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G.C. HSIAO (*) - W.L. WENDLAND (**)

Boundary Integral Equations Recast as Pseudodifferential Equations (***)

ABSTRACT. — It is known that the treatment of boundary value problems based on variational principles often leads to corresponding boundary integral equations in weak formulations. Their mapping properties can then be derived from those of the associated partial differential equations. However, this approach is restricted only to those boundary value problems which can be formulated in terms of general variational principles based on Gårding's inequality. On the other hand, boundary integral equations can also be recast as special classes of pseudodifferential equations. In this paper, we are concerned with the latter approach by applying pseudodifferential operator theory to elliptic boundary value problems. In particular, a scalar model problem and the Lamé system in linear elasticity will serve as concrete examples illustrating the basic ideas how one can apply the theory of pseudodifferential operators and their calculus to obtain basic properties for the corresponding boundary integral operators.

1. - INTRODUCTION

The class of pseudo-differential operators is essentially the smallest algebra of operators which contains all differential operators, all fundamental solutions of elliptic differential operators and all integral operators with pseudohomogeneous kernel expansions . The linear pseudo-differential operators can be characterized by generalized Fourier multipliers, the so-called symbols. The development of the theory of pseudo-differential operators has made possible to provide a unified treatment of differential and integral operators. For boundary element methods, we are generally dealing with variational solutions of linear integral operators arising from Green representations for the solutions of elliptic boundary value problems. Therefore, it is natural to recast them as

(*) Indirizzo dell'Autore: Department of Mathematical Sciences, University of Delaware Newark, Delaware 19716, U.S.A; e-mail: hsiao@math.udel.edu
(**) Indirizzo dell'Autore: Institut für Angewandte Analysis und Numerische Simulation Universität Stuttgart, Pfaffenwaldring 57, 70569, Germany;
e-mail: wendland@mathematik.uni-stuttgart.de
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elliptic pseudo-differential operators from which we may utilize all the developed calculus of pseudo-differential operators to study their properties from algebraic calculations of the corresponding symbols.

There are many excellent existing books on pseudo-differential operators (see, e.g., [8], [7], [9], [12], [14], [15], [16], [17], [21], [22], [28], and [29], to name a few). However, most of them seem to be focused on pseudo-differential operators with applications to differential operators in mind. On the other hand, in [1], [2], [4], [6], [24], [25], and [27], we may find some applications of pseudo-differential operators to integral operators. Yet in our opinion, the approaches there are either too general or too special for treating general boundary integral equations arising in applications. For our purpose, there seem to be some gaps particularly in applying the standard calculus of pseudo-differential operators to integral operators on closed boundary manifolds. It is the purpose of this paper to bridge these gaps and to give, in particular, a simple procedure for analyzing a class of boundary integral operators in most of applications, including classical boundary potentials, which can be followed by people in practice without too much deep knowledge on pseudo-differential operators. Our presentation follows [19] and is originally motivated by the work of [20] , [23] and [25]. Moreover, it can be seen as a very simple case of pseudo-differential calculus in [3] for treating elliptic boundary problems. We present the results without the proofs and refer to the details and proofs in our forthcoming monograph [19].

In order to give an idea what kind of integral operators we have in mind, we consider a model problem of the form:

(1.1) $$Pu := -\Delta u + q(x)u = f \quad \text{in} \quad \Omega \subset \mathbb{R}^3,$$

with $q(x) \geq 0, x \in \overline{\Omega}$; and $supp\, f \subset \Omega \cup \widetilde{\Omega}$, where Ω is a bounded domain with C^∞ boundary Γ, and $\widetilde{\Omega}$ is a tubular neighborhood of $\Gamma = \partial\Omega$ (see Fig. 3). Here q and f are given functions with regularity to be specified later. Our staring point is the representation of the solution of (1.1)

$$u = -\int_\Omega E(x,y)q(y)u(y)dy + \int_\Omega E(x,y)f(y)dy$$
$$+ \int_\Gamma E(x,y)\tau(y)ds_y - \int_\Gamma \frac{\partial}{\partial n_y} E(x,y)\varphi(y)ds_y \quad \text{for} \qquad x \in \Omega,$$

where $E(x,y)$ is the fundamental solution of $-\Delta$:

$$E(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|},$$

and φ, τ are the Cauchy data of the solution,

$$\gamma_0 u = u|_\Gamma = \varphi, \ \gamma_1 u = \partial u/\partial n|_\Gamma = \tau.$$

The representation of u can be rewritten as a domain integral equation in the form

(1.2) $$u - \mathcal{T}u = \mathcal{V}\tau - \mathcal{W}\varphi + \mathcal{N}f \quad \text{in} \quad \Omega,$$

where $\mathcal{V}, \mathcal{W}$ are, respectively, the simple and double layer potential operator for $-\Delta$, and $\mathcal{N}$ the Newton potential operator. Here the domain integral operator $\mathcal{T}$ is a weighted Newton operator defined by

$$\mathcal{T}u(x) := -\int_{\Omega} E(x,y)q(y)u(y)dy \quad \text{on} \quad \Omega.$$

By taking the trace of the domain integral equation (1.2), we obtain the following two boundary integral equations:

$$\varphi - \gamma_0 \mathcal{T}u = \gamma_0 \mathcal{V}\tau - \gamma_0 \mathcal{W}\varphi + \gamma_0 \mathcal{N}f \quad \text{on} \quad \Gamma \tag{1.3}$$

$$\tau - \gamma_1 \mathcal{T}u = \gamma_1 \mathcal{V}\tau - \gamma_1 \mathcal{W}\varphi + \gamma_1 \mathcal{N}f \quad \text{on} \quad \Gamma \tag{1.4}$$

The solution u of the partial differential equation (1.1), and its Cauchy data φ, τ are related by the coupled domain-boundary integral equations, (1.2)-(1.4). If one of the Cauchy data is prescribed, then the solution of (1.1) and the remaining unknown Cauchy data can be in principle determined from the domain integral equation (1.2) together with either one of the boundary integral equations (1.3) and (1.4). This simple model problem is rather special. Nevertheless, it leads to typical integral operators which we may encounter in the study of boundary value problems by boundary element methods. It is these integral operators, whose mapping properties in particular, we believe can be best obtained by using the theory of pseudo-differential operators.

The main results concerning boundary integral operators generated by pseudo-differential operators on domains (ψdOs on domains) will be presented in Section 3. Sections 2 contains definitions and basic properties of pseudo-differential operators. In addition, we show that domain integral operators of the Newton potential type belong to a special class of pseudo-differential operators of negative order on domains. In the last section, Section 4, we introduce the concept of strong ellipticity for the boundary pseudo-differential operators. The latter then leads to the Gårding inequality of the strongly elliptic boundary integral operators. As a consequence, this provides the Fredholm alternatives for the solvability of the variational formulations of the corresponding integral equations as well as straightforward stability properties of finite- and boundary element approximations in related computational methods, see [18]. We conclude the paper by applying ψdO results to simple and hypersingular boundary integral operators in linear elasticity.

