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A Memory of Gaetano Fichera

It is a pleasure to remember Gaetano Fichera and my visits to Rome in the sixties. Particularly I remember the spring of 1964 when Olga Oleinik and I enjoyed together the Roman scene. We were visiting the University of Rome as guests of Gaetano but the visit was interrupted by the first of the student strikes of the sixties. I think it was the first for Europe. A student had been killed in a political demonstration of left versus right and the whole student body erupted. The gates of the university were locked shut and tanks stood at the ready, I think inside. I had already given my talks and Oleinik was scheduled to be next. She was continuously informed on the state of the strike by the Russian embassy. I relied on radio news and the graffiti on the walls of the university. Fichera kept telling us not to worry: Olga would give her talk as scheduled and there would be an audience. And so it was. Fichera who was very adept at making things happen, sneaked Olga, me and the audience into the university by some obscure gate in the university wall. He had somehow obtained a key and for that day of the strike, for us, life was normal.

Gaetano Fichera became well known in the fifties for his work in elliptic equations and in applied mathematics. A problem of great interest at that time was the behavior of steady two dimensional compressible potential flow. If $(u, v) = \nabla\phi$ are the two components of velocity and c, the sound speed, is a function of the flow speed $|\nabla\phi|$ which is given by Bernoulli's law, the equation for ϕ, the potential, is

$$(c^2 - u^2)\phi_{xx} - 2uv\phi_{xy} + (c^2 - v^2)\phi_{yy} = 0$$

which is strictly elliptic if $u^2 + v^2 = |\nabla\phi|^2 < c^2$, i.e. at subsonic speeds. Back then it was not known how the flow broke down when the speed became supersonic in some regions i.e. when the equation was elliptic in some regions and hyperbolic in others.

The model example of F. Tricomi [5]

$$yf_{xx} + f_{yy} = 0$$

had only been studied in one case.

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Fichera had the idea to study simple problems where the elliptic equation degenerated on the boundary, as, for example, in the equation

$$(x^2 + y^2) f_{xx} + f_{yy} = 0$$

in $x^2 + y^2 \leq 1$ with $f(x, y)$ given on $x^2 + y^2 = 1$. His results, [1], inspired Olga Oleinik to seek sharp conditions for existence and uniqueness and general singular equations. Her paper where she credits Fichera with much inspiration remains the standard on the subject.

In fluid dynamics problems it is easy to show that if a flow is varied by changing a simple parameter, e.g. the flow past a closed airfoil with prescribed speed at infinity as the parameter, then the flow first becomes sonic at the boundary airfoil. The linearized flow equations display exactly the phenomenon first considered by Fichera.

A second problem that has importance in mixed equations and fluid flow involves a steady flow that has a supersonic region. These flows are only exceptionally without shocks, see Morawetz [4]. One expects nearby flows to have discontinuities in the velocity. What kind of linear boundary value problem determines the perturbation of this flow? First the perturbation velocity, i.e. the gradient of the perturbation potential, must be normal to the boundary. It is easy to show that the equation for the perturbation potential is elliptic in the domain where the original flow is subsonic and hyperbolic where it is supersonic. So we have a linear elliptic-hyperbolic problem with Neumann data. The data is unnatural on the hyperbolic part of the boundary. Fichera in his paper [2] considered a model problem where the equation is purely hyperbolic in a lens shaped domain and on the boundary a Dirichlet condition is imposed. The simplest example is: Find $f(x, y)$ if

$$f_{xx} - f_{yy} = 0 \text{ in } \mathcal{D} \tag{1}$$

Here $\mathcal{D}$ is the lens-shaped domain,

$$y_-(x) \leq y(x) \leq 0,$$

$$y_-(\pm 1) = 0, \quad \frac{d^2 y_-}{dx^2} > 0.$$

The angle of the lens at the corners is less than $\frac{\pi}{4}$

$$f(x, 0) = f_-(x) \tag{2}$$

$$f(x, y_+(x)) = f_+(x) \tag{3}$$

with $f_-(+1) = f_+(+1), \ \ f_-(-1) = f_+(-1)$.

The problem is easily solved from left to right using the characteristics pointing to the left to obtain $f(x, y)$ in terms of f_- and f_+ using the condition on the lens angle. Clearly the solution can behave singularly as one approaches the point $(1, 0)$. One can do the same from right to left. The solution is not unique unless one specifies the nature of the singularity at one end or the other of the lens.

We now consider a mixed problem by adding another lens in $y > 0$ so that the final domain $\mathcal{D}$ is

$$-1 \leq x \leq +1, \;\; y_-(x) \leq y \leq y_+(x)$$

Where $y_\pm(x)$ satisfies

$$y_-(x) \leq 0 \text{ and } y_+(x) \geq 0$$

We require that $f(x, y)$ satisfy the Tricomi equation,

$$yf_{xx} + f_{yy} = 0,$$

In $\mathcal{D}$ and Dirichlet data on $y = y_-(x)$ and $y = y_+(x)$. And of course the solution $f(x, y)$ must be continuous across the parabolic line, $y = 0$. This is very close to the transonic problem except for the limitations on the corners of the domain at the parabolic line, $y = 0$.

For the physical perturbation problem, one needs to let the tangents to the boundary be smooth at the parabolic points. However Lupo, Payne and Morawetz [3] have determined some existence and uniqueness properties for this problem as long as the corners at the parabolic boundary are lens like.

Open problems still remain but it should be remembered that Fichera was the pioneer in the area of degenerate elliptic equations and hyperbolic problems in closed domains with Dirichlet or Neumann boundary conditions.

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