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The Fichera Function and Nonlinear Equations

Dedicated to my father, Nathan Keyfitz,
in fond recollection of our shared love of Italy

ABSTRACT. — Fichera's well-posedness results on boundary value problems for linear partial differential equations of elliptic-parabolic type mark the beginning of modern research in this area. The past decade has seen renewed interest in quasilinear partial differential equations that change type, spurred by a new approach to multidimensional conservation laws. Recent research suggests a nonlinear analogue of the indicator function developed by Fichera, as well as revealing an elegant interpretation of the Fichera function.

1. - INTRODUCTION

In the 1950's, Gaetano Fichera published two papers [15, 16] that addressed an important question: How can one determine the correct boundary conditions for a linear equation with nonnegative characteristic form? Stated differently, the question can be posed: Is it possible to give a unified theory for equations that might be elliptic in one part of their domain and parabolic in another? Fichera noted the relation between this question and work of Friedrichs [17]; Friedrichs' work in turn was inspired in part by the study of transonic flow and work of Tricomi. Although Fichera does not refer to Tricomi, it is tempting to guess that Tricomi's research might have provided some motivation.

With my co-author Suncića Čanić, I came across Fichera's work through the volume of Oleĭnik and Radkević [25], who looked at a large class of boundary value problems for second-order linear partial differential equations of the form

$$L(u) = \sum a^{ij}(x)\partial_i\partial_j u + \sum b^k(x)\partial_k u + c(x)u = f(x), \tag{1.1}$$

with $\sum a^{ij}\xi_i\xi_j \geq 0$. Equations like this appear in a number of contexts. For example, the heat operator $\partial_x^2 - \partial_t$ is of this form in space-time. The principal part of the heat operator is semi-definite at every point in space-time.

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Following Fichera, we consider a domain Ω with boundary Σ divided into four subsets. Recall that a surface with normal $\nu = (\nu_1, \ldots, \nu_n)$ is *characteristic* for the operator L in (1.1) if $\sum a^{ij}\nu_i\nu_j = 0$. We write

$$\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$$

with Σ_3 consisting of the noncharacteristic part of the boundary. (The set Σ_3 may be empty. For example, this is the case if the second-order part of L vanishes identically, a possibility considered explicitly by Fichera and in [25].) On the complement, we take ν to be an inward normal, for definiteness, and define the *Fichera function*

$$b(x) = \sum \left(b^k(x) - \sum \partial_j a^{kj}(x)\right)\nu_k. \tag{1.2}$$

We define Σ_0, Σ_1 and Σ_2 as the parts of $\Sigma \backslash \Sigma_3$ where b is zero, positive and negative, respectively. Then $\Sigma_2 \cup \Sigma_3$ is the part of the boundary on which Dirichlet data (or more general data) should be prescribed to obtain a well-posed problem.

REMARK 1: Consider the heat equation on the rectangular domain $(x, t) \in (0, 1) \times (0, T)$. The vertical sides are noncharacteristic, while the bottom forms Σ_2 and the top Σ_1. Thus one recovers the well-known correct boundary conditions for this elementary problem.

We note that, for the heat operator, the principal part of the operator makes no contribution to the Fichera function. On the characteristic part of the boundary, the determination of when it is appropriate to pose a boundary condition rests on the first-order terms in L. Such is the case also for the Tricomi operator, $y\partial_x^2 + \partial_y^2$.

Now, one way in which the Tricomi equation arises in the partial differential equations of inviscid compressible flow is through the hodograph transform applied to steady transonic flow [12]. A quasilinear system whose coefficients do not contain the independent variables explicitly becomes linear under this transformation, which interchanges the roles of dependent and independent variables. As the independent variables in the transformed equation are no longer space variables, the significance of particular boundary conditions is not completely clear. In addition, the usefulness of the hodograph tranform is confined to steady problems. When Čanić and I began to look at self-similar problems arising in two-dimensional inviscid compressible flow, we found degenerate elliptic equations arising in a natural way, but now in the physical plane (or, rather, in the physical similarity variables). The equations were quasilinear. Hence we were led to examine whether Fichera's classification could be extended to nonlinear equations. That is the topic of this paper.

