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Plane Problems in Non-Linear Theory of Elasticity for Hardening Media (**)

ABSTRACT. — The conditions for the weak plane stress and plane strain solutions in the theory of elastic media with hardening to exist and be unique are considered.

Consider the cylindrical domain T with the axe x_3 and cross-section $\Omega = \{(x_1, x_2)\}$. The elastic equilibrum system may be written in the form

$$L_j(\boldsymbol{u}) + \rho f_j \equiv D_i \sigma_{ij} + \rho f_j = 0 \quad (i, j = 1, 2, 3), \tag{1}$$

where f_j ($j = 1, 2, 3$) are given body (mass) forces and $\rho = \text{const}$ is the media density. Assume also that on the lateral area of the cylinder $\partial\Omega$ the boundary condition

$$\boldsymbol{u}\Big|_{\partial\Omega} = 0 \tag{2}$$

is given for the displacement field $\boldsymbol{u} = (u_1, u_2, u_3)$. If the cylinder is bounded along the axe $Ox_3 \equiv Oz$, so that $0 < z < h$, then we consider the analogous condition on the end-walls:

$$\boldsymbol{u}\Big|_{z=0} = \boldsymbol{u}\Big|_{z=h} = 0. \tag{3}$$

We shall write the basic relations of the hardening elastic media in the form

$$\sigma_{ij} = a_{ij}(x, \varepsilon_{kl}), \tag{4}$$

where $x = (x_1, x_2, x_3)$ and $\varepsilon_{kl} = (D_k u_l + D_l u_k)/2$. Suppose that the functions $a_{ij}(x, p)$ are continuously differentiable with respect to p for almost all x, the matrix

$$A = \left\{ \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \right\} \quad (i, j, k, l = 1, 2, 3) \tag{5}$$

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is symmetric and its eigenvalues λ_i satisfy the inequalities

(6) $$\sup \lambda_i = \Lambda_0 < +\infty, \quad \inf \lambda_i = \lambda_0 > 0.$$

Consider the iterative process of elastic solutions [1]

(7) $$\Delta u_j^{(n+1)} + \operatorname{grad}\operatorname{div} u_j^{(n+1)} = \Delta u_j^{(n)} + \operatorname{grad}\operatorname{div} u_j^{(n)} - \varepsilon\Big(L_j(\boldsymbol{u}^{(n)}) + \rho f_j\Big)$$

$(j = 1, 2, 3)$ with boundary conditions (2) and (3).

In the monograph [1] (p. 165) it is proved that if $f \in L_2(T), \varepsilon = 2(\Lambda_0 + \lambda_0)^{-1}$, then the process (7) converges in $W_2^{(1)}(T)$ to the solution of the problem (1)-(3) as a geometric progression with the ratio $(\Lambda_0 - \lambda_0)(\Lambda_0 + \lambda_0)^{-1}$, starting from the arbitrary $\boldsymbol{u}^{(0)} \in W_2^{(1)}(T)$. This proves the uniqueness and existence of the weak solution of the space problem (1)-(3).

Consider now the plane stress problem when the conditions

(8) $$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$$

are proposed to be satisfied in all points of the cylindrical domain T. In the same time the condition (3) is omitted. Suppose also that $f_3 = 0$ and the components f_1 and f_2 depend only on x_1, x_2. In this case the third equation from (1) is satisfied identically and other two equations can be written in the form

(9) $$\begin{cases} D_1\sigma_{11} + D_2\sigma_{12} + \rho f_1 = 0, \\ D_1\sigma_{12} + D_2\sigma_{22} + \rho f_2 = 0. \end{cases}$$

For the homogenous isotropic elastic media the relations (4), as it is well known, take the form

(10) $$\sigma_{ij} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\delta_{ij} + 2\mu\varepsilon_{ij}$$

where λ and μ are Lame constants. In this case it follows from (8) and (10) that

(11) $$\varepsilon_{13} = 0, \quad \varepsilon_{23} = 0, \quad \varepsilon_{33} = -\frac{\lambda}{\lambda + 2\mu}(\varepsilon_{11} + \varepsilon_{22})$$

so

(12) $$\sigma_{ij} = \lambda^*(\varepsilon_{11} + \varepsilon_{22})\delta_{ij} + 2\mu\varepsilon_{ij} \quad (i, j = 1, 2),$$