2. - Domain integral Operators as ψdOs on Domains

As the linear pseudo-differential operators can be characterized by generalized Fourier multipliers, the so-called symbols, we begin with the basic definition of symbols classes for the standard pseudo-differential operators $\boldsymbol{S}^m(\Omega \times \mathbb{R}^n)$ on functions and distributions defined on some domain $\Omega \subset \mathbb{R}^n$.

Definition 2.1: *For* $m \in \mathbb{R}$, *the symbol class* $\boldsymbol{S}^m(\Omega \times \mathbb{R}^n)$ *of order* m *is defined to consist of the set of functions* $a \in C^\infty(\Omega \times \mathbb{R}^n)$ *with the property that, for any compact set* $K \Subset \Omega \subset \mathbb{R}^n$ *and multiple-indices* α, β *there exist positive constants* $c(K, \alpha, \beta)$ *such that*

$$\left| \left(\frac{\partial}{\partial x}\right)^\beta \left(\frac{\partial}{\partial \xi}\right)^\alpha a(x, \xi)\right| \leq c(K, \alpha, \beta)(1 + |\xi|)^{m-|\alpha|} \tag{2.1}$$

for all $x \in K$ *and* $\xi \in \mathbb{R}^n$.

The elements of $\boldsymbol{S}^m(\Omega \times \mathbb{R}^n)$, i.e, $a(x, \xi)$, are called *symbols* of order m.
In connection with the symbols $a \in \boldsymbol{S}^m(\Omega \times \mathbb{R}^n)$, we define the associated standard ψdO of order m.

Definition 2.2: *For* $a \in \boldsymbol{S}^m(\Omega \times \mathbb{R}^n)$, *the standard* ψdO *of order* m *is defined by*

$$\begin{aligned} A(x, D)u :=& \mathcal{F}^{-1}_{\xi \mapsto x}(a(x, \xi)\mathcal{F}_{y \mapsto \xi} u(y)) \\ =&(2\pi)^{-n} \int\limits_{\mathbb{R}^n} \int\limits_{\Omega} e^{i(x-y)\cdot\xi} a(x, \xi) u(y) dy d\xi \end{aligned} \tag{2.2}$$

for $u \in C_0^\infty(\Omega)$ *and* $x \in \Omega$.

The set of all standard ψdOs of order m is denoted by $OPS^m(\Omega \times \mathbb{R}^n)$.
One of the basic theorems is the following.

Theorem 2.3: *The operator* $A \in OPS^m(\Omega \times \mathbb{R}^n)$ *is a continuous operator*

$$A : C_0^\infty(\Omega) \to C^\infty(\Omega).$$

The operator A *can be extended to a continuous linear mapping from* $\widetilde{H}^s(K)$ *into* $H^{s-m}_{loc}(\Omega)$ *for any compact subset* $K \Subset \Omega$. *Furthermore, in the framework of distributions,* A *can also be extended to a continuous linear operator*

$$A : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega).$$

As a simple example, let us consider the differential operator $P = -\Delta + q(x)$ in the model problem (1.1). Since the Fourier transform of $-\Delta u$,

$$-\widehat{\Delta u} = |\xi|^2 \hat{u},$$

we see that

$$Pu = \mathcal{F}^{-1}_{\xi \mapsto x}(- \widehat{\Delta u} + q(x)\hat{u}) = \mathcal{F}^{-1}_{\xi \mapsto x}(|\xi|^2 + q(x))\mathcal{F}_{x \mapsto \xi} u).$$

Hence for bounded smooth function q,

$$a_P(x, \xi) = |\xi|^2 + q(x)$$

is the symbol of P and the P is a pseudodifferential operator in $OPS^2(\Omega \times \mathbb{R}^3)$.

On the other hand, let us consider the integral operator on Ω:

$$\mathcal{N}f := \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} f(y)dy.$$

If we denote the symbol of the $-\Delta$ by $a_{-\Delta}(x,\xi) = |\xi|^2$, then we could write

$$\mathcal{N}f = \mathcal{F}^{-1}_{\xi\mapsto x}(a_{-\Delta}(x,\xi)^{-1}\mathcal{F}_{y\mapsto\xi}f(y)),$$

and take

$$a_{\mathcal{N}}(x,\xi) = a_{-\Delta}(x,\xi)^{-1} = \frac{1}{|\xi|^2}$$

as the symbol of the integral operator $\mathcal{N}$ so that $\mathcal{N}$ belongs to the class $OPS^{-2}(\Omega\times\mathbb{R}^3)$. However $\frac{1}{|\xi|^2}$ is singular at $\xi = 0$, and can never be a symbol according to the definition (2.1). In order to circumvent this difficulty, we take a cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi(\xi) = 0$ for $|\xi| \le 1/2$, and $\chi(\xi) = 1$ for $|\xi| \ge 1$, and write

$$\begin{aligned}\mathcal{N}f =& \mathcal{F}^{-1}_{\xi\mapsto x}\big(\chi(\xi)a_{-\Delta}(x,\xi)^{-1}\mathcal{F}_{y\mapsto\xi}f(y)\big)\\ &+\mathcal{F}^{-1}_{\xi\mapsto x}\big((1-\chi(\xi))a_{-\Delta}(x,\xi)^{-1}\mathcal{F}_{y\mapsto\xi}f(y)\big) =: \mathcal{N}_0 f + \mathcal{R}f.\end{aligned} \tag{2.3}$$

It follows from the fundamental Paley-Wiener-Schwartz theorem [13] that the operator $\mathcal{R}$ is a smoothing operator. The operator $\mathcal{N}_0$ is a pseudodifferential operator in $OPS^{-2}(\Omega\times\mathbb{R}^3)$. In fact, the decomposition leads to $\mathcal{N}\in\mathcal{L}^m(\Omega)$, a class larger than $OPS^m(\Omega\times\mathbb{R}^n)$ (with $m=-2, n=3$ in this case).

DEFINITION 2.4: *The class of operators $\mathcal{L}^m(\Omega)$ consists of Fourier integral operators of the form*

$$Au(x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\Omega} e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)dyd\xi \tag{2.4}$$

for $u\in C_0^\infty(\Omega)$ and $x,y\in\Omega$ with the so-called amplitude function $a\in\boldsymbol{S}^m(\Omega\times\Omega\times\mathbb{R}^n)$ and with the special phase function $\varphi(x,y,\xi) = (x-y)\cdot\xi$.

The following theorem is relevant to domain integral operators.

THEOREM 2.5: *Every operator $A\in\mathcal{L}^m(\Omega)$ can be written as*

$$A = A_0(x,D) + R$$

where $A_0(x,D)\in OPS^m(\Omega\times\mathbb{R}^n)$ is properly supported and $R\in\mathcal{L}^{-\infty}(\Omega)$ with

$$\mathcal{L}^{-\infty}(\Omega) := \bigcap_{m\in\mathbb{R}}\mathcal{L}^m(\Omega) = OPS^{-\infty}(\Omega\times\Omega\times\mathbb{R}^n)$$

Some clarifications of this theorem are in order.

• *Smoothing operator.* The operator $R \in \mathcal{L}^{-\infty}(\Omega)$ is called a *smoothing operator* and it has a $C^\infty(\Omega \times \Omega)$ Schwartz kernel.