2. - SELF-SIMILAR REDUCTION OF CONSERVATION LAWS

For the benefit of readers who are curious about the impact of Fichera's work in this area, but do not have a background in conservation laws, we give a brief summary of recent developments in the field. We begin with a general system of hyperbolic conservation laws in two space dimensions and time:

$$\partial_t J(U) + \partial_x F(U) + \partial_y G(U) = 0, \tag{2.1}$$

where $U \in \boldsymbol{R}^n$ is the *state variable* and F and G are nonlinear *flux functions*. The vector $J(U)$ represents the set of conserved quantities.

The most important example of a system of the form (2.1) in fluid dynamics is the Euler equations of ideal, inviscid compressible gas dynamics, either isentropic (a system of three equations for density and velocity) or adiabatic (a system of four equations obtained by considering the pressure as an additional state variable). The adiabatic system looks like

$$\begin{aligned} \rho_t + (\rho u)_x + (\rho v)_y &= 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0 \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0 \\ (\rho E)_t + (\rho uH)_x + (\rho vH)_y &= 0. \end{aligned} \tag{2.2}$$

With the notation $q^2 = u^2 + v^2$ and the relation $p = (\gamma - 1)e\rho$, we have

$$E = e + \frac{1}{2}q^2, \quad H = \gamma e + \frac{1}{2}q^2.$$

The parameter γ characterises the medium and is usually taken to be a constant greater than unity; for example, $7/5$ is the value for air. The *primitive variables* are ρ (density), u, v (velocity), and p (pressure), while the conserved quantities are ρ, the two components of momentum $(\rho u, \rho v)$ and the specific energy ρE. In this example, except at the vacuum state $\rho = 0$ the mapping between primitive and conserved variables is smooth and invertible. Thus it is reasonabe to assume, as we shall do, that J in (2.1) is either the identity or a projection.

The *isentropic* system is a simplification of (2.2) obtained by dropping the fourth equation and assuming that pressure is a function of density, usually of the form $p = A\rho^\gamma$. The *sonic speed* or *acoustic wave speed* in (2.2), $c = \sqrt{\gamma p/\rho}$, is one of the characteristic speeds of that system. A simplified system, known as the *unsteady transonic small disturbance* (UTSD) system, can be derived from (2.2) using perturbation theory, assuming that the flow lies principally in one direction, x, and that its speed q is close to sonic. In variables representing perturbation from a uniform sonic flow, the system takes the form

$$\begin{aligned} u_t + uu_x + v_y &= 0 \\ v_x - u_y &= 0. \end{aligned} \tag{2.3}$$

The reason for the absence of a time-derivative in the second equation has to do with the details of the perturbation. In this system, x represents distance along a ray, rather than the spatial variable. It is easy to verify that (2.3) is hyperbolic in appropriate directions, but that t is not a time-like variable. Writing our systems in the form (2.1) allows us to include this example.

The system (2.2) (and its isentropic reduction) is hyperbolic (symmetric or symmetrizable), and one anticipates that the Cauchy problem is well-posed. For sufficiently smooth data a short-time existence theorem can be proved using energy methods [14]. However, for almost all data, shock waves (discontinuities in the solution) form in finite time, and it is likely that entirely new techniques will be needed to establish existence of solutions of (2.1), even for specific systems such as (2.2). Even for quasilinear hyperbolic conservation laws in a single space variable, well-posedness has been established only recently, through work of Bressan and colleagues, building on earlier work of Glimm, Lax, Smoller and others; see [1, 13] for background. It is worth mentioning as well that for general systems of conservation laws in a single space variable, existence of solutions has been proved only for data that is close to a constant in some sense. This continues to be an active research area.