$$\lambda^* = \frac{2\lambda\mu}{\lambda + 2\mu}.$$

Suppose that the values $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$, and therefore the stresses $\sigma_{11}, \sigma_{22}, \sigma_{12}$ do not depend on x_3. Then (9) with the help of (12) reduces to a two-dimensional system with the Lame constants λ^* and μ. For this system the existence and uniqueness theorems are valid in the class $W_2^{(1)}(\Omega)$[1]. The iterative process (7) also converges as in the space problem case. But the Saint-Venant' compatibility conditions are not satisfied in general for such solution [2]. It is easy to see that these conditions will be satisfied only if ε_{33} from (11) is a linear function of x_1, x_2. Therefore our assumption concerning the strain and stress independence from the coordinate x_3 in the cylindrical domain T in the case of pure plane stress state is valid only for restricted cases. This was the reason to consider the generalized plane stress state for thin plates in the elasticity theory when the dependence on the coordinate x_3 may be omitted by averaging.

Consider now the nonlinear problem for a hardening media. Let the intensities of strains and stresses be

$$e_0 = \frac{\sqrt{2}}{3}\sqrt{\sum_{j<l}\left[(D_j u_j - D_l u_l)^2 + \frac{3}{2}(D_j u_l + D_l u_j)^2\right]}, \tag{13}$$

$$\sigma_0 = \frac{\sqrt{2}}{2}\sqrt{\sum_{j<l}\left[(\sigma_{jj} - \sigma_{ll})^2 + 6\sigma_{jl}^2\right]}. \tag{14}$$

Following the postulates of deformation plasticity theory [4] we assume that

$$\sigma_0 = \Phi(e_0) \equiv 3\mu(1 - \omega(e_0))e_0, \tag{15}$$

and the continuously differentiable function ω satisfies the inequalities

$$\inf\,(1 - \omega - \omega' e_0) > 0, \quad \omega \geqslant 0, \quad \omega' \geqslant 0. \tag{16}$$

The basic relations of the nonlinear theory may be presented in the form

$$\sigma_{ij} = \left(\lambda + \frac{2}{3}\mu\omega\right)(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\delta_{ij} + 2\mu(1 - \omega)\varepsilon_{ij}. \tag{17}$$

If the conditions (8) are satisfied then the equalities

$$\varepsilon_{13} = 0, \quad \varepsilon_{23} = 0 \tag{18}$$

are valid and the expression (13) leads to the formula

$$e_0 = \frac{\sqrt{2}}{3}\sqrt{(\varepsilon_{11} - \varepsilon_{22})^2 + (\varepsilon_{11} - \varepsilon_{33})^2 + (\varepsilon_{22} - \varepsilon_{33})^2 + 6\varepsilon_{12}^2}. \tag{19}$$

From the relations (17) we shall get the equality

$$\varepsilon_{33} = -\frac{\lambda}{\lambda + 2\mu}(\varepsilon_{11} + \varepsilon_{22}) - \frac{2\mu\omega}{3(\lambda + 2\mu)}(\varepsilon_{11} + \varepsilon_{22} - 2\varepsilon_{33}) \tag{20}$$

Denote the right-hand side of (20) by Ω_3. Then

$$\frac{\partial\Omega_3}{\partial\varepsilon_{33}} = -\frac{2\mu}{3(\lambda + 2\mu)}\left[\omega' \frac{\partial e_0}{\partial\varepsilon_{33}}(\varepsilon_{11} + \varepsilon_{22} - 2\varepsilon_{33}) - 2\omega\right] =$$

$$= \frac{2\mu}{3(\lambda + 2\mu)}\left[\frac{\sqrt{2}}{3}\omega' \frac{(\varepsilon_{11} + \varepsilon_{22} - 2\varepsilon_{33})^2}{\sqrt{(\varepsilon_{11} - \varepsilon_{22})^2 + (\varepsilon_{11} - \varepsilon_{33})^2 + (\varepsilon_{22} - \varepsilon_{33})^2 + 6\varepsilon_{12}^2}} + 2\omega\right].$$

With the help of (19) we come to the expression

$$\frac{\partial\Omega_3}{\partial\varepsilon_{33}} = \frac{4\mu}{3(\lambda + 2\mu)}\left[\frac{1}{9}\omega' \frac{(\varepsilon_{11} + \varepsilon_{22} - 2\varepsilon_{33})^2}{e_0} + \omega\right].$$