• *Symbol of a properly supported operator.* It $A_0 \in OPS^m(\Omega \times \mathbb{R}^n)$ is properly supported then

$$a(x,\xi) = e^{-ix\cdot\xi}(A_0 e^{i\xi\cdot})(x)$$

is the symbol.

• *Complete symbol.* If $A \in \mathcal{L}^m(\Omega)$ is properly supported, then we have

$$a(x,\xi) = e^{-ix\cdot\xi}(Ae^{i\xi\cdot})(x) \in \boldsymbol{S}^m(\Omega \times \mathbb{R}^n),$$

and $A = A(x,D)$. Furthermore, if $a(x,y,\xi) \in \boldsymbol{S}^m(\Omega \times \Omega \times \mathbb{R}^n)$ is an amplitude for A, we have the asymptotic expansion

$$a(x,\xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!}\left(\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(-i\frac{\partial}{\partial y}\right)^{\alpha} a(x,y,\xi)\right)_{|_{y=x}}.$$

The symbol $a(x,\xi)$ is called the *complete symbol* of A.

• *Complete symbol class.* The operator A_0 in the Theorem 2.5 is not unique. Hence for $A \in \mathcal{L}^m(\Omega)$, we choose a properly supported operator $A_0 \in \mathcal{L}^m(\Omega)$ such that $A - A_0 \in \mathcal{L}^\infty(\Omega)$, and we define

$$\begin{aligned}\sigma_A &:= \text{the equivalence class of complete symbols of } A_0\\ &\text{in } \boldsymbol{S}^m(\Omega \times \mathbb{R}^n)/\boldsymbol{S}^{-\infty}(\Omega \times \mathbb{R}^n).\end{aligned}$$

This equivalence class is called the *complete symbol class* of $A \in \mathcal{L}^m(\Omega)$.

• *Principle symbol class.* The equivalence class defined by the complete symbols of A_0 in $\boldsymbol{S}^m(\Omega \times \mathbb{R}^n)/\boldsymbol{S}^{m-1}(\Omega \times \mathbb{R}^n)$ is called the *principal symbol class* of A and denoted by σ_{mA}.

We remark that for equivalence classes in general one often uses just one representative of the class σ_A or σ_{mA}, respectively, to identify the whole class in $\boldsymbol{S}^m(\Omega \times \mathbb{R}^n)$.

We also need a subclass of $\mathcal{L}^m(\Omega)$, the so-called classical ψdO class which is very important in connection with elliptic boundary value problems and boundary integral equations. First, we need the definition of the subclass of symbols.

Definition 2.6: *A symbol $a \in \mathbf{S}^m(\Omega \times \mathbb{R}^n)$ is called classical symbol, if there exists a sequence of functions $a_{m-j} \in \mathbf{S}^{m-j}(\Omega \times \mathbb{R}^n)$, $j \in \mathbb{N}_0$ which are of homogeneous degree $m_j = m - j$ such that*

$$a \sim \sum_{j=0}^{\infty} a_{m_j}$$

in the sense that if for every $k > 0$ *there holds*

$$a - \sum_{j=0}^{k-1} a_{m_j} \in \mathbf{S}^{m_k}(\Omega \times \mathbb{R}^n).$$

The set of all classical symbols of order m *will be denoted by* $\mathbf{S}^m_{c\ell}(\Omega \times \mathbb{R}^n)$.

In Definition 2.6, by a *homogeneous degree* m_j *function* a_{m_j}, we mean a_{m_j} has the property:

$$a_{m_j}(x, t\xi) = t^{m_j} a_{m_j}(x, \xi) \ \text{ for } \ t \geq 1 \qquad \text{and} \quad |\xi| \geq 1.$$

We remark that for $a \in \mathbf{S}^m_{c\ell}(\Omega \times \mathbb{R}^n)$, the homogeneous functions $a_{m_j}(x, \xi)$ for $|\xi| \geq 1$ are uniquely determined. The leading term $a_{m_0} \in \mathbf{S}^{m_0}(\Omega \times \mathbb{R}^n)$ is called the *principal symbol*.

DEFINITION 2.7: *A* ψdO $A \in \mathcal{L}^m(\Omega)$ *is said to be classical, if its complete symbol* σ_A *has a representative in the class* $\mathbf{S}^m_{c\ell}(\Omega \times \mathbb{R}^n)$. *We denote by* $\mathcal{L}^m_{c\ell}(\Omega \times \mathbb{R}^n)$ *the set of all classical* ψ*dOs of order* m.

For $A \in \mathcal{L}^m_{c\ell}(\Omega)$, the principal symbol class σ_{mA} has a representative

$$a^0_m(x, \xi) := |\xi|^m a_m\left(x, \frac{\xi}{|\xi|}\right) \tag{2.5}$$

which belongs to $C^\infty(\Omega \times (\mathbb{R}^n \setminus \{0\}))$ and is positively homogeneous of degree m with respect to ξ; i.e.,

$$a^0_m(x, t\xi) = t^m a^0_m(x, \xi) \ \text{ for every } \ t > 0 \ \text{ and for all } \ \xi \in \mathbb{R}^n \setminus \{0\}.$$

The function $a^0_m(x, \xi)$ is called the *homogeneous principal symbol* of $A \in \mathcal{L}^m_{c\ell}(\Omega)$. If we denote by

$$a^0_{m-j}(x, \xi) := |\xi|^{m-j} a_{m-j}\left(x, \frac{\xi}{|\xi|}\right) \tag{2.6}$$

the homogeneous parts of the asymptotic expansion of the classical symbol σ_{mA}, which have the properties

$$\begin{aligned} a^0_{m-j}(x, \xi) &= a_{m-j}(x, \xi) && \text{for } |\xi| \geq 1 \text{ and} \\ a^0_{m-j}(x, t\xi) &= t^{m-j} a^0_{m-j}(x, \xi) && \text{for all } t > 0 \text{ and } \xi \neq 0, \end{aligned}$$

then σ_{mA} may be represented asymptotically by the formal sum $\sum_{j=0}^{\infty} a^0_{m-j}(x, \xi)$.