Based on experience with one-dimensional problems, several groups of researchers have begun a study of self-similar solutions for systems (2.1) in two space variables. Specifically, we have considered *two-dimensional Riemann data*, data that is constant in sectors of the x, y plane, and we look for solutions

$$U(x, y, t) = U\left(\frac{x}{t}, \frac{y}{t}\right) = U(\xi, \eta)$$

depending on the similarity variables ξ and η. Introducing the notation

$$A = \frac{\partial F}{\partial U} \quad \text{and} \quad B = \frac{\partial G}{\partial U}$$

for the Jacobians of the flux functions, and taking J to be linear, $J(U) = JU$ for a projection matrix J, we are led to consider reduced systems of the form

$$(A - \xi J)U_\xi + (B - \eta J)U_\eta = (F - \xi JU)_\xi + (G - \eta JU)_\eta + 2JU = 0. \tag{2.4}$$

A distinctive feature of systems like (2.2) and (2.3) is that, because of the acoustic wave cones, their self-similar reductions (2.4) change type. When (2.4) is linearized about a constant state, a pair of characteristic speeds becomes complex as one crosses into the interior of the acoustic wave cone corresponding to that state. In this respect, self-similar systems (2.4) resemble the equations of steady transonic flow mentioned in the introduction. Whether one considers similarity solutions in $(x/t, y/t)$ or steady solutions that are functions of (x, y), the reduced equations change type. Hence it is not surprising that some of the same mathematical analysis proves useful. However, in the former case the similarity space (ξ, η) still exhibits some influence of the time variable. Specifically, in regions where the reduced system is hyperbolic, it is possible to define a forward time-like direction, and it points toward the origin. We shall see that in the non-hyperbolic region this is expressed by the Fichera function.

The following proposition results from an elementary calculation, discussed in greater generality in [4].

PROPOSITION 2.1: *When J is the identity, the system* (2.4) *is hyperbolic far from the origin.*

COROLLARY 2.2: *Sectorially constant Riemann data for* (2.1) *can be turned into data for* (2.4) *imposed on a circle far from the origin.*

Hence, at least in principle, one can solve such a Cauchy problem for (2.4) outside a bounded set in which (2.4) is no longer hyperbolic.

Another example we have studied, the *nonlinear wave system*, which is loosely based on the gas dynamics equations, illustrates this. It can be reduced to a second-order equation whose self-similar form is

$$(2.5)\qquad \Big((c^2-\xi^2)\rho_\xi-\xi\eta\rho_\eta\Big)_\xi+\Big((c^2-\eta^2)\rho_\eta-\xi\eta\rho_\xi\Big)_\eta+\xi\rho_\xi+\eta\rho_\eta=0$$

where we chose $c^2=c^2(\rho)=\rho^{\gamma-1}$ in our work. This equation is hyperbolic or elliptic according as $\xi^2+\eta^2>c^2(\rho)$ or $\xi^2+\eta^2<c^2(\rho)$. It is convenient, in both this equation and the UTSD equation, that one can work with a second-order equation for a single variable. Because of the circular symmetry of (2.5), the nature of the change of type in the nonlinear wave system becomes more apparent if we write the equation in polar coordinates $x=r\cos\theta$, $y=r\sin\theta$:

$$(2.6)\qquad \Big((c^2-r^2)\rho_r\Big)_r+\frac{c^2}{r}\rho_r+\Big(\frac{c^2}{r^2}\rho_\theta\Big)_\theta=0.$$

In the case of the UTSD equation, a second-order equation with diagonal principal part is obtained by eliminating v from the self-similar reduction of equation (2.3) and changing variables $(\xi,\eta)\mapsto(-\xi-\eta^2/4,\eta)$ to obtain

$$(2.7)\qquad \big((u+x)u_x-u/2\big)_x+u_{yy}=0.$$

This equation changes type at the curve $u+x=0$.

3. - BOUNDARY VALUE PROBLEMS FOR THE SELF-SIMILAR EQUATION

The study of general initial conditions for (2.4) would be interesting, but so far we have focussed on some particular problems, especially those connected to shock reflection, a self-similar problem much studied by experimentalists and in numerical simulations. Of particular interest is the topic of weak shock reflection and the so-called 'von Neumann paradox'. This term refers to the fact that for certain angles of incidence and wave strengths there is no solution of the equations, locally around a reflection point, that can be found, even approximately, by elementary analysis. Stated in different words, the scenario that emerges from simulations and from experiments has not as yet been described by functions that satisfy the equations.