Now it is useful to apply an elementary inequality

$$|\varepsilon_{11} + \varepsilon_{22} - 2\varepsilon_{33}| \leqslant \sqrt{2}\sqrt{(\varepsilon_{11} - \varepsilon_{33})^2 + (\varepsilon_{22} - \varepsilon_{33})^2} \leqslant 3e_0$$

and get the apriory estimate for the derivative

$$\left|\frac{\partial \Omega_3}{\partial \varepsilon_{33}}\right| \leqslant \frac{4\mu}{3(\lambda + 2\mu)}[\omega' e_0 + \omega] \leqslant \frac{2}{3}[\omega' e_0 + \omega] < 1.$$

It follows from our estimate that the equation (20) has the unique solution $\varepsilon_{33} = \zeta(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})$. The derivatives of the expression (20) on $\varepsilon_{ij}(i, j = 1, 2)$ may be presented in the form

$$\frac{\partial \varepsilon_{33}}{\partial \varepsilon_{ij}} = -\frac{\lambda}{\lambda + 2\mu}\delta_{ij} + O(\max\{\sup \omega, \sup \omega'\}) \quad (i, j = 1, 2).$$

For stresses $\sigma_{11}, \sigma_{22}, \sigma_{12}$ we get

(21) $$\sigma_{ij} = \lambda^*(\varepsilon_{11} + \varepsilon_{22})\delta_{ij} + 2\mu\varepsilon_{ij} + \varphi_{ij}(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}) \quad (i, j = 1, 2),$$

where functions φ_{ij} and their derivatives have the order

(22) $$\kappa = O(\max\{\sup \omega, \sup \omega'\}).$$

The matrix (5) for $i, j, k, l = 1, 2$ may be written as

$$A = \begin{pmatrix} \lambda^* + 2\mu & 0 & \lambda^* \\ 0 & 2\mu & 0 \\ \lambda^* & 0 & \lambda^* + 2\mu \end{pmatrix} + B,$$

where all elements of the symmetric matrix B have the same order (22). If the order (22) is small enough the eigenvalues of A are approximately equal to $\Lambda_0 = 2(\lambda^* + \mu)$ and $\lambda_0 = 2\mu$. Now we can use Lemma (4.3.1) from the book [1] to prove that our problem (9) with the boundary conditions (2) and non-linear elastic relations (21) possesses the unique solution in $W_2^{(1)}(\Omega)$. The iteration process (7) converges to this solution if $\varepsilon = 2(\Lambda_0 + \lambda_0)^{-1}$ as a geometric progression with the ratio $(\Lambda_0 - \lambda_0)(\Lambda_0 + \lambda_0)^{-1}$.

In the case of the plane deformation it is assumed that the displacements field $\boldsymbol{u}(x_1, x_2)$ and the mass forces $\boldsymbol{f}(x_1, x_2)$ don't depend on x_3 and $u_3 = f_3 = 0$. Therefore

(23) $$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$$

and $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$ don't depend on x_3. The expression (19) for the strains intensity takes the form

(24) $$e_0 = \frac{\sqrt{2}}{3}\sqrt{(\varepsilon_{11} - \varepsilon_{22})^2 + \varepsilon_{11}^2 + \varepsilon_{22}^2 + 6\varepsilon_{12}^2} = \frac{2}{3}\sqrt{\varepsilon_{11}^2 + \varepsilon_{22}^2 - \varepsilon_{11}\varepsilon_{22} + 3\varepsilon_{12}^2}.$$

The relations (17)together with (23) give

(25) $$\begin{aligned} \sigma_{11} &= (\lambda + 2\mu)\varepsilon_{11} + \lambda\varepsilon_{22} - \frac{2}{3}\mu\omega(2\varepsilon_{11} - \varepsilon_{22}), \\ \sigma_{22} &= (\lambda + 2\mu)\varepsilon_{22} + \lambda\varepsilon_{11} - \frac{2}{3}\mu\omega(2\varepsilon_{22} - \varepsilon_{11}), \\ \sigma_{12} &= 2\mu(1 - \omega)\varepsilon_{12}, \end{aligned}$$

