Istituto Nazionale di Alta Matematica. I was soon involved in both these activities, as his assistant in the undergraduate course and charged with writing up

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the notes of his lectures at INdAM. The first task was not particularly demanding: Segre was lecturing on algebra, following essentially some chapters of the book *Moderne Algebra* by Van der Waerden, and I had to prepare exercises on elimination theory. But my job connected with the course at INdAM was much harder. There, Segre was lecturing on the De Rham cohomology, and the required background in general and algebraic topology had sometime to be supplied on the spot to an audience a part of which did only have a very superficial knowledge of both fields. Besides, Segre's presentation of algebraic topology did not go any further than the simplicial approach as it was exposed in Lefschetz' 1930 notes, and looked sometime as a reminiscence of the Conferenze di geometria algebrica, held by Severi in 1927/28 and 1928/29, whose notes were prepared by Segre himself.

In 1951 the question of the desingularization of complex projective varieties was aroused once again by an article by L. Derwidué, which had just appeared in the Mathematische Annalen, announcing the solution of the problem in any dimension. The proof was wrong – in some parts almost transparently wrong – and got a scathing review by Oscar Zariski in the Mathematical Reviews. One of the crucial points in Derwidué's argument, and one of the basic mistakes, that got immediately the attention of Beniamino Segre, concerned the behaviour of the polar hypersurfaces at the singular points of an algebraic hypersurface: a behaviour that Derwidué had assumed as an extension to higher dimensions of a "classical" result concerning the polar curves of a plane algebraic curve, contained in the second volume of the treatise by F. Enriques and O. Chisini on algebraic curves [10]. B. Segre constructed counterexamples to this latter result, and raised the question how to correct the statement in the one-dimensional case [22], [23]. The answer turned out to be not too difficult to find, as it came out of a careful inspection of the Newton diagram of a plane algebraic curve.

The article containing that result pleased Segre and pleased also Zariski, who in 1953 had come to Rome and lectured at INdAM on valuation theory [51], [52]. Looking now back at those years, in the first half of the fifties, one realizes that, as a matter of fact, a considerable amount of mathematical activity was going on in Rome at that time. Almost simultaneously with Zariski's visit, Heinz Hopf was lecturing on algebraic topology, while E. Witt spent an entire year at INdAM and Erich Kaehler gave a course on arithmetic geometry whose notes absorbed an entire issue of the "Annali di Matematica" and caught the attention of André Weil and, some time later, of Alexandre Grothendieck. Beside those visitors, Luigi Fantappié lectured on topological groups and on analytic functionals [13], Beniamino Segre on De Rham cohomology [24], Fabio Conforto on modular abelian functions [8], Francesco Severi on integrals on algebraic surfaces, ...

In retrospect, one may say now that the output was quite modest. It was an activity essentially uncoordinated but, at the same time, formal and academic, that, as such, lacked some sort of follow up, did not provoke any spontaneous seminarial activity, and was therefore scarcely productive in terms of research and in terms of training a new generation of young mathematicians.

Training a new generation of young mathematicians, putting new blood into mathematical research was the main objective of Beniamino Segre, who was one of the chief proposers of many of the new initiatives that were going on in the Department of Mathematics and in the Istituto Nazionale di Alta Matematica. As he told me years later, at the origin of his efforts were his reflections on the narrowness and the provinciality of the panorama of italian mathematics: reflections that he had made already during the eight years, from 1938 to 1946, he spent in England, after having been expelled from his chair in the University of Bologna by the racial laws enacted by the fascist regime.

This vision found a quite receptive humus in the eclecticity of Segre's scientific interests. In fact, looking at his copious bibliography, one discovers a peculiar aspect of Segre's personality: the capacity to keep abreast simultaneously with different research fields, quite apart from each other: projective and metric differential geometry, algebraic geometry, complex function theory, differential topology, projective geometries in their various extensions, combinatorics ... Under this aspect, Segre's work is quite far from that of other mathematicians, as, for example, Kiyoshi Oka or Harish Chandra, who built their own obelisk on a single research topic, or from that of John von Neumann or Jake Schwartz whose research interests pertain in fact to different areas, but are active in time periods quite apart from each other.

2. - Beniamino Segre in Rome and Bologna

Segre's *années d'apprentissage*, which had begun in 1923 in the University of Turin with a dissertation written under the direction of Corrado Segre, were concluded in 1926 by a *stage* with Elie Cartan, in Paris, supported by a Rockfeller fellowship. This first contact with Elie Cartan was revitalised six years later, in 1932, when Cartan solved a problem posed by Beniamino Segre within the theory of holomorphic functions of two complex variables, and produced a complete system of invariants characterizing the equivalence class of a real hypersurface in $\boldsymbol{C}^2$ (1) with respect to local biholomorphic transformations of $\boldsymbol{C}^2$. Fifty years later, in 1975, Elie Cartan's results were extended by S. S. Chern and J. K. Moser [7] to real hypersurfaces in $\boldsymbol{C}^n$, and, in 1983, by J. K. Moser and S. M. Webster [20] to real analytic surfaces in $\boldsymbol{C}^2$.