Upon applying the relatively simple initial conditions for shock reflection, which by Corollary 2.2 become boundary conditions on a curve far from the origin, one finds that the far-field problem is hyperbolic, and the boundary conditions act like initial conditions, at least locally in this part of similarity space. The procedure is detailed in

[20]. (Recall that directions pointing toward the origin correspond to forward time.) This part of the problem turns out to be straightforward. In fact, for a wide set of self-similar problems, the solution is constant, or piecewise constant, in the hyperbolic region, and only the nonhyperbolic part is unknown. Thus mathematical interest concentrates on the region where change of type occurs.

One type of boundary value problem that arises in connection with (2.5), (2.6) or (2.7) corresponds to a transonic shock, which separates a known solution in the supersonic region from a subsonic region in which the flow and the shock position must be determined simultaneously by solving a free boundary problem, with boundary values given by the Rankine-Hugoniot conditions along the shock. In several papers, we have considered variants of this problem, including a consideration of steady transonic shocks. See [6, 7, 9, 10, 18, 19]. The subsonic problem in the neighborhood of the shock is typically strictly elliptic, and the need to invoke the Fichera function does not arise. However, we also find situations in which we solve an equation like (2.5), (2.6) or (2.7) in a region where part of the boundary is degenerate. This leads to an examination of a nonlinear version of Fichera's theory. There appear to be two cases, depending on whether the degenerate portion of the boundary is fixed – prescribed by the known solution in the supersonic/hyperbolic region – or is a priori unknown.

3.1. *Fixed Degenerate Boundaries*

It is often the case that linearizing a free boundary problem results in a fixed boundary problem, coupled with perturbation equations for the free boundary. Hence one might imagine that the natural extension of Fichera's criterion to nonlinear problems takes the form of a free boundary problem. This may indeed happen, but it is not the only natural case. In fact, the first problems we studied, weak regular reflection of shocks in the UTSD and nonlinear wave system models, led to fixed boundary problems [7, 18]. Specifically, behind a regular reflection point, such as occurs when a shock travels up a wedge, one finds a constant supersonic flow immediately behind the reflection point. Imagine ρ fixed in (2.6) or u fixed in (2.7) in some domain where the equations are hyperbolic. This is the supersonic region right behind the reflection point. But now, as r decreases in (2.6) or x increases in (2.7) (meaning, in both cases, that t increases), there will be a curve, necessarily a straight line, at which the equation degenerates.

In our examples, what is the Fichera function on this line? We make progress by observing that when the equation is linear and the second-order part is in divergence form, the Fichera function simplifies. Specifically, using the summation convention, divergence form

$$\partial_j\left(a^{ij}\partial_i u\right) + \widetilde{b}^i\partial_i u = a^{ij}\partial_i\partial_j u + \big(\underbrace{\partial_j a^{ij} + \widetilde{b}^i}\big)\partial_i u$$

results in the simple formula

$$b(x) = \widetilde{b}^j \nu_j \tag{3.1}$$

for the Fichera function. This emphasizes, again, the connection between $b(x)$ and the first-order terms $\widetilde{b^j}$ of the operator. Now, it is easy to see that in (2.7) the degenerate boundary, at $u = const$, is $x = -u$, with Ω lying to the right of this vertical line. Hence the inward normal is $\nu = (1, 0)$, and $b = -\frac{1}{2} < 0$. Similarly, in (2.6), the degenerate boundary is $r = c(\rho)$, again constant. This time, $\nu = (-1, 0)$ and $b = -\frac{c^2}{r} = -c$ is again negative.

This result is reassuring, since, as a matter of logic, if one does not impose the boundary condition $\rho = c^{-1}(r)$ at the degenerate boundary in (2.6), then one has no a priori guarantee that the boundary is even degenerate. (And if it is not degenerate, then of course one expects to impose a Dirichlet or other boundary condition.) In order to obtain the tidy formula (3.1), which can be calculated equally well for linear or nonlinear equations, it is necessary to take the step of rewriting the principal part of the linear equation in divergence form, calculating what would be the definition of the Fichera function, and using that definition for a nonlinear problem (also in divergence form). The standard framework for linearizing a nonlinear operator $Q(u)$; that is,

$$L[w](u) = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} Q(w + \varepsilon u),$$

gives a different, and less satisfactory, answer. Using this approach, the Fichera function b depends upon the function w at which one has linearized. For the UTSD equation, one obtains

$$b(x) = w_x + \frac{1}{2}.$$

Now, the estimates in [3] suggest that $w_x = -\frac{1}{2}$ at the degenerate boundary in this case, leading to the conclusion $b = 0$ (and hence, incorrectly, that boundary data should not be imposed). A similar inconclusive result is found for the nonlinear wave system. On the basis of this example, we postulate that a correct generalization of the Fichera function to nonlinear equations requires rewriting the system in divergence form.

