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Variational Problems with Fractal Layers (**)

ABSTRACT. — We construct suitable energy forms with singular terms on fractal layers and state related variational principles.

Problemi variazionali con strati frattali

SUNTO. — Si costruiscono alcuni funzionali dell'energia con termini singolari su certi strati frattali e si formulano i relativi principi variazionali.

1. - INTRODUCTION

The aim of this note is to describe some variational principles for transmission problems, which involve a highly conductive layer of fractal type imbedded in an Euclidean domain. Broadly speaking, the layer is characterized with respect to the surrounding space by the property of possessing a much greater conductivity, or permeability. The flow is absorbed by the layer and it starts to diffuse within the layer much more «efficiently» than in the surrounding space, giving rise to a second order transmission condition across the layer boundary.

Boundary value problems for elliptic and parabolic equations with second order boundary conditions on a layer have been investigated since the early 70's, in connection with various engineering problems of the kind arising, for example, in the flow of oil in a fractured medium (T.R. Cannon and G. H. Meyer, 1971, [4]), or in electrostatics and magnetostatics in presence of highly conductive layers (H. Pham Huy and E.

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Sanchez-Palencia, 1974, [28]). We also refer to the recent survey of D. E. Apushkinskaya and A. I. Nazarov, 2000, [2], where the b.v.p.'s mentioned before are seen in the more general perspective of so–called *Venttsel problems*, which go back to the late 50's (Venttsel, 1959, [30]).

In many applications, like the ones studied by Cannon and Meyer, one is indeed interested in enhancing the layer absorption and diffusion. In principle – for a given conductivity of the layer material – this could be also achieved by raising as much as possible the *surface* of the layer with respect to the surrounding volume. In this respect, suitable layers of fractal type – as the one considered in this paper – may provide a geometric *surface vs volume* relation adequate to the preceding goal.

We consider here only the stationary version of the layer problem and, in addition, we confine ourselves to the formulation of a *variational principle*, involving both the volume and layer contributions to the total energy.

The method we use, in order to construct the energy functional, is new and is based on the product construction of a local energy, the Lagrangean. This is the analogue of the classical Fubini theorem for measures, applied now to the measure-valued Lagrangean forms, see [23]. This method – which we think is interesting in itself – applies, in particular, to product fractals of the type $K \times I$, where $I = [0, 1]$ and K is, *e.g.*,the planar Koch or Sierpinski curve (see [26]). If K is the Koch curve – or a piecewise Koch curve, like the snow-flake – the weak solution can be proved to be a «strong» solution, which satisfies a second order transmission condition on the layer $K \times I$. In this paper however, we shall not deal with the strong formulation of the b.v.p. satisfied by the weak solution, which is of course of great interest in the applications. This study is carried out, for K the snow-flake, in the paper of M. R. Lancia in this volume (see [19]).

Brownian motions penetrating fractals and related Dirichlet forms have been also studied in [13], [22] and [16]. We notice in this regard, that the layer surface S considered in this paper is not a nested fractal, in the sense of [21], and has spectral dimension 2. For energy functionals with singular terms, as in (2.2) below, see also [8] and [24].

In Section 2 we define the energy in the pre-fractal case, in Section 3 the energy in the fractal case. Our main results are Theorem 3.2 and its corollary Theorem 3.3 (the variational principle). Section 4 is entirely devoted to the proof of Theorem 3.2. Some important technical tools are given in the Appendix.

2. - Energy forms

We consider a 3-dimensional Euclidean domain Q containing a fractal subset S, the *layer*. Our basic model refers to the geometry illustrated in Fig. 1. Here the layer is of the type

$$S = K \times I, \tag{2.1}$$

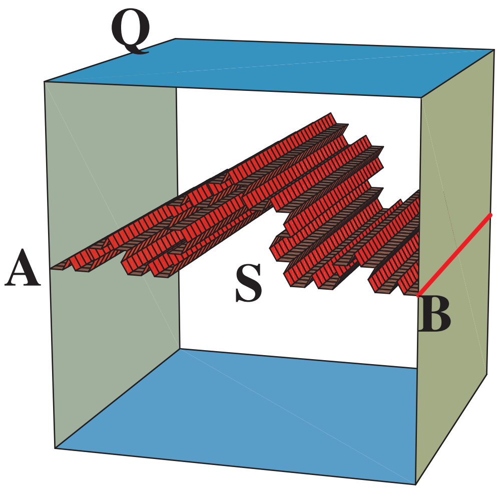

Fig. 1. – Basic model.

where K is the so-called *Koch curve* in the plane, whose endpoints are A and B:

$$\overset{\circ}{K} = K \backslash \{A, B\}$$

and $I = [0, L]$ is a real interval (for simplicity we take $L = 1$). However, K could be any finitely ramified fractal like a Sierpinski curve; (see *e.g.* [11] and [5]).