3. - BOUNDARY INTEGRAL OPERATORS GENERATED BY ψdOs ON DOMAINS

This section is devoted to the connection between classical ψdOs and the boundary integral operators. A large class of boundary integral operators, as appeared in the

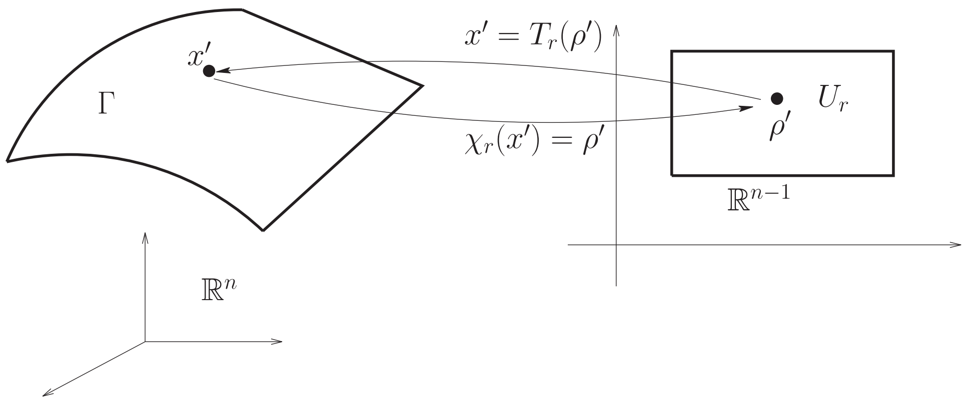

FIG. 1. - A local surface representation.

model problem, belong to the special class of classical ψdOs on compact manifolds having symbols of the so–called rational type. We are particularly interested in strongly elliptic systems of ψdOs providing the Gåring inequality (see Theorem 4.2 in Section 4). The particular class of operators having symbols of rational type enjoys many special properties such as their relation to Newton potentials which define genuine ψdOs in $\mathbb{R}^n$. The traces of their composition with tensor product distributions involving $\delta_\Gamma^{(k)}$ generate in a natural way boundary operators ψdOs on the boundary manifold.

We consider the boundary Γ of Ω as a manifold immersed into $\mathbb{R}^n$ in the sense of differential geometry and associate Γ with a so-called atlas $\mathfrak{A}$ which is a family of local charts $\{(O_r, U_r, \chi_r) | r \in I\}$. We recall that each of the local charts is a triplet: $U_r = \chi_r(O_r) \subset \mathbb{R}^{n-1}$, an open subset of the parametric space of Γ in $\mathbb{R}^{n-1}$; The parametric representation $x = T_r(\rho') = \chi_r^{(-1)}(\rho')$ for $\rho' \in U_r$ defines a parameterized patch $O_r := T_r(U_r)$ of the surface Γ (or, respectively, $U_r = \chi_r(O_r)$). The mappings T_r and χ_r are both bijective and bi-continuous, hence $T_r = \chi_r^{(-1)}$ is a homeomorphism. For an atlas we require $\Gamma = \bigcup_{r \in I} O_r$. Moreover, if $O_{rt} := O_r \cap O_t = O_t \neq \emptyset$ then the mapping

$$\Phi_{rt} := \chi_t \varnothing T_r = \chi_t \circ \chi_r^{(-1)} \circ \chi_r(O_{rt}) \to \chi_t(O_{rt}) \tag{3.1}$$

is supposed to be a sufficiently smooth diffeomorphism.

We begin with the class $\mathcal{L}^m(\Gamma)$:

DEFINITION 3.1: *Let $A : \mathcal{D}(\Gamma) \to \mathcal{E}(\Gamma)$ be a continuous linear operator. Then A is said to be in the class $\mathcal{L}^m(\Gamma)$ of ψdOs, if for every local chart $\{(O_r, U_r, \chi_r) | r \in I\}$ the associated local operator*

$$A_{\chi_r} := \chi_{r*} A \chi_r^* : \mathcal{D}(U_r) \mapsto \mathcal{E}(U_r) \tag{3.2}$$

belongs to $\mathcal{L}^m(U_r)$, where χ_{r^*} and χ_r^* are pushforward and pullback respectively.

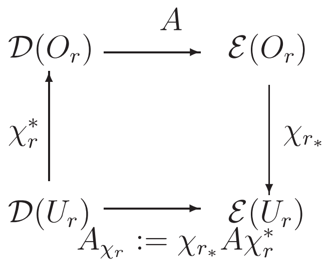

FIG. 2. - ψdOs on boundary manifold.

We are interested in boundary operators on functions given on Γ, in some n-dimensional domain that contains Γ in its interior. This amounts to study the trace of a ψdO A given on some tubular neighborhood of Γ. To this end, let $\{(O_r, U_r, \chi_r) | r \in I\}$ be any local chart of a atlas $\mathfrak{A}$ for Γ with the parametrization $T_r = \chi_r^{(-1)}$. Then for every point $x \in \widetilde{O}_r \subset \mathbb{R}^n$ where $\widetilde{O}_r$ is an open set containing O_r, we define the mapping

$$x = \Psi_r(\rho) := T_r(\varrho') + \varrho_n n(\varrho'), \;\; \varrho = (\varrho', \varrho_n) \tag{3.3}$$

for $\varrho' \in U_r \subset \mathbb{R}^{n-1}$ and $\varrho_n \in (-\varepsilon, \varepsilon)$ with $\varepsilon > 0$. Then, $\widetilde{O}_r = \Psi_r\big(U_r \times (-\varepsilon, \varepsilon)\big) \subset \mathbb{R}^n$. Note that for a smooth surface Γ and appropriate $\varepsilon > 0$, the inverse mapping $\Phi_r = \Psi_r^{(-1)}$ exists,

$$\varrho = \Phi_r(x) \quad \text{for} \quad x \in \widetilde{O}_r\,, \tag{3.4}$$

which maps $\Omega \cap \widetilde{O}_r$ onto $(\varrho', \varrho_n) \in U_r \times (-\varepsilon, 0)$. The boundary patch O_r is mapped to $U_r \times \{0\}$, i.e. $\varrho_n = 0$. We call $\widetilde{\Omega} := \bigcup_{r \in I} \widetilde{O}_r \subset \mathbb{R}^n$ a *tubular neighborhood* of Γ (see Figure 3).

Now we are in position to state the main result.

THEOREM 3.2: *Let $A \in \mathcal{L}^m_{c\ell}(\Omega \cup \widetilde{\Omega})$ with $m \in \mathbb{R}$. Then the limit*

$$Q_\Gamma v(x') = \lim_{0 > \rho_n \to 0} (\widetilde{Q}_\Gamma v)(x), \quad x = x' + \rho_n n(\chi_r(x')) \in \widetilde{Q}$$

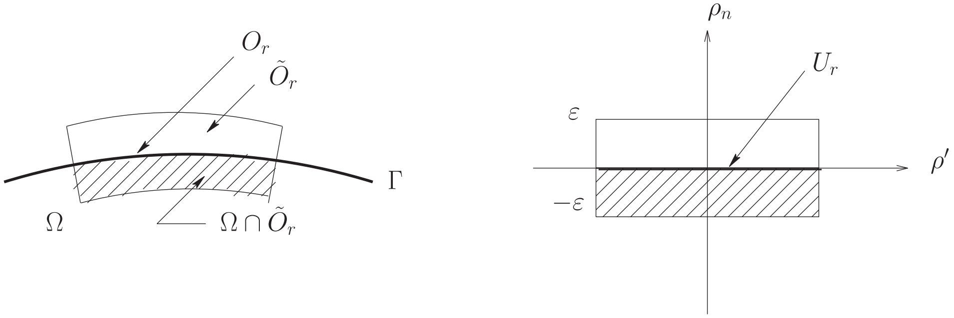

FIG. 3. - A tubular neighborhood of O_r.