Since the sign of the Fichera function for an equation in divergence form rests entirely on the first-order derivative terms, an interesting problem for our examples would be to provide an explanation for why b has the correct sign in both these model equations. It was noted in Remark 1 that comparison with the heat equation suggests that moving in from the boundary may correspond to moving forward or backward in time. Moving into the subsonic region (which is typically close to the origin) is indeed moving in a forward timelike direction in these model problems, as the similarity variables, x/t and y/t, decrease with increasing t.

Before leaving this topic, we mention another property of the fixed degenerate boundaries that appear in self-similar reductions of the examples we have considered. Based on an expansion of leading order terms near a degenerate boundary along which the solution is constant, it appears that there are two qualitatively different types of behaviour: the solution may be Lipschitz, or it may contain a derivative singularity (typically, growth like the square root of distance from the boundary). This dichotomy was noted in our first papers on the subject and it was noted in [5] that the more singular solution characterizes linear problems, while the Lipschitz solution appears only in genuinely nonlinear systems. Furthermore, the singular solution, as found in [2] and also in [31], does not extend to a weak solution of the full problem across the sonic line, thus limiting its usefulness for the application to conservation laws [9, 20].

3.2. *The Sonic Line as a Free Boundary*

We have been using the terms supersonic and subsonic loosely to refer to the regions in physical (that is, similarity) space where the reduced equation (2.4) is, respectively, hyperbolic or not hyperbolic. A self-similar flow may contain sonic boundaries – separating one type of solution from the other – across which the solution is continuous, but not constant. One familiar example arises in steady flows when a steady but nonuniform subsonic upstream flow interacts with an airfoil and produces a supersonic region over a portion of the airfoil. Such a supersonic region usually terminates with a shock at its trailing edge, but a part of the boundary at least consists of a continuous transition.

For self-similar problems (2.4), initial (or boundary) conditions that are likely to produce continuous sonic boundaries are those that contain rarefaction waves. An example was given in [8], and a related example discussed in more detail in [21], where it was first recognized that the sonic boundary will need to be determined simultaneously with both the supersonic and the subsonic flow, as a two-phase Stefan problem. See also [20, Example 5.1]. This somewhat artificial problem becomes more interesting when one notes that it may be related to a phenomenon that has been exhibited numerically (first by Tesdall and Hunter [30], more recently by our group [28, 29]) and experimentally, by Skews and Ashworth [27]. This phenomenon, Guderley Mach reflection, consists of a very complicated and singular flow in confined to a tiny region. It is currently a candidate for an explanation of the von Neumann paradox, in the sense that it might provide a mathematical solution (albeit a very complicated one) to the equations. To put this in perspective, though, a much better understanding of this solution and its stability will be necessary if one wants to truly resolve the paradox.

Recently, Mary Chern [11] has demonstrated existence of solutions for a model problem of this type, consisting of the self-similar UTSD equation (2.7) in a bounded (subsonic) domain contained in the half-space $x + y \geq 0$, with part of its boundary on the line $x + y = 0$, and with boundary conditions $u + x = 0$ there, $u + x > 0$ on the remainder of the boundary. Chern, using a method similar to [3], shows that this problem has a solution in a weighted Sobolev space, and that the solution is Lipschitz near the degenerate boundary. Once again, one can verify that the Fichera function b is negative on the degenerate boundary, provided that one works with the operator in divergence form.

A positive result on the fixed boundary problem does not contradict the statements at the beginning of this section that one expects to find free boundary problems. The full problem will involve nonuniform flows in both the supersonic and the subsonic regions, coupled by the condition of continuity at the sonic boundary. We are beginning an investigation of prototype problems of this kind.