After the year spent in Paris, Beniamino Segre moved to Rome, where he became an assistant of Francesco Severi (who at that time held the chair of analysis in the university),

(1) The problem – which arose from a remark of H. Poincaré [21] whereby two pieces of real analytic hypersurfaces of $\boldsymbol{C}^2$ are not necessarily mapped into each other by a holomorphic automorphism of $\boldsymbol{C}^2$, and was posed by B. Segre [25], [26] – consisted in the search of all invariants associated to a real analytic hypersurface of $\boldsymbol{C}^2$ with respect to the group of all holomorphic automorphisms of $\boldsymbol{C}^2$. The problem was solved by E. Cartan [3], [4], applying a general method on the equivalence of Pfaff systems, developed in E. Cartan [5]. See also [6]

and four or five years later moved to the University of Bologna, where he was appointed professor of geometry.

If until 1927 the scientific interests of Segre had ranged mainly around themes of projective geometry and of differential geometry, the presence in Rome of Guido Castelnuovo, Federigo Enriques and Francesco Severi had a profound influence on him and concentrated on algebraic geometry his main scientific interests. The extension of Riemann's existence theorem of algebraic functions from one to two variables or, equivalently, the representation as a multiple plane of an algebraic surface embedded in a three dimensional complex projective space – together with its consequences on the number of moduli of the surface – was, since the beginning of the last century, at the center of the interests of F. Enriques, M. De Franchis, A. Comessatti, O. Chisini, L. Campedelli and of some of Enriques' younger students. A memoir by B. Segre on the characterization of the branch curve of a multiple plane, that appeared in 1930 and was inspired by a paper of ENRIQUES [11], followed shortly a paper by O. Zariski on the same topic. But a critical remark made by O. Zariski (2) on an infinitesimal method used by Enriques in earlier papers on the moduli of an algebraic surface, set some doubts on the validity of Enriques' argument and, consequently on the papers that Segre devoted to this topics.

A gap pointed out by Severi in 1921 in Enriques' proof of the completeness of the characteristic series cut on the generic element of a continuous system of curves in an algebraic surface – which Zariski linked to his own critical remark – was then at the origin of a bitter and long controversy between Enriques and Severi, that damaged not only the image but also the development of what would later be called the "Italian school". On the problem of the completeness of the characteristic series, SEGRE published in 1938 and 1939 two articles [27], (see also [28]), [29] – which turned out not to be exempt from critical remarks by Enriques – and a third, still not conclusive paper, in 1942, when he was in England [30], [31]. That was in fact his last contribution to the question, and one is lead to wonder whether he somehow anticipated the pessimistic views on the possibility of getting to the result through the way explored by Enriques and Severi that Castelnuovo would express in the introduction to Enriques' posthumous book *Le superficie algebriche* (3).

In those years in Rome, Beniamino Segre was attracted also by a different research topic. Very early in his investigations Francesco Severi had realized how the study of the local geometry of an algebraic complex manifold could benefit from the theory of

(2) See O. ZARISKI [53]. For an exhaustive history on this topic, see pp. 97-98 and 160-175 of Zariski's monograph.

(3) Alluding to the way suggested by Enriques in Chapter IX of the book, Castelnuovo wrote, in 1949: *Debbo confessare che non vedo come quella via possa tradursi in un procedimento irreprensibile*. See: F. ENRIQUES [12]. For some indications on the subsequent contributions to this topic, see, e.g. [9], and, for further details, the appendices, by S. S. Abhyankar, D. Mumford and J. Lipman, to O. ZARISKI [54].

holomorphic functions, offering what might now be considered an *ante litteram* motivation to the future development of the geometry of complex manifolds. Severi began to work intensely in complex analysis associating soon in his work Beniamino Segre (4) and, later on, Enzo Martinelli who would bring to holomorphic function theory the important contribution of the Cauchy-Martinelli integral formula [15], [16], [17], [18], [19] (or the Martinelli-Bochner formula [2]). One should stress the local character of these investigations, that ignored (except for a revisitation of E. E. Levi results on the Levi form) the global aspects of complex analysis that in those years were already coming to the attention of French and Japanese mathematicians.

Besides the questions concerning multiple planes and moduli, other relevant problems of algebraic geometry attracted the attention of Segre in the years he spent in Rome as an assistant of Francesco Severi. The most prominent of them stemmed from a series of papers in which Severi tried to extend the classical equivalence theory for divisors on a smooth complex projective variety to the equivalence of algebraic cycles of any dimension [50]. As a first step he introduced a notion of equivalence in dimension zero on a projective surface, and defined the canonical series, which is invariant under the action of biregular transformations and whose "order" equals the Euler-Poincaré characteristic of the surface. Thus, as was pointed out by Severi, the arithmetic genus of the surface can be expressed in terms of the order of the canonical series and of the self-intersection of the canonical divisor.