The layer is embedded in a 3-dimensional box:

$$Q := (0, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times (0, 1),$$

with coordinates (x_1, x_2, y) and the boundary of S belongs to the boundary of Q. We put

$$\Omega := (0, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right),$$

with coordinates (x_1, x_2).

In order to state the variational principle, we need to define an energy functional E of the type

$$E = E_Q + E_S \tag{2.2}$$

where E_Q gives the volume energy and E_S the energy of the layer.

We assume that the energy E_Q is simply the usual Dirichlet integral

$$E_Q[u] = \int_Q |\nabla u|^2 \, dQ \tag{2.3}$$

where $dQ = dx_1 \, dx_2 \, dy$ is the Lebesgue volume measure on $\mathbb{R}^3$.

The space of functions of finite energy on Q, vanishing on ∂Q, is the usual Sobolev space $H_0^1(Q)$. It is well known that these functions have a well defined *q.e.* representative in Q (see *e.g.* Adams-Hedberg [1]). In the following we still denote by u the *q.e.*-representative of a function $u \in H_0^1(Q)$.

We now describe the construction of the layer energy E_S, by first considering the case where K is the pre-fractal Lipschitz curve occurring in the construction of the Koch curve, see Fig. 2.

We set

$$E_S[u] = \int_I dy \int_K |D_\ell u|^2 \, d\ell + \int_K d\ell \int_I |D_y u|^2 \, dy. \tag{2.4}$$

The domain of the total energy form E is the space

$$D_0[E] = \{ u \in H_0^1(Q) : u_{|S} \in H_0^1(S) \}. \tag{2.5}$$

In (2.5), $H_0^1(Q)$ denotes the usual Sobolev space in Q and $H_0^1(S)$ the Sobolev space on S according to Necǎs definition, [27] (see also Grisvard [9]). We note that integrals in the right-hand side of (2.4) turn out to be the sum of integrals over the «faces» S_j

$$\sum_j \left(\int_{S_j} |D_\ell u|^2 + |D_y u|^2 \right) dS$$

where D_ℓ denotes the tangential derivative along the pre-fractal K and D_y the usual partial derivative in the y direction; $d\ell$ denotes the one-dimensional measure on K relative to the arc-length ℓ and dS the surface measure on S, that is, $dS = d\ell \; dy$.

By E, in the following, we shall denote both the quadratic functional and the associated (symmetric) bilinear form.

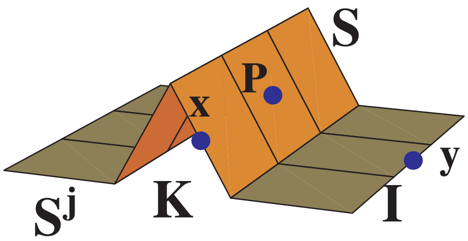

Fig. 2. – $P \in S$, $P = (x, y)$, $x = (x_1, x_2) \in K$, $y \in I$.

Proposition 2.1: *The space $D_0[E]$ given by* (2.5) *is a Hilbert space under the norm*

$$\|u\|_{D_0[E]} = (E[u])^{1/2}$$

and E, with domain $D_0[E]$, is a regular, strongly local Dirichlet form in $L^2(Q)$.

The proof follows from Proposition 5.1 of the Appendix. We do not give details here, because the proof is similar, but simpler, than the proof given in Section 4 of the following Theorem 3.2.

For definitions and main properties of Dirichlet forms see *e.g.* [6], [24] and [8].

The construction of E_S when K is fractal will be carried on in the following section.