The equilibrium equations do not change their form (9). For the existence of weak solution of the problem (9), (25), (2) it is sufficient that the conditions (4.2.3) and (4.2.4) from [1] would be satisfied. In the space problem these conditions are satisfied under restrictions (16). In particular the parameter a from the inequality (4.2.3) may be taken in the form

$$a = \frac{\mu}{2}\left(1 - \sup_{e_0} (\omega + \omega' e_0)\right).$$

For the existence of weak solution for the plane deformation problem for the hardening media it is sufficient to show that the inequality

$$\sum \overline{\frac{\partial \sigma_{ik}}{\partial \varepsilon_{jl}}} \varepsilon_{ik}\varepsilon_{jl} \geqslant a_2(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{12}^2) \quad (i, j, k, l = 1, 2) \tag{26}$$

is valid where $a_2 > 0$ and the bar over the derivatives shows that they are calculated for some intermediate values of the arguments [1].

Finding the required derivatives from (25) and taking into account that according to (24) we have

$$\frac{\partial e_0}{\partial \varepsilon_{11}} = \frac{2(2\varepsilon_{11} - \varepsilon_{22})}{9e_0}, \quad \frac{\partial e_0}{\partial \varepsilon_{22}} = \frac{2(2\varepsilon_{22} - \varepsilon_{11})}{9e_0}, \quad \frac{\partial e_0}{\partial \varepsilon_{12}} = \frac{4\varepsilon_{12}}{3e_0},$$

we come to the expression

$$\begin{aligned}
\sum \overline{\frac{\partial \sigma_{ik}}{\partial \varepsilon_{jl}}} \varepsilon_{ik}\varepsilon_{jl} = \lambda(\varepsilon_{11} + \varepsilon_{22})^2 + 2\mu(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{12}^2) - \\
- \frac{4}{3}\mu\frac{\bar{\omega}'}{\bar{e}_3}\left[\frac{1}{9}(2\bar{\varepsilon}_{11} - \bar{\varepsilon}_{22})^2\varepsilon_{11}^2 + \frac{1}{9}(2\bar{\varepsilon}_{22} - \bar{\varepsilon}_{11})^2\varepsilon_{22}^2 + \right. \\
+ \frac{2}{9}(2\bar{\varepsilon}_{11} - \bar{\varepsilon}_{22})(2\bar{\varepsilon}_{22} - \bar{\varepsilon}_{11})\varepsilon_{11}\varepsilon_{22} + \\
\left. + (2\bar{\varepsilon}_{11} - \bar{\varepsilon}_{22})\bar{\varepsilon}_{12}\varepsilon_{11}\varepsilon_{12} + (2\bar{\varepsilon}_{22} - \bar{\varepsilon}_{11})\bar{\varepsilon}_{12}\varepsilon_{22}\varepsilon_{12} + 2\bar{\varepsilon}_{12}^2\varepsilon_{12}^2\right] - \\
- \frac{2}{3}\mu\bar{\omega}(2\varepsilon_{11}^2 + 2\varepsilon_{22}^2 + 3\varepsilon_{12}^2 - 2\varepsilon_{11}\varepsilon_{22}).
\end{aligned}$$

After denoting

$$\zeta_{11} = \frac{2\bar{\varepsilon}_{11} - \bar{\varepsilon}_{22}}{3\bar{e}_0}\varepsilon_{11}, \quad \zeta_{22} = \frac{2\bar{\varepsilon}_{22} - \bar{\varepsilon}_{11}}{3\bar{e}_0}\varepsilon_{22}, \quad \zeta_{12} = \frac{\bar{\varepsilon}_{12}}{\bar{e}_0}\varepsilon_{12}$$

we can write for the quadratic form in (26) the expression

$$\begin{aligned}
\sum \overline{\frac{\partial \sigma_{ik}}{\partial \varepsilon_{jl}}} \varepsilon_{ik}\varepsilon_{jl} = \lambda(\varepsilon_{11} + \varepsilon_{22})^2 + 2\mu(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{12}^2) - \\
- \frac{4}{3}\mu\overline{\omega' e_3}(\zeta_{11}^2 + 3\zeta_{11}\zeta_{12} + 2\zeta_{11}\zeta_{22} + 3\zeta_{22}\zeta_{12} + \zeta_{22}^2 + 2\zeta_{12}^2) - \\
- \frac{4}{3}\mu\bar{\omega}\left(\varepsilon_{11}^2 + \varepsilon_{22}^2 - \varepsilon_{11}\varepsilon_{22} + \frac{3}{2}\varepsilon_{12}^2\right).
\end{aligned}$$