always exists if $m < -1$, and $Q_\Gamma \in \mathcal{L}^{m+1}_{c\ell}(\Gamma)$ is a ψdO of order $m+1$ on Γ, where

$$(\widetilde{Q}_\Gamma v)(x) := A(v \otimes \delta_\Gamma)(x) \quad \text{for} \quad x = x' + \rho_n n(\chi_r(X'))$$

with $x \in \widetilde{O}_r \setminus \Gamma$ and $v \in C_0^\infty(O_r)$, $(x', 0) \in \Gamma$. Furthermore, the homogeneous principal symbol q^0_{m+1} of Q_Γ is given by the contour integral

$$q^0_{m+1}(x', \xi') = \frac{1}{2\pi} \int_{\mathfrak{c}} a^0_m((x', 0), (\xi', z))dz,$$

where a^0_{m+1} is the homogeneous principal symbol of A, and the contour $\mathfrak{c} \subset \mathbb{C}$ consisting of points

$$\mathfrak{c} = \big\{ z \in [-c_0, c_0] \cup \{ z = c_0 e^{i\vartheta} : 0 \geq \vartheta \geq -\pi \} \big\}$$

in the lower–half plane is clockwise oriented where $c_0 > 0$ is chosen sufficiently large so that all the poles of $a^0_{m+1}\big((x', 0), (\xi', z)\big)$ in the lower half–plane are enclosed in the interior complex domain with boundary $\mathfrak{c} \subset \mathbb{C}$.

REMARK 3.3: *We remark that Theorem 3.2 remains valid for $m \geq -1$, including the case when $m + 1 \in \mathbb{N}_0$, provided additional conditions are satisfied for $\widetilde{Q}$ (to be precise, the so-called extension and Tricomi conditions, see Theorem* 8.5.1 *in* [19]).

As a simple illustration, we consider the simple layer operator $V := \gamma_0 \mathcal{V}$ for the Laplacian in $\mathbb{R}^3$. It is clear that the Newton potential operator $\mathcal{N}$,

$$\mathcal{N}f(x) := \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} f(y)dy$$

belongs to $\mathcal{L}^m_{cl}(\Omega \cup \widetilde{Q})$ with $m = -2$. The corresponding operator $\widetilde{Q}_\Gamma$ according to Theorem 3.2 with A replaced by $\mathcal{N}$ is defined by

$$\begin{aligned} \widetilde{Q}_\Gamma \tau(x) &:= \mathcal{N}(\tau \otimes \delta_\Gamma)(x) \\ &= \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} (\tau(y') \otimes \delta_\Gamma) dy = \int_\Gamma \frac{1}{4\pi|x-y'|} \sigma(y') ds_{y'}. \end{aligned}$$

This implies that

$$V\tau(x') := Q_\Gamma \tau(x') = \lim_{x \to \Gamma} (\widetilde{Q}_\Gamma \sigma)(x) = \int_\Gamma \frac{1}{4\pi|x'-y'|} \tau(y') ds_{y'}, \; x' \in \Gamma,$$

and hence V is in $\mathcal{L}^{m+1}_{cl}(\Gamma)$ with $m + 1 = -1$.

To conclude this section, we now return to the model problem in Section 1. Based on Theorem 3.2 the mapping properties of those domain and boundary operators for q sufficiently smooth can now be summarized as follows.

• *ψdOs on Ω*. The operators $\mathcal{N} \in \mathcal{L}^{-2}_{c\ell}(\Omega \cup \widetilde{\Omega})$ and $\mathcal{T} \in \mathcal{L}^{-2}_{c\ell}(\Omega \cup \widetilde{\Omega})$ map

$$\mathcal{N} : \widetilde{H}^{-1+s}(\Omega) \to \widetilde{H}^{1+s}(\Omega)$$
$$\mathcal{T} : H^{1+s}(\Omega) \to H^{1+s+2}(\Omega), \; s > -3/2$$

• *Surface potential operators*. The operators $\mathcal{V}$ and $\mathcal{W}$ define surface potentials and map

$$\mathcal{V} : H^{-1+s+1/2}(\Gamma) \to H^{1+s}(\Omega)$$
$$\mathcal{W} : H^{1+s-1/2}(\Gamma) \to H^{1+s}(\Omega)$$

• *Operators defined on Γ by traces of ψdOs on domains*. They have the mapping properties:

$$\gamma_0\mathcal{N} : H^{-1+s}(\Omega) \to H^{1+s-1/2}(\Gamma), s-1 > -1/2$$
$$\gamma_1\mathcal{N} : H^{-1+s}(\Omega) \to H^{-1+s+1/2}(\Gamma), s-1 > 1/2$$
$$\gamma_0\mathcal{T} : H^{1+s}(\Omega) \to H^{1+s+2-1/2}(\Gamma), 1+s > -1/2$$
$$\gamma_1\mathcal{T} : H^{1+s}(\Omega) \to H^{-1+s+2+1/2}(\Gamma), 1+s > 1/2$$

• *ψdOs on Γ*. These are basic boundary integral operators belonging to the class $\mathcal{L}^{m}_{c\ell}(\Gamma)$ with $m = -1, 0, 0, 1$, respectively:

$$\gamma_0\mathcal{V} : H^{-1+s-1/2}(\Gamma) \to H^{-1+s+1/2}(\Gamma)$$
$$\gamma_0\mathcal{W} : H^{1+s-1/2}(\Gamma) \to H^{1+s-1/2}(\Gamma)$$
$$\gamma_1\mathcal{V} : H^{-1+s+1/2}(\Gamma) \to H^{-1+s+1/2}(\Gamma)$$
$$\gamma_1\mathcal{W} : H^{1+s-1/2}(\Gamma) \to H^{-1+s+1/2}(\Gamma)$$

4. - Strong Ellipticity

One of the advantages of considering integral operators as ψdOs is that the mapping properties of the boundary integral operators can be deduced by examining the symbols of the ψdos. On the other hand, Gårding's inequality for the integral operators plays a fundamental role in the variational formulation of the integral equations. The latter follows from the definition of *strong ellipticity* of ψdOs. Since the ψdOs on the boundary Γ are characterized by their representations with respect to an atlas $\mathfrak{A}$ of Γ and its local charts $= \{(O_r, U_r, \chi_r) | r \in I\}$, the concept of the *strong ellipticity* can be introduced in accordance with [26] with respect to local charts. The strong elliptic yields immediately stability properties for approximations of boundary integral equations via Galerkin methods and is in this regards fundamentally important. The definition of the strong ellipticity reads.

Definition 4.1: *Let $A = ((A_{jk}))_{p\times p}$ be a system of ψdOs $A_{jk} \in \mathcal{L}^{s_j+t_k}_{c\ell}(\Gamma)$ on Γ. We call the system $((A_{jk}))_{p\times p}$ strongly elliptic, if to the principal symbol matrices $a^0(x,\xi) = ((a^{jk0}_{s_j+t_k}(\chi_r(x), \xi)))_{p\times p}$ on the charts (O_r, U_r, χ_r) of the atlas $\mathfrak{A}$ there exists a C^∞ matrix–*

valued function $\Theta(x) = ((\Theta_{j\ell}))_{p\times p}$ *on* Γ, *and a constant* $\gamma_0 > 0$ *such that*

$$Re\zeta^\top \theta(x) a^0(x,\xi')\overline{\zeta} \geq \gamma_0 |\zeta|^2$$

is satisfied for all $x \in \Gamma, \zeta \in \mathbb{C}^p$ *and* $\xi' \in \mathbb{R}^{n-1}$ *with* $|\xi'| = 1$.