3. - Fractal energy

We now consider the case of the layer $S = K \times I$, whose section is the fractal Koch curve K. We introduce the coordinates described in Fig. 2, where every $P \in S$ is uniquely described by its projection $x = (x_1, x_2)$ on the plane (x_1, x_2) and by its projection y on the interval $[0, 1]$, hence, $P = (x, y)$. It is well known that on the Koch curve K there exists an invariant measure μ, that is a regular positive Borel measure μ, that, after normalization, coincides with the restriction to K of the d_f-dimensional Hausdorff measure $\mathcal{H}^{d_f}$ of $\mathbb{R}^2$:

$$\mu = (\mathcal{H}^{d_f}(K))^{-1}\mathcal{H}^{d_f}|_K, \tag{3.1}$$

where

$$d_f = \frac{\ln 4}{\ln 3} \tag{3.2}$$

is the Hausdorff dimension of K. The measure μ has the property that there exists two positive constants c_1, c_2

$$c_1 r^{d_f} \leq \mu(B_e(x, r) \cap K) \leq c_2 r^{d_f}, \qquad \forall x \in K, \tag{3.3}$$

(see Hutchinson [11] and Falconer [5]), where $B_e(x, r)$ denote the Euclidean ball in $\mathbb{R}^2$. According to Jonsson and Wallin [15], we say that K is a *d_f-set.*

We recall that a *Lagrangean* $\mathcal{L}_x(u, v)$ is defined on K and

$$E_K(u, v) = \int_K \mathcal{L}_x(u, v)(dx) \tag{3.4}$$

is a regular strongly local Dirichlet form with domain $D[E_K]$ dense in $L^2(K, \mu)$. For the notion of Lagrangean in the present context see [23] and [25].

$D[E_K]$ is a Hilbert space with respect to the norm

$$\|u\|_{D[E_K]} = (E_K(u, u) + \|u\|^2_{L^2(K, \mu)})^{1/2}. \tag{3.5}$$

The functions in $D[E_K]$ possess a continuous representative, which is actually Hölder continuous on K. For these properties we refer *e.g.* to [17] and [25].

We now consider the subspace

$$D_0[E_K] = \{u \in D[E_K] : u = 0 \ \text{ on } A \text{ and } B\}. \tag{3.6}$$

$D_0[E_K]$ is a closed subspace of $D[E_K]$, hence a Hilbert space with respect to the norm (3.5). From the Poincaré inequality:

$$\|u\|_{L^2(K,\mu)} \leq c(E_K(u, u))^{1/2}, \quad u \in D_0[E_K] \tag{3.7}$$

(see *e.g.* [7] and [17]), we deduce that

$$\|u\|_{D_0[E_K]} = (E_K(u, u))^{1/2} \tag{3.8}$$

is an equivalent norm in $D_0[E_K]$. The form E_K, with domain $D_0[E_K]$ is a regular, closed and densely defined in $L^2(K, \mu)$.

We define the product Lagrangean $\mathcal{L}_{x,y}(\cdot, \cdot)$ on the fractal $S = K \times I$ by setting

$$\mathcal{L}_{x,y}(u, v)(dx, dy) = \mathcal{L}_x(u, v)(dx)\, dy + D_y u D_y v dy \mu(dx) \tag{3.9}$$

on the set

$$D_{\mathcal{L}_{x,y}} = C_0(S) \cap L^2(I; D_0[E_K]) \cap L^2(K; H_0^1(I)) \tag{3.10}$$

where

$$L^2(K) = L^2(K, \mu). \tag{3.11}$$

Here $\mathcal{L}_x(\cdot, \cdot)(dx)$ denotes the measure-valued Lagrangian of the energy for E_K of K (see (3.4)), now acting on $u(x, y)$ and $v(x, y)$ as a functions of $x \in K$ for a.e. $y \in I$ and $\mu(dx)$ is the measure defined in (3.1) acting on each section K of S for a.e. $y \in I$.

We use the notation

$$\mathcal{L}_x[u] = \mathcal{L}_x(u, u), \qquad u \in D_{\mathcal{L}_{x,y}}.$$

The following functional is well defined for every $u \in D_{\mathcal{L}_{x,y}}$

$$E_S[u] = \int_I dy \int_K \mathcal{L}_x[u](dx) + \int_K d\mu(x) \int_I |D_y u|^2\, dy. \tag{3.12}$$

We then define $D_0[E_S]$ as the completion of $D_{\mathcal{L}_{x,y}}$ in the intrinsic norm:

$$\|u\|_{D_0[E_S]} = \big(E_S[u]\big)^{1/2}, \tag{3.13}$$

and we remark that the functional in (3.12) is also well defined on $D_0[E_S]$.