The eigenvalues of the matrix corresponding to the quadratic form of the variables ζ, are equal to $0, 2 \pm 3/\sqrt{2}$. The eigenvalues of the form with respect to ε are equal to $3/2, 3/2, 1/2$. As a result we get the estimate

$$\sum \overline{\frac{\partial \sigma_{ik}}{\partial \varepsilon_{jl}}} \varepsilon_{ik}\varepsilon_{jl} \geqslant 2\mu(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{12}^2) - \frac{4}{3}\mu\overline{\omega' e_3}\left(2 + \frac{3}{\sqrt{2}}\right)(\zeta_{11}^2 + \zeta_{22}^2 + \zeta_{12}^2) -$$

$$-2\mu\overline{\omega}\left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{12}^2\right).$$

One can easily see that

$$|\zeta_{11}| \leqslant \frac{|2\bar{\varepsilon}_{11} - \bar{\varepsilon}_{22}|}{3\bar{e}_0}|\varepsilon_{11}|, \quad |\zeta_{22}| \leqslant \frac{|2\bar{\varepsilon}_{22} - \bar{\varepsilon}_{11}|}{3\bar{e}_0}|\varepsilon_{22}|, \quad |\zeta_{12}| = \frac{|\bar{\varepsilon}_{12}|}{\bar{e}_0}|\varepsilon_{12}|.$$

Since

$$|2\bar{\varepsilon}_{11} - \bar{\varepsilon}_{22}|^2 = (\bar{\varepsilon}_{11} - \bar{\varepsilon}_{22} + \bar{\varepsilon}_{11})^2 \leqslant 2\left[(\bar{\varepsilon}_{11} - \bar{\varepsilon}_{22})^2 + \bar{\varepsilon}_{11}^2 + \bar{\varepsilon}_{22}^2 + 6\bar{\varepsilon}_{12}^2\right] \leqslant$$

$$\leqslant 2\left(\frac{3}{\sqrt{2}}\right)^2\left[\frac{\sqrt{2}}{3}\sqrt{(\bar{\varepsilon}_{11} - \bar{\varepsilon}_{22})^2 + \bar{\varepsilon}_{11}^2 + \bar{\varepsilon}_{22}^2 + 6\bar{\varepsilon}_{12}^2}\right]^2 = 9\bar{e}_0^2$$

and similarly

$$|2\bar{\varepsilon}_{22} - \bar{\varepsilon}_{11}|^2 \leqslant 9\bar{e}_0^2,$$

$$|\bar{\varepsilon}_{12}| \leqslant \frac{1}{\sqrt{6}}\frac{3}{\sqrt{2}}\frac{\sqrt{2}}{3}\sqrt{(\bar{\varepsilon}_{11} - \bar{\varepsilon}_{22})^2 + \bar{\varepsilon}_{11}^2 + \bar{\varepsilon}_{22}^2 + 6\bar{\varepsilon}_{12}^2} = \frac{\sqrt{3}}{2}\bar{e}_0,$$

we have

$$|2\bar{\varepsilon}_{22} - \bar{\varepsilon}_{11}| \leqslant 3\bar{e}_0, \quad |2\bar{\varepsilon}_{11} - \bar{\varepsilon}_{22}| \leqslant 3\bar{e}_0,$$

$$|\zeta_{11}| \leqslant |\varepsilon_{11}|, \quad |\zeta_{22}| \leqslant |\varepsilon_{22}|, \quad |\zeta_{12}| \leqslant \frac{\sqrt{3}}{2}|\varepsilon_{12}|.$$

Therefore,

$$\sum \overline{\frac{\partial \sigma_{ik}}{\partial \varepsilon_{jl}}} \varepsilon_{ik}\varepsilon_{jl} \geqslant 2\mu(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{12}^2) - \left[4\mu\overline{\omega' e_3}\left(2 + \frac{3}{\sqrt{2}}\right) + 2\mu\overline{\omega}\right](\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{12}^2)$$

and for the existence of the solution for the plane deformation problem in a cylindrical domain T it is sufficient to have the inequality

$$2\left(2 + \frac{3}{\sqrt{2}}\right)\omega' e_0 + \omega < 1 - \delta$$

be true with some constant $\delta > 0$.

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