In terms of the Bessel potential on Γ defined by

$$\Lambda^\alpha = (-\Delta_\Gamma + 1)^{\alpha/2},$$

where Δ_Γ is the Laplace-Beltrami operator for the Laplacian on Γ, the following Gårding inequality holds.

THEOREM 4.2: *If* $A = ((A_{jk}))_{p\times p}$ *is a strongly elliptic system of ψdOs on* Γ, *then there exist constants* $\gamma_0 > 0$ *and* $\gamma_1 \geq 0$ *such that Gårding's inequality holds in the form*

$$Re(w, \Lambda_\Gamma^\sigma \Theta \Lambda_\Gamma^{-\sigma} A w)_{\prod_{\ell=1}^p H^{(t_\ell - s_\ell)/2}(\Gamma)} \geq \gamma_0 \|w\|^2_{\prod_{\ell=1}^p H^{t_\ell}(\Gamma)} - \gamma_1 \|w\|^2_{\prod_{\ell=1}^p H^{t_\ell - 1}(\Gamma)} \tag{4.1}$$

for all $w \in \prod_{\ell=1}^p H^{t_\ell}(\Gamma)$, *where* $\Lambda_\Gamma^\sigma = ((\Lambda_\Gamma^{s_j}\delta_{j\ell}))_{p\times p}$. *The last term in* (4.1) *defines a linear compact operator*

$$C : \prod_{\ell=1}^p H^{t_\ell}(\Gamma) \to \prod_{\ell=1}^p H^{-s_\ell}(\Gamma),$$

which is given by

$$(v, Cw)_{\prod_{\ell=1}^p H^{(t_\ell - s_\ell)/2}(\Gamma)} = \gamma_1 (v, w)_{\prod_{\ell=1}^p H^{t_\ell - 1}(\Gamma)}.$$

With this compact operator C, the Gårding inequality (4.1) *takes the form*

$$Re\big(w, (\Lambda_\Gamma^\sigma \Theta \Lambda_\Gamma^{-\sigma} A + C) w\big)_{\prod_{\ell=1}^p H^{(t_\ell - s_\ell)/2}(\Gamma)} \geq \gamma_1 \|w\|^2_{\prod_{\ell=1}^p H^{t_\ell}(\Gamma)}. \tag{4.2}$$

REMARK 4.3: *As a consequence of* (4.2), *any strongly elliptic ψdO or any strongly elliptic system of ψdOs defines a Fredholm operator of index zero and the classical Fredholm alternative holds for the corresponding sesqulinear form*

$$a(v, w) := \big(v, (\Lambda_\Gamma^\sigma \Theta \Lambda_\Gamma^{-\sigma} A w\big)_{\prod_{\ell=1}^p H^{(t_\ell - s_\ell)/2}(\Gamma)}$$

REMARK 4.4: *In the special case when* $\alpha_{jk} = 2\alpha$ *is a constant, where* 2α *is the same order for all* A_{jk}, *we may choose* $\alpha = s_j = t_k$, *and if the system is strongly elliptic, then Gårding's inequality* (4.2) *reduces to the familiar form:*

$$Re\big(w, (\Theta A + C) w\big)_{L^2(\Gamma)} \geq \gamma_1 \|w\|_\alpha \quad \text{for all} \quad w \in H^\alpha(\Gamma).$$

As a concrete example, we now consider the simplest nontrivial elliptic system of equations in linear elasticity, the Lamé system:

$$Pu = -\Delta^* \vec{u} := -\mu \Delta \vec{u} - (\mu + \lambda)\,\mathrm{grad}\,\mathrm{div}\,\vec{u} = \vec{f} \;\text{ in }\; \Omega \cup \widetilde{\Omega} \subset \mathbb{R}^3, \tag{4.3}$$

where $\vec{u}$ is the displacement field, $\vec{f}$ a given body force. The parameters μ and λ are the *Lamé constants* which characterize the elastic material (see e.g., [10], [11]), and satisfy the relation $\lambda = 2\mu\upsilon/(1-2\upsilon)$ in terms of the Poisson ratio υ. It is required that $0 \leq \upsilon < \frac{1}{2}$. The four basic boundary integral operators are strongly elliptic. In particular, we will carry out the analysis for the simple and the hyper-singular boundary operators.

The quadratic symbol matrix of the partial differential operator P can be calculated

$$\sigma_P(\xi) = \left(\left(\mu|\xi|^2\delta_{jk} + (\mu+\lambda)\xi_j\xi_k\right)\right)_{3\times 3}, \tag{4.4}$$

and hence, the characteristic determinant is given by

$$det\,\sigma_P(\xi) = \mu^2(\lambda+2\mu)|\xi|^6 .$$

Since $det\,\sigma_P(\xi) \neq 0$ for all $x \in \mathbb{R}^3$ and for all $\xi \in \mathbb{R}^3 \setminus \{0\}$, P is elliptic in the sense of Agmon-Douglis-Nirenberg and the inverse to $\sigma_P(\xi)$ defines the symbol of the Newton potential operator $\mathcal{N}$,

$$\left(\sigma_P(\xi)\right)^{-1} = \frac{1}{\mu|\xi|^4}\left(\left(|\xi|^2\delta_{jk} - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\,\xi_j\xi_k\right)\right)_{3\times 3}. \tag{4.5}$$

The Fourier inverse of $(\sigma_P)^{-1}$ defines the fundamental matrix $E(x,y)$ for P,

$$\begin{aligned} E(x,y) &= (2\pi)^{-3}\,p.f.\int_{\mathbb{R}^3}\left(\sigma_P(\xi)\right)^{-1}e^{i(x-y)d\xi}d\xi \\ &= \frac{1}{\mu(2\pi)^3}\Bigg(\Bigg(p.f.\int_{\mathbb{R}^3}\frac{1}{|\xi|^2}e^{i(x-y)\cdot\xi}d\xi\delta_{jk} \\ &\qquad + \frac{\lambda+\mu}{\lambda+2\mu}\,\frac{\partial^2}{\partial x_j\partial x_k}\,p.f.\int_{\mathbb{R}^3}\frac{1}{|\xi|^4}e^{i(x-y)\cdot\xi}d\xi\Bigg)\Bigg)_{3\times 3} \\ &= \frac{1}{8\pi\mu}\,\frac{(3\mu+\lambda)}{(\lambda+2\mu)}\left(\left(\frac{\delta_{jk}}{|x-y|} + \frac{\lambda+\mu}{3\mu+\lambda}\,\frac{(x_j-y_j)(x_k-y_k)}{|x-y|^3}\right)\right)_{3\times 3} \end{aligned} \tag{4.6}$$