In two memoirs published between 1933 and 1936 [34], [35], [36] Segre developed an extension of Severi's ideas to three dimensional smooth complex projective varieties introducing, beside the canonical divisor, canonical cycles of complex dimension zero and one and expressing, in terms of them, the arithmetic genus of a three dimensional projective variety.

This pioneering work of Segre was extended to higher dimensional smooth complex projective varieties by M. Eger and by J. A. Todd, who conjectured that the arithmetic genus of the variety might be expressed by universal formulas in terms of the canonical cycles. The problem of extending his own and Severi's results to higher dimensions was one of the main lines of research carried out by Segre during the war years he spent in England, and, later on, first in Bologna, where, in 1946 he resumed his academic position in the university, and then in the University of Rome, where he was appointed professor of geometry in 1950, and where he stayed until his retirement in 1973.

The outcome of his reflections on the construction of canonical cycles of any dimension materialized in a long memoir which appeared in the Annali di Matematica in 1953 [37], [38] (5). Although very difficult to read, this paper offered a cue to the

(4) Severi's and Segre's main contributions to holomorphic function theory are summarized in the reports [47], [48], [32], [33]. See also [49].

(5) This memoir was followed by a paper [39], [40], where the canonical cycles were defined in terms of a blowing up process. See also [41].

solution of a problem that was posed (and partially solved) by W. V. D. Hodge in 1951 [14] about the coincidence (up to the sign) of the Chern classes and of the canonical cycles, as heuristically defined by J. A. Todd.

3. - Combinatorial geometries

Coming back from England in 1946, Beniamino Segre brought with him new mathematical research topics and a more mature and wider vision of mathematics. In the years he had spent teaching in London, in Cambridge and in Manchester he developed a lively attention to the combinatorial aspects of geometry. This new attitude can be already perceived in a book on non-singular cubic surfaces that he wrote in Cambridge in 1941, which is mainly devoted to the combinatorial aspects of the structure of the twenty seven lines of the surface [42]. These combinatorial aspects became more prominent in a note on the configuration of the lines in a cubic surface over any, possibly finite, field, that he wrote in 1949 [43] stressing the radically new phenomena that appear according to the properties of the ground field. Just one year before, the first volume of his *Lezioni di geometria moderna* (6) had appeared, which was devoted to the foundations of geometry on an arbitrary field and was originated in a course of lectures delivered in the University of Bologna in the academic year 1946/47. In Segre's programs, this first volume should have been followed by two more books devoted respectively to non-linear projective geometry, with its arithmetical applications, and to the group of birational transformations. Although this ambitious editorial program did not materialize, the first volume of the *Lezioni* gave rise, thirteen years later, to a substantially enlarged english version [44], which more than duplicated the size of the 1948 text with a large expansion of the chapters on Galois spaces and on non-desarguesian graphic planes, and the addition of entirely new chapters. But in those thirteen years, and in the years to follow, that part of the scientific program which, in Segre's words, would be largely concerned *with finite fields and their application to the study of finite linear spaces*, had taken quite an unexpected and successful turn.

The starting point of these new developments was Segre's characterization of conics in finite planes over a field of odd characteristic [45], [46] (7), which answered a conjecture by P. Kustaanheimo and was later generalized to three dimensional spaces by A. Barlotti and G. Panella. Time and again in the following years Segre came back to the progress made by the theory of Galois spaces and, more in general, to what was called *combinatorial geometry*, a vital component of *algebraic combinatorics* that, in Gian-

(6) Zanichelli, Bologna, 1948.

(7) Many years later, the case in which the characteristic is two was dealt with by B. Segre in a joint paper with U. Bartocci [1] which was at the outset of the classification of the ovals in Galois planes of even characteristic.

Carlo Rota's words, *at last found* (...) *a firm place in the mathematics of our time*, due also to factors external to mathematics.

Beniamino Segre was one of the founders and became rapidly one of the leaders of combinatorial geometry.

According to one's personal background, it is open to discussion which part of the research work of Beniamino Segre – whether in "classical" algebraic geometry or in combinatorial geometry – is more valuable than the other.

What in my opinion is undisputable is that combinatorial geometry was the field in which he realized himself more freely, perhaps also because in his work in classical algebraic geometry he felt the shadow of Francesco Severi.

Perhaps, to gain a better vision of geometry in Beniamino Segre' times, I should have paid attention also to other research fields in mathematics that were going on in those years. But, to use a sentence by Oscar Zariski, *whatever happens in mathematics, happens in algebraic geometry first*.

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