By m we denote the product measure on S,

$$dm = d\mu(x)\, dy.$$

The measure m has the property that there exists two positive constants c_1, c_2 such that

$$c_1 r^d \leqslant m(B_e(P, r) \cap S) \leqslant c_2 r^d, \qquad \forall P \in S, \tag{3.14}$$

where

$$d = d_f + 1 = \frac{\log 12}{\log 3} \tag{3.15}$$

and where $B_e(P, r)$ denote the Euclidean ball in $\mathbb{R}^3$. As before, S is a *d-set* where now $d = d_f + 1$. From the Poincaré inequality on K, see (3.7), we derive easily the Poincaré inequality on S:

$$\|u\|_{L^2(S, m)} \leqslant c(E_S[u])^{1/2}, \qquad u \in D_0[E_S]. \tag{3.16}$$

PROPOSITION 3.1: *$D_0[E_S]$ is a Hilbert space under the intrinsic norm* (3.13) *and the form E_S, with domain $D_0[E_S]$, is a regular Dirichlet form in $L^2(S, m)$.*

PROOF: We first prove that E_S is a closed form in $L^2(S, m)$. Let $\{v_h\}_{h \in \mathbb{N}}$ be a Cauchy sequence in the norm

$$\|u\|_1 = \|u\|_{L^2(S, m)} + (E_S[u])^{1/2} \tag{3.17}$$

than there exists a function u in $L^2(S, m)$ that is the limit of v_h in $L^2(S, m)$. The sequence v_h is also a Cauchy sequence in the space $L^2(I; D_0[E_K])$ therefore it converges (strongly) in this space to a function that can be identified with u. Similarly the sequence v_h is a Cauchy sequence in the space $L^2(K; H_0^1(I))$ therefore it converges in this space to a function which, again, can be identified with u. This show the completeness. The form inherits the Markovian property from the form E_K on K.

We now prove that this form is regular. As the density of $C_0(S) \cap D_0[E_S]$ in $D_0[E_S]$ (in the intrinsic norm) follows from the definition of $D_0[E_S]$, then it suffices to show that $C_0(S) \cap D_0[E_S]$ is dense in $C_0(S)$ in the norm of $C_0(S)$. By Stone-Weierstrass theorem, we only need to prove that for any choice of $\widehat{P}$ and $\tilde{P}$ in $S \setminus \partial S$ with $\widehat{P} \neq \tilde{P}$ there exist f in $C_0(S) \cap D_0[E_S]$ such that $f(\widehat{P}) \neq f(\tilde{P})$. Let $\widehat{P} = (\widehat{x}, \widehat{y})$ and $\tilde{P} = (\tilde{x}, \tilde{y})$. If $\widehat{x} = \tilde{x}$ and $\widehat{y} \neq \tilde{y}$ then we choose $h \in C_0^1(I)$ such that $h(\widehat{y}) \neq h(\tilde{y})$ and we put:

$$f(x, y) = g(x)\, h(y) \tag{3.18}$$

where $g(x)$ is the capacitary potential of $\widehat{K}$ in $D_0[E_K]$, where $\widehat{K}$ is a closed subset of $\overset{\circ}{K}$ contained in K and containing the point $\widehat{x}$. If instead $\widehat{x} \neq \tilde{x}$ we choose g in $D_0[E_K]$ such that $g(\widehat{x}) \neq g(\tilde{x})$, (any function of $D_0[E_K]$ is Hölder continuous as previously mentioned), and we put

$$f(x, y) = g(x)\, h(y) \tag{3.19}$$

where h is a function in $C_0^1(I)$ such that $h(\hat{y}) = h(\tilde{y}) = 1$. ■

We now come back to the total energy functional (2.2), S being the fractal layer. The domain of the form E is the space

$$D_0[E] = \{u \in H_0^1(Q) : u_{|S} \in D_0[E_S]\}. \tag{3.20}$$

THEOREM 3.2: *The space $D_0[E]$ given by* (3.20) *is a Hilbert space under the intrinsic norm*

$$\|u\|_{D_0[E]} = (E[u])^{1/2} \tag{3.21}$$

and the form E, with domain $D_0[E]$, is a regular Dirichlet form in $L^2(Q)$.

The proof of Theorem 3.2 will be carried out in the following section.

Let us conclude this section by formulating the variational principle, for both the pre-fractal and the fractal problem.

THEOREM 3.3: *Let f be a given function in $L^2(Q)$. Let E_Q be as in* (2.3) *and E_S be given by* (2.4) *in the pre-fractal case and by* (3.12) *in the fractal case. Let E be the energy functional* (2.2) *with domain $D_0[E]$ given by* (2.5) *in the pre-fractal case and by* (3.20) *in the fractal case. Then there exists a unique $u \in D_0[E]$ that minimizes the functional*

$$\frac{1}{2} E[u] - \int_Q f u dQ$$

in $D_0[E]$.