We remark that since the symbol in (4.5) is homogeneous, the fundamental solution defines the Schwartz kernel of the Newton potential. Although one may still define a parametrix as in our model problem by multiplying the homogeneous symbol in (4.5) by a cut-off function $\chi(\xi)$. Alternatively, the Newton potential $\mathcal{N} \in \mathcal{L}^{-2}_{c\ell}(\mathbb{R}^3)$ can be decomposed in the form

$$\begin{aligned} \mathcal{N}\mathbf{f} &= \int_{\mathbb{R}^3} E(x,y)\mathbf{f}(y)dy \\ &= \int_{\mathbb{R}^3} E(x,y)\psi(|x-y|)\mathbf{f}(y)dy + \int_{\mathbb{R}^3} E(x,y)\left(1-\psi(|x-y|)\right)\mathbf{f}(y)dy \\ &=: \mathcal{N}_0\mathbf{f} + \mathcal{R}\mathbf{f}, \end{aligned}$$

where $\mathcal{N}_0 \in OPS^{-2}(\mathbb{R}^3 \times \mathbb{R}^3)$ is properly supported and $\mathcal{R}$ is smoothing. Here, $\psi(z) \in C_0^\infty(\mathbb{R}^3)$ is a cut-off function with $\psi(z) = 1$ for all $|z| \leq 1/2$ and $\psi(z) = 0$ for all $|z| \geq 1$.

As for boundary integral operators, we begin with the *simple-layer boundary integral operator*

$$V\tau(x') = \int_\Gamma E(x', y')\tau(y')ds_{y'}, \quad x' \in \Gamma. \tag{4.7}$$

Similar to the model problem, we define

$$\begin{aligned}\widetilde{Q}_\Gamma \tau(x) &:= \mathcal{N}(\tau \otimes \delta_\Gamma)(x) \\ &= \int_{\mathbb{R}^3} E(x, y)(\tau(y') \otimes \delta_\Gamma)dy \\ &= \int_\Gamma E(x, y')\tau(y')ds_{y'}, \quad x \notin \Gamma\end{aligned}$$

with $E(x, y)$ given by (4.6). For $x \to \Gamma$, the simple boundary integral operator

$$V\tau(x') = \lim_{\Omega \ni x \to \Gamma} \widetilde{Q}_\Gamma \tau(x) = \int_\Gamma E(x', y')\tau(y')ds_\Gamma(y') \text{ for } x' \in \Gamma$$

defines a ψdO, $V \in \mathcal{L}^{-1}_{c\ell}(\Gamma)$.

To show $V \in \mathcal{L}^{-1}_{c\ell}(\Gamma)$ is strongly elliptic, we now compute the symbol $q_{-1}(x', \xi')$. For simplicity, we identify Γ with $x_3 = 0$. For general Γ, we refer to our monograph [19]. It can be shown that the *symbol* of V can be calculated form $\sigma_P^{-1}(\xi)$ in (4.5) by the contour integral as in Theorem 3.2,

$$\begin{aligned}q_{-1}(x', \xi') &= \frac{1}{2\pi}\int_c \left(\sigma_p^{-1}(\xi', z)\right)dz = \frac{1}{2\pi}\int_c \frac{\delta_{jk}}{\mu\{z^2 + |\xi'|^2\}}dz \\ &- \frac{1}{2\pi}\int_c \frac{\lambda+\mu}{\mu(\lambda+2\mu)} \frac{1}{\{z^2+|\xi'|^2\}^2}\begin{pmatrix} \xi_1^2 & , & \xi_1\xi_2 & , & \xi_1 z \\ \xi_1\xi_2 & , & \xi_2^2 & , & \xi_2 z \\ \xi_1 z & , & \xi_2 z & , & z^2\end{pmatrix}dz\,.\end{aligned}$$

By using the residue formulas, this yields the complete homogeneous symbol matrix of V on Γ,

$$\begin{aligned}q_{-1}(\xi') &= \frac{1}{2\mu|\xi'|}\delta_{jk} - \frac{\lambda+\mu}{4\mu(\lambda+2\mu)|\xi'|^3}\begin{pmatrix} \xi_1^2 & , & \xi_1\xi_2 & , & 0 \\ \xi_1\xi_2 & , & \xi_2^2 & , & 0 \\ 0 & , & 0 & , & |\xi'|^2\end{pmatrix}, \\ &= \frac{(\lambda+3\mu)}{4|\xi'|^3\mu(\lambda+2\mu)}\begin{pmatrix} |\xi'|^2+\kappa\xi_2^2 & , & -\kappa\xi_1\xi_2 & , & 0 \\ -\kappa\xi_2\xi_1 & , & |\xi'|^2+\kappa\xi_1^2 & , & 0 \\ 0 & , & 0 & , & |\xi'|^2\end{pmatrix}\end{aligned}$$

where $\kappa = \dfrac{\lambda+\mu}{\lambda+3\mu}$. Obviously, since $0 < \kappa < 1$, this symmetric matrix is positive definite, therefore, $q_{-1}(\xi')$ satisfies the estimate

$$Re\zeta^\top q_{-1}(\xi')\overline{\zeta} \geq \gamma_0|\xi'|^{-1}|\zeta|^2 \text{ for all } 0 \neq \xi' \in \mathbb{R}^2 \text{ and } \zeta \in \mathbb{C}^3$$

with $\gamma_0 = \dfrac{1}{2(\lambda+2\mu)} > 0$. This shows that the simple-layer boundary integral operator is strongly elliptic.

Next, we consider the hypersingular boundary integral operator D,

$$D\varphi(x') := -T_{x'} \int_\Gamma T_{y'}E(x',y')^\top \varphi(y')ds_{y'} \quad \text{for} \quad x' \in \Gamma, \tag{4.8}$$

where T is the traction operator on Γ,

$$T\vec{u}_\Gamma = \left(\lambda(\text{div } \vec{u})\vec{n} + 2\mu\frac{\partial\vec{v}}{\partial n} + \mu\vec{n}\times curl\,\vec{u}\right)\Big|_\Gamma$$

We remark that this is exact the case where $m+1 \in \mathbb{N}_0$. As we mentioned earlier, additional conditions are needed in order to apply Theorem 2.5. Nevertheless, we shall carry through the analysis. We define similarly

$$\begin{aligned} \widetilde{Q}_\Gamma\varphi(x) := A(\varphi\otimes\delta_\Gamma)(x) &:= -\int_\Gamma T_x\big(T_{y'}E(x,y')\big)^\top\varphi(y')ds_{y'} \\ &= \int_{\mathbb{R}^3} k_A(x,x-y)\big(\varphi(y')\otimes\delta_\Gamma\big)dy \;\text{ for }\; x \notin \Gamma\,. \end{aligned} \tag{4.9}$$