4. - Tricomi versus Keldysh

Geometrically, the operators $y\partial_x^2 + \partial_y^2$ (Tricomi) and $\partial_x^2 + y\partial_y^2$ (Keldysh) differ in their characteristic behaviour on the hyperbolic side of the degenerate line $y = 0$. Where the system is degenerate, the characteristic curves of opposite families become parallel, so there is a single characteristic direction. However, in the case of the Tricomi equation, this direction is transverse to the degenerate curve, while the characteristics of the Keldysh equation are tangent to the degenerate curve. This contrasting behaviour may have some implications (as yet unexplored) for the solution of the free boundary problems mentioned at the end of the previous section. The fact that the geometry of characteristics depends only on the principal part of the operator, while the Fichera function b is sensitive to the first-order part, means that, in principle, any combination of boundary behaviours could be found. This raises the interesting question of what nonlinear extensions one might expect to find in applications, such as the self-similar conservation law systems that motivated our investigation.

A model for our nonlinear problems is the *linear* wave equation in two space dimensions and time. Reduced to self-similar coordinates and transformed to polar coordinates, it has exactly the form (2.6), but now with c constant. The principal part looks like the Keldysh operator. Outside the sonic circle $r = c$, the equation is hyperbolic, and its characteristic lines/surfaces are the straight lines tangent to the sonic circle (corresponding to planes tangent to the wave cone). The linear version of (2.6), like the nonlinear equation, produces a value $b < 0$ of the Fichera function.

Questions of well-posedness for the Tricomi equation were answered by Morawetz [23, 22, 24], by looking at domains that contain both hyperbolic and elliptic subsets. This work in some sense complements Fichera's approach, as domains are chosen so that the equations change type in the interior rather than on the boundary. If one examines only the subdomain in which the characteristic form is non-negative, then the Fichera function for the linear operators considered by Morawetz, $K(y)\partial_x^2 + \partial_y^2$, is zero, as there are no lower order terms. (This operator could be written in divergence form but no new terms would be introduced.) Morawetz's results have been extended by a number of authors. See for example Payne [26], who allows lower order terms and obtains results via a construction that is independent of these terms. However, these results all require the presence of a hyperbolic region, and it is possible that singularities would appear if one attempted to take a limit that eliminated that region.

Returning to the UTSD model, we note that in the problem solved by Chern, as in any problems for (2.7) or (2.6) in which the solution is not constant on the degenerate boundary, the characteristics in the hyperbolic region are not tangent to that boundary. Hence, if one takes this condition to be the significant defining feature of the distinction between the Tricomi and Keldysh equation, then problems for (2.6) or (2.7) in which the solution is not constant on the boundary generally give us an equation of Tricomi type, coupled with a value $b < 0$ of the Fichera function.

5. - Conclusions

Manipulation of Fichera's formula for the indicator function, b, of boundary conditions for degenerate boundaries of elliptic-parabolic equations results in an expression that can be extended to quasilinear equations. Furthermore, in the (few) examples for which existence of solutions has been proved, the results are consistent with Fichera's linear theory. Use of the Fichera function motivates a solution procedure for problems which change type: Solve separately in the hyperbolic and elliptic regions, matching boundary values at the free boundary. Such a method would not be simple to implement. However, it provides an alternative to the existing theory for Tricomi-type equations, which seems to be restricted to linear and semi-linear equations.

In addition to its practical usefulness, the Fichera function contains the insight that the characteristic boundaries of degenerate elliptic (or elliptic-parabolic) equations come in two flavours: 'incoming' and 'outgoing'. The interpretation of the Tricomi equation in gas dynamics (which may have motivated Fichera's analysis) does not give rise to any straightforward interpretation involving dynamics. Boundary value problems for the Tricomi equation are notoriously difficult. Unlike problems that can be posed for the Tricomi equation, for which it makes no sense to speak of time or of motion, the degenerate boundaries of elliptic regions in the self-similar formulation of conservation laws are in some sense inflow boundaries. The Fichera function, suitably calculated, tells one exactly that.

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