PROOF: The proof follows from Theorem 3.2 by applying Lax-Milgram theorem to the form E.

4. - PROOF OF THEOREM 3.2

We extend the form E defined in (2.2) to the whole space $L^2(Q)$, by defining

$$E[u] = +\infty \quad \text{for every } u \in L^2(Q) \setminus D_0[E].$$

In the same way, we extend the form E_K defined in (3.4) to the whole space $L^2(K, \mu)$.

We recall, see [24], that E is closed in $L^2(Q)$ if and only if the extended quadratic functional $E(u, u)$ is lower semicontinuous on $L^2(Q)$.

We prove that the extended functional $E[u]$ is lower semicontinuous in $L^2(Q)$.

We suppose that

$$v_h \to u \quad \text{in } L^2(Q) \text{ strongly as } h \to +\infty \tag{4.1}$$

It is not restrictive to assume that there exists a subsequence, still denoted by v_h, and a constant c, independent of h, such that

$$v_h \in D_0[E], \tag{4.2}$$

where $D_0[E]$ is the domain (3.20) of the form E defined in (2.2), moreover,

$$\int_Q |\nabla v_h|^2 \, \mathrm{d}Q \leq c \tag{4.3}$$

$$\int_I dy \int_K \mathcal{L}_x[v_h](dx) \leq c \tag{4.4}$$

$$\int_K d\mu(x) \int_I |D_y v_h|^2 \, dy \leq c. \tag{4.5}$$

From (4.3) there exists a subsequence of v_h weakly converging in $H_0^1(Q)$ and hence strongly converging in $H_0^s(Q)$ for all $0 < s < 1$, in particular strongly converging in $L^2(Q)$. By (4.1), the whole sequence v_h weakly converges to u in $H_0^1(Q)$. Hence

$$\int_Q |\nabla u|^2 \, dQ \leq \underline{\lim} \int_Q |\nabla v_h|^2 \, dQ. \tag{4.6}$$

By Proposition 5.3 in the Appendix and taking into account that S is a d-set with $d = 1 + d_f$, we have that $v_{h|_S}$ and $u_{|_S}$ belong to the Besov space $B^{2,2}_{d_f/2}(S)$ and

$$v_{h|_S} \to u_{|_S} \quad \text{in } B^{2,2}_\alpha(S) \text{ strongly} \tag{4.7}$$

with $\alpha < \dfrac{d_f}{2}$. In particular, $v_{h|_S} \to u_{|_S}$ strongly in $L^2(S, m)$.

In a similar way and taking into account that K is a d_f-set (see (3.3)), we have that $v_{h|_K}$ and $u_{|_K}$ belong to the Besov space $B^{2,2}_{(d_f-1)/2}(K)$ and

$$v_{h|_K} \to u_{|_K} \quad \text{in } B^{2,2}_\alpha(K) \text{ strongly} \tag{4.8}$$

with $\alpha < \dfrac{d_f - 1}{2}$. In particular, we have strong convergence in $L^2(K, \mu)$.

From now on, for simplicity, we denote by the same symbol w the function w, its trace $w_{|_S}$ on S and also its trace $w_{|_K}$ on K, when the meaning is clear from the context.

From the closedness of the form E_K in $L^2(K, \mu)$ we deduce:

$$\int_K \mathcal{L}_x[u](dx) \leq \underline{\lim} \int_K \mathcal{L}_x[v_h](dx). \tag{4.9}$$

By making use of Fatou's Lemma,

$$\int_I dy \int_K \mathcal{L}_x[u](dx) \leq \underline{\lim} \int_I dy \int_K \mathcal{L}_x[v_h](dx). \tag{4.10}$$

Similarly, we deduce from (4.5), that there exists a subsequence of $D_y v_h$ weakly converging in $L^2(S, m)$ to a function w. By (4.1) and the uniqueness of weak derivatives, the whole sequence $D_y v_h$ converges to $D_y u$ weakly in $L^2(S, m)$ hence

$$\int_S |D_y u|^2 \, dm \leq \underline{\lim} \int_S |D_y v_h|^2 \, dm. \tag{4.11}$$

Summing up inequalities (4.11), (4.10) and (4.6), we have proved the lower semicontinuity of the functional E and this concludes the proof of the completeness of the domain $D_0[E]$.