Here the Schwartz kernel is given by

$$\begin{aligned} k_A(x,x-y) = \frac{1}{4\pi}\,\frac{\mu}{\lambda+2\mu}\,\frac{1}{|z|^5}\Bigg(\Bigg(&3z\cdot n(y)\Big\{2\mu n_j(x)z_k \\ &+ \lambda n_k(x)z_j + \lambda z\cdot n(x)\delta_{jk} - \frac{5}{|z|^2}z\cdot n(x)z_jz_k\Big\} \\ &+ 3\lambda\big\{n(x)\cdot n(y)z_jz_k + z\cdot n(x)n_j(y)z_k\big\} \\ &+ 2\mu\big\{3z\cdot n(x)z_jn_k(y) + |z|^2n_j(y)n_k(x) + |z|^2n(x)\cdot n(y)\delta_{jk}\big\} \\ &- 2(\mu-\lambda)n_j(x)n_k(y)|z|^2\Bigg)\Bigg)_{3\times3} \end{aligned} \tag{4.10}$$

where $z = x-y$. Since the hypersingular boundary integral operator is defined by

$$D\varphi(x') := \lim_{\Omega\ni x\to\Gamma}\widetilde{Q}_\Gamma\varphi(x) = \lim_{\Omega\ni x\to\Gamma}A(\varphi\otimes\delta_\Gamma)(x),$$

we need to compute the symbol of A. We note that the composition with the fundamental solution matrix given by

$$(T_yE(x,y))^\top = \mathcal{N}_W(x,y)$$

defines in the tubular neighborhood $\widetilde{\Omega} \subset \mathbb{R}^3$ of Γ a pseudo-homogeneous Schwartz kernel generating a ψdO , A_W of order -1 with symbol of rational type, since both T_y and the elastic Newton potential operator $\mathcal{N}$ are of rational type. Hence the ψdO A is the composition of $-T_x$ and A_W. The idea here is to use standard formula of ψdOs to compute the σ_A in terms of the composition of σ_{-T_x} and σ_{A_w}. In order to facilitate the computation, σ_{A_W} can be calculated in terms of its transposed operator $A_W^\top$ which has the kernel $T_xE(x,y)$. This yields the complete symbol of A:

$$(4.11)\qquad \sigma_{-T_x\circ A_W}(x,\xi) =$$

$$-\sum_{|\beta|\geq 0}\Big(\frac{1}{\beta!}\Big(\frac{\partial}{\partial\xi}\Big)^\beta i((\lambda n_j(x)\xi_k + \mu\xi_j n_k(x) + \mu\big(\xi\cdot n(x)\big)\delta_{jk}))_{3\times 3}\Big)\times$$

$$\times\Big(-i\frac{\partial}{\partial x}\Big)^\beta \sum_{|\alpha|\geq 0}\frac{1}{\alpha!}\Big(\frac{\partial}{\partial\xi}\Big)^\alpha\Big(-i\frac{\partial}{\partial x}\Big)^\alpha\times$$

$$\times\left\{\frac{i}{\mu|\xi|^4}\,\frac{\lambda+\mu}{\lambda+2\mu}\Big(\Big(\lambda|\xi|^2 n_\ell(x)\xi_m + 2\mu\big(n(x)\cdot\xi\big)\xi_\ell\xi_m\Big)\Big)_{3\times 3}\right.$$

$$\left.-\frac{i}{\mu|\xi|^2}\Big(\Big(\lambda n_\ell(x)\xi_m + \mu\xi_\ell n_m(x) + \mu\big(n(x)\cdot\xi\big)\delta_{\ell m}\Big)\Big)_{3\times 3}\right\}^\top .$$

We now confine ourselves to the computation of the principal symbol of $-T_x\circ A_W$ and of D on Γ by choosing $\alpha=\beta=0$ in (4.11). A straightforward computation yields

$$(4.12)\qquad \sigma^0_{-T_x\circ A_W}(x,\xi) = \frac{\lambda+\mu}{\mu(\lambda+2\mu)}\,\frac{1}{|\xi|^4}((\lambda^2|\xi|^4 n_j(x)n_k(x)$$

$$+\,2\lambda\mu|\xi|^2\big(n(x)\cdot\xi\big)\big(\xi_j n_k(x) + n_j(x)\xi_k\big) + 4\mu^2\big(n(x)\cdot\xi\big)^2\xi_j\xi_k))_{3\times 3}$$

$$-\,\frac{1}{\mu|\xi|^2}((\lambda^2|\xi|^2 n_j(x)n_k(x) + \mu(2\lambda+\mu)\big(n(x)\cdot\xi\big)\big(\xi_j n_k(x) + n_j(x)\xi_k\big)$$

$$+\,\mu^2\xi_j\xi_k + \mu^2\big(n(x)\cdot\xi\big)^2\delta_{jk}))_{3\times 3}\,.$$

Again we identify Γ with $x_3=0$ by setting $n_1(x)=n_2(x)=0$, $n_3(x)=1$, and $\xi_3=z$, we obtain

$$\sigma^0_{\alpha\beta} = -\mu \frac{z^2}{z^2+d_0^2}\delta_{\alpha\beta} + \Big(\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\ \frac{z^2}{(z^2+d_0^2)^2}\ -\mu\frac{1}{z^2+d_0^2}\Big)\xi_\alpha\xi_\beta\,,$$

$$\sigma^0_{\alpha 3} = \sigma^0_{3\alpha} = 4\mu\frac{\lambda+\mu}{\lambda+2\mu}\Big(\frac{z^3}{(z^2+d_0^2)^2} - \frac{z}{z^2+d_0^2}\Big)\xi_\alpha \ \text{ for } \alpha,\beta = 1,2\,;$$

$$\sigma^0_{33} = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\Big(\frac{z^4}{(z^2+d_0^2)^2} - 2\frac{z^2}{z^2+d_0^2}\Big) - \frac{\lambda^2}{\lambda+2\mu}\,.$$

By employing the contour integral formula

$$q^0_{D,1}(\xi') = \frac{1}{2\pi}\int\limits_{\mathfrak{C}} ((\sigma^0_{\ell m}(\xi',z))_{3\times 3}dz$$

in view of

$$\frac{1}{2\pi}\int\limits_{\mathfrak{C}} \frac{z^4}{(z^2+d_0^2)^2}dz = -\frac{3}{4}d_0,$$

we obtain the principal symbol of the hypersingular boundary integral operator $D \in \mathcal{L}^1_{c\ell}(\Gamma)$:

$$q^0_{D,1} = \frac{\mu}{2|\xi'|}\begin{pmatrix} |\xi'|^2+\varepsilon_1\xi_1^2 & , & \varepsilon_1\xi_1\xi_2 & , & 0 \\ \varepsilon_1\xi_2\xi_1 & , & |\xi'|^2+\varepsilon_1\xi_2^2 & , & 0 \\ 0 & , & 0 & , & (1+\varepsilon_1)|\xi'|^2 \end{pmatrix}, \tag{4.13}$$

where $\varepsilon_1 = \dfrac{\lambda}{\lambda+2\mu}$ and, hence, $|\varepsilon_1| < 1$.

Obviously, for any $\xi' \in \mathbb{R}^2$ with $|\xi'| = 1$, the matrix $q^0_{D,1}$ is symmetric and positive definite. Consequently, D satisfies the condition of strong ellipticity with $\Theta = ((\delta_{jk}))_{3\times 3}$ and $\gamma_0 = (1-|\varepsilon_1|) > 0$. Moreover, the hypersingular boundary integral operator matrix D satisfies on Γ the Gårding inequality (4.2), where $t_\ell = s_\ell = \dfrac{1}{2}$.

In closing, we remark that for details and more general results on integral equations recast as pseudodifferential equations, we refer to our forthcoming monograph [19].

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