The form is Markovian and strongly local, as it follows easily from the analogous properties which hold in $H_0^1(Q)$ and $D_0[E_S]$.

Hence to achieve the proof of Theorem 3.2 we have only to show that the form $E(\cdot, \cdot)$ is regular.

Step 1

We first see that the space $C_0(\overline{Q}) \cap D_0[E_S]$ is dense in $C_0(Q)$ with respect to the uniform norm *i. e.*

$$\overline{C_0(\overline{Q}) \cap D_0[E]}^{\|\cdot\|_\infty} = C_0(\overline{Q}). \tag{4.12}$$

By Stone-Weierstrass theorem it suffices to show that for every pair of points $(\widehat{P}, \tilde{P})$ – with $\widehat{P} \neq \tilde{P}$, $\widehat{P}$ and $\tilde{P} \in Q$ – there exists a function in $C_0(\overline{Q}) \cap D_0[E]$ such that: $f(\widehat{P}) \neq f(\tilde{P})$.

Without loss of generality we can assume $\widehat{P}, \tilde{P} \in S \setminus \partial S$. In the notation of Section 3 we set $\widehat{P} = (\widehat{x}, \widehat{y})$ and $\tilde{P} = (\tilde{x}, \tilde{y})$. If $\widehat{x} \neq \tilde{x}$, we can choose g in $D_0[E_K]$ such that $g(\widehat{x}) \neq g(\tilde{x})$. We recall that such a function exists and it is continuous. If instead $\widehat{x} = \tilde{x}$, we choose $g(x)$ to be the capacitary potential of $\widehat{K}$ in $D_0[E_K]$, where $\widehat{K}$ is closed, $\widehat{K} \subset K$ and $\widehat{x} \in \widehat{K}$. In both cases ($\widehat{x} \neq \tilde{x}$ and $\widehat{x} = \tilde{x}$), we set

$$\tilde{g} = \begin{cases} g & \text{on } K \\ 0 & \text{on } \partial\Omega \setminus K \end{cases} \tag{4.13}$$

as a function on the (closed) subset $K \cup \partial\Omega$ of $\mathbb{R}^2$. We want to extend $\tilde{g}$ to a function $\widehat{g}$ in $\mathbb{R}^2$ such that $\widehat{g}$ belongs to $C_0(\overline{\Omega}) \cap H_0^1(\Omega)$. We apply Theorem 1 of [14], (see also

Remark 4.1 below). This theorem provides, in particular, a continuous linear operator $\widehat{E}xt$,

$$\widehat{E}xt : \widetilde{B}^{2,2}_{1+\varepsilon}(K \cup \partial\Omega) \to H^{1+\varepsilon}(\mathbb{R}^2) \qquad \varepsilon \in \left(0,\, 1 - \frac{d_f}{2}\right) \tag{4.14}$$

where $\widetilde{B}^{2,2}_{1+\varepsilon}(\cdot)$ is the Besov space defined in [14] page 356. We now observe that g belongs to the Besov space $B^{2,2}_{d_f/2+\varepsilon}(K)$, (see definition 5.2 in the Appendix and Proposition 4.4 of [20]) and $\tilde{g}$ belongs to $\widetilde{B}^{2,2}_{1+\varepsilon}(K \cup \partial\Omega)$. Therefore the function $\widehat{E}xt\tilde{g}$ belongs to $H^{1+\varepsilon}(\mathbb{R}^2)$ and $\hat{g} = \widehat{E}xt\tilde{g}_{|\Omega}$ belongs to $C_0(\overline{\Omega}) \cap H^1_0(\Omega)$. Now we set

$$f(x, y) = \begin{cases} \hat{g}(x)\ \hat{f}(y) & \text{if } \hat{x} \neq \tilde{x} \\ \hat{g}(x)\ \tilde{f}(y) & \text{if } \hat{x} = \tilde{x} \end{cases} \tag{4.15}$$

where $\hat{f}$ and $\tilde{f}$ belong to $C^1_0(I)$ and are such that:

$$\begin{cases} \hat{f}(\hat{y}) = \hat{f}(\tilde{y}) = 1 \\ \tilde{f}(\hat{y}) \neq \tilde{f}(\tilde{y})\,. \end{cases} \tag{4.16}$$

This concludes the proof of (4.12).

Step 2

Now we show that the space $C_0(\overline{Q}) \cap D_0[E]$ is dense in $D_0[E]$ with respect to the intrinsic norm (3.21). Consider $u \in D_0[E]$: then $u_{|S}$ belongs to $D_0[E_S]$ and there exists a sequence of functions h_n in $C_0(S) \cap D_0[E_S]$ converging towards $u_{|S}$ in the intrinsic norm $\|\cdot\|_{D_0[E_S]}$ (see (3.13)).

As in *Step 1*, we set

$$\tilde{h}_n = \begin{cases} h_n & \text{on } S \\ 0 & \text{on } \partial Q \backslash S \end{cases} \tag{4.17}$$

as a function on the (closed) subset $S \cup \partial Q$ of $\mathbb{R}^3$.

We want to extend $\tilde{h}_n$ to a function $\hat{h}_n$ in $\mathbb{R}^3$ such that $\hat{h}_n$ belongs to $C_0(\overline{Q}) \cap H^1_0(Q)$. We apply again Theorem 1 of [14], (see also Remark 4.1 below), with a different choice of the spaces. More precisely, $\widehat{E}xt$ denotes now a continuous linear operator

$$\widehat{E}xt : \widetilde{B}^{2,2}_1(S \cup \partial Q) \to H^1(\mathbb{R}^3) \tag{4.18}$$

($\widetilde{B}^{2,2}_1(\cdot)$ is the Besov space defined in [14] page 356) and c_0 is the norm of this operator,

$$c_0 = \|\widehat{E}xt\|_{L(\widetilde{B}^{2,2}_1(S \cup \partial Q),\, H^1(\mathbb{R}^3))}\,. \tag{4.19}$$

We observe that h_n belongs to the Besov space $B^{2,2}_{d_f/2}(S)$, (see definition 5.2 in the Appendix and Proposition 3.1 of [19]) and $\tilde{h}_n$ belongs to $\widetilde{B}^{2,2}_1(S \cup \partial Q)$. Therefore the function $\widehat{E}xt\, \tilde{h}_n$ belongs to $H^1(\mathbb{R}^3)$. From the definition of the extension operator (see section 3 of [14]) we have that $\widehat{E}xt\, \tilde{h}_n$ is smooth on $\mathbb{R}^3 \setminus S$. From the properties of the Whitney decomposition and of the extension operator and taking into account that h_n is continuous on S we deduce that, for every n, $\widehat{E}xt\, \tilde{h}_n$ is continuous on $\mathbb{R}^3$.

We set

$$\widehat{h}_n = \widehat{E}xt\, \tilde{h}_n\,|_Q \tag{4.20}$$

and we have $\widehat{h}_n \in C_0(\overline{Q}) \cap H_0^1(Q)$.

We prove that the sequence $\widehat{h}_n$ converges towards a function $\widehat{u}$ in $H_0^1(Q)$. In fact $D_0[E_S]$ is continuously injected in $B^{2,2}_{d_f/2}(S)$ (see Proposition 3.1 of [19]), then we have

$$\int_0^1 \|\widehat{h}_m - \widehat{h}_n\|^2_{H_0^1(\Omega)}\, dy \leq cc_0^2\, \|h_m - h_n\|^2_{D_0[E_S]}. \tag{4.21}$$

Since the sequences h_m converges in the norm $\|\cdot\|_{D_0[E_S]}$, the right hand side of (4.21) tends to 0 as $n, m \to \infty$. We call $\widehat{u}$ the limit of $\widehat{h}_n$ in $L^2(I; H_0^1(\Omega))$ as $n \to \infty$.

Analogously, we have

$$\int_0^1 \|D_y\widehat{h}_m - D_y\widehat{h}_n\|^2_{L^2(\Omega)}\, dy \leq cc_0^2\, \|h_m - h_n\|^2_{D_0[E_S]}. \tag{4.22}$$

Therefore the sequences $D_y\widehat{h}_n$ converges towards $D_y\widehat{u}$ as $n \to \infty$.

We now observe that $u - \widehat{u}$ belongs to $H_0^1(Q\setminus S)$. In fact, $u - \widehat{u}$ belongs to $H_0^1(Q)$; moreover $\widehat{u}\,|_S = \lim \widehat{h_n}\,|_S$ in $B^{2,2}_{d_f/2}(S)$, $u|_S = \lim h_n$ in $D_0[E_S]$, and $\widehat{h}_n\,|_S = h_n$, therefore $u|_S = \widehat{u}\,|_S$. It follows that there exist a sequence f_n in $C_0^1(Q\setminus S)$ converging to $u - \widehat{u}$ in $H_0^1(Q)$. Since $(u - \widehat{u})\,|_S \equiv f_n\,|_S = 0$, the sequence converges also in $D_0[E]$ (see (3.20)). On the other hand, the sequences $\widehat{h}_n$ converges in $D_0[E]$ towards $\widehat{u}$, because $\widehat{h}_n$ converges in $H_0^1(Q)$ and $\widehat{h}_n\,|_S = h_n$ converges in $D_0[E_s]$. Hence the sequences $f_n + \widehat{h}_n$ converges towards u in $D_0[E]$. ■

REMARK 4.1: *We note that $K \cup \partial\Omega$ and $S \cup \partial Q$ are not d-sets in the sense of the definition of Section* 3 (*see* (3.3) *and* (3.14)), *so we cannot apply Proposition* 5.3 *of the Appendix to extend the functions defined in $K \cup \partial\Omega$ or in $S \cup \partial Q$. Therefore, in the Step 1 and Step 2 of the previous proof, we relay on the more sophisticated tools given in Jonsson* (*see* [14] *Theorem* 1).

5. - APPENDIX

5.1. - *Traces in Sobolev spaces*

We recall some trace results specialized to our case, by referring to [9], [27], [12] and [3] for a more general theory.

Let S be defined in (2.1) where K is the pre-fractal curve approximating the Koch curve. We define the trace operator

$$\gamma_0 u = u|_S$$

according to the following proposition (see *e.g.* [12]):

PROPOSITION 5.1: *Let S be defined in* (2.1) *where K is the pre-fractal curve approximating the Koch curve. Let* $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$. *Then* $H^{s-1/2}(S)$, *is the trace space to S of* $H^s(Q)$ *in the following sense: There exists* γ_0, *such that*

(i) γ_0 *is a continuous and linear operator from* $H^s(Q)$ *to* $H^{s-1/2}(S)$,

(ii) *there is a continuous linear operator Ext from* $H^{s-1/2}(S)$ *to* $H^s(Q)$, *such that* $\gamma_0 \circ \mathrm{Ext}$ *is the identity operator in* $H^{s-1/2}(S)$.

5.2. - *Traces in Besov spaces*

We now introduce the definition of Besov spaces on the Koch curve K and on the surface S defined in (2.1). Let us recall that S and K are *d-sets* (see (3.14), (3.3)) with $d = \dfrac{\log 12}{\log 3}$ and $d = \dfrac{\log 4}{\log 3}$ (respectively).

There are actually many equivalent definitions of the Besov spaces $B_\alpha^{p,q}$ (see for instance [29] and [15]). We recall here the one which best fits our aims and we restrict ourselves to the case $0 < \alpha < 1$, $p = q = 2$. The general setting is more involved, see [15], but it is not needed here.

Let Γ denote S or K. We define the Besov space $B_\alpha^{2,2}(\Gamma)$.

DEFINITION 5.2: *By* $B_\alpha^{2,2}(\Gamma)$, $0 < \alpha < 1$, *we denote the space of all functions f such that*

$$\|f\|_{B_\alpha^{2,2}(\Gamma)} = \|f\|_{2,\nu} + \left(\int \int_{|P-P'|<1} \frac{|f(P) - f(P')|^2}{|P - P'|^{d+2\alpha}} \, d\nu(P)\, d\nu(P') \right)^{1/2} < \infty ,$$

where ν denotes the restriction to Γ of the (normalized) d-dimensional Hausdorff measure of $\mathbb{R}^3$.

By applying Theorem 7.1 in [15] to our case, see also [29] and [31] we define the trace operator

$$\gamma_0 u = u|_\Gamma$$

according to the following:

PROPOSITION 5.3: *Let Γ denote S or K. Let* $s \in \left(\frac{3-d}{2}, \frac{5-d}{2}\right)$, $\beta = s - \frac{3-d}{2}$. *Then* $B_\beta^{2,2}(\Gamma)$ *is the trace space of* $H^s(Q)$ *on Γ in the following sense: There exists* γ_0, *such that*

(i) γ_0 *is a continuous linear operator from* $H^s(Q)$ *to* $B_\beta^{2,2}(\Gamma)$.

(ii) *there is a continuous linear operator* Ext *from* $B_\beta^{2,2}(\Gamma)$ *to* $H^s(Q)$ *such that* $\gamma_0 \circ \mathrm{Ext}$ *is the identity operator in* $B_\beta^{2,2}(\Gamma)$.

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