ABSTRACT. — A second order transmission problem across either a fractal layer $S$ or the corresponding prefractal layer $S_h$ is studied. Existence, uniqueness and regularity results for the weak solution, in both cases, are established.

**Problemi di trasmissione del secondo ordine attraverso una superficie frattale**

SUNTO. — Si studia un problema di trasmissione del secondo ordine in cui lo strato è una superficie frattale $S$ oppure la corrispondente superficie prefrattale $S_h$. Si provano risultati di esistenza, unicità e regolarità della soluzione variazionale sia nel caso frattale che prefrattale.

**INTRODUCTION**

There is a huge literature dealing with transmission problems. Transmission problems arise naturally in various fields (see [44]). For instance, in electrostatics and magnetostatics the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem (in this regard see the paper by Pham Huy and Sanchez Palencia [42] and the references listed in).

Another completely different field is that of «hydraulic fracturing» (see the paper of Cannon and Meyer [9]) used in order to increase the flow of oil from a reservoir into a producing oil well. We refer to [9] for more details. Further examples can be found in Dautray and Lions [11].

In all these applications, the mathematical model is an elliptic or parabolic bound-
ary value problem involving a transmission condition on the interface (layer) either of order zero, one or two.

In this paper we will deal with a model problem considered in [42]. This is a second order transmission problem with a «flat» smooth layer, formally stated as

\[
(P_0) \quad \begin{cases} 
-\Delta u = f & \text{in } \Omega^i, \ i = 1, 2 \ j \\
-c_0 \Delta_i u = \left[ \frac{\partial u}{\partial n} \right] & \text{on } S, \ j \\
u = 0 & \text{on } \partial\Omega, \ j \\
u^1 = u^2 & \text{on } S, \ jv \\
u = 0 & \text{on } \partial S \ jv 
\end{cases}
\]

where \(\Omega\) is a regular bounded open set in \(\mathbb{R}^3\), say a regular cylinder and \(S\) is a cross section, say a disk. \(S\) divides \(\Omega\) in two subsets \(\Omega^1\) and \(\Omega^2\), \(u^i\) denotes the restriction of \(u\) to \(\Omega^i\), \([u]\) = \(u^1 - u^2\) denotes the jump of \(u\) across \(S\) and \(\left[ \frac{\partial u}{\partial n} \right] = \frac{\partial u}{\partial n_1} + \frac{\partial u}{\partial n_2}\) the jump of the normal derivatives across \(S\), \(n_i\) being the outward normal vector; \(\Delta\) is the Laplace operator in \(\mathbb{R}^3\) and \(\Delta_i\) denotes the tangential Laplacian on \(S\), \(f\) is a given function in \(L^2(\Omega)\).

The existence and the uniqueness of the weak solution of problem \(P_0\) are proved in [42]. No regularity results are proved. As far as we know, the question of the regularity of solutions of this kind of transmission problems has been firstly addressed in [32], where it has been proved that the solution \(u\) is continuous in \(\overline{\Omega}\), the restrictions \(u^i\) are in \(H^2(\Omega^i)\) and the transmission condition \(jj\) holds in the \(L^2\)-sense.

Problems of this type – with a transmission condition of the second order as in \((P_0)\) – come out naturally from electrostatics and magnetostatics or from problems of «hydraulic fracturing». In the context of electrostatics or magnetostatics condition \(jj\) is typical of infinitely conductive layers and \(c_0\) is a positive constant representing the dielectric constant or the magnetic permeability; in hydraulic fracturing \(c_0\) depends on the permeability of the reservoir and on the fluid properties.

From the point of view of the applications, mentioned before, it would be interesting to study those model-problems in which the absorption of the tensions, electric conduction or flows is the relevant effect and hence the surface effects have to be enhanced. In this context fractal layers provide new interesting settings.

In the present paper we study problem \((P_0)\) when the layer \(S\) is not flat but is a fractal surface. In this case, the presence of a fractal interface changes drastically the nature of the problem. While in the problems considered in [42], [9] and in [32] the layer has the same Hausdorff dimension of the boundary of \(\Omega\), now in the fractal case, the layer has Hausdorff dimension greater than that of \(\partial\Omega\).

Problem \((P_0)\) can be seen as the Euler conditions satisfied by the minimizer of a suitable energy functional. Variational principles of this kind have been stated in [39]...
both in the case when $S$ is a pre-fractal or a fractal surface of the Koch type. In particular in [39] the existence of a variational (weak) solution (minimizer) is proved. In this paper we will consider the case in which $S$ is a fractal surface of the von Koch snowflake type.

We shall be mainly concerned with the «strong» problem $(P)$ associated with this geometry. We will also study the approximating problems $(P_h)$ – with layer the pre-fractal interface $S_h$ approximating $S$.

The paper is organized as follows. In Section 1 we recall the definition of the von Koch snowflake $F$ and we describe the geometry of both the fractal and pre-fractal layer $S = F \times I$ and $S_h = F_h \times I$, $I = [0, 1]$. In section 2 we describe the relevant functional spaces which will be used as well as the trace theorems on $S$ and $S_h$ respectively. In section 3 we consider the variational formulation for the fractal transmission problem $(P)$; existence and uniqueness for the weak solution of problem $(P)$ in a «convenient» space are proved (see Proposition 3.5) and the transmission condition is interpreted in a suitable duality sense (see Theorem 3.7). In section 4 we consider the variational formulation for the pre-fractal transmission problem $(P_h)$; existence and uniqueness for the weak solution of problem $(P_h)$ in a convenient space are proved (see Proposition 4.2), regularity results for the weak solution of problem $(P_h)$ and a strong interpretation of the transmission condition in the $L^2$ - sense are established in Theorem 4.3.

We think that Theorem 4.3 can be useful to build approximating numerical schemes for the solutions of transmission problems with fractal layers.

1. Geometry of the Fractal Layers $S$ and $S_h$

In the paper, by $|P-P_0|$ we denote the Euclidean distance in $\mathbb{R}^D$ and the Euclidean balls by $B(P_0, r) = \{P \in \mathbb{R}^D: |P - P_0| < r\}$, $P \in \mathbb{R}^D$, $r > 0$. By the Von Koch snowflake $F$, we will denote the union of three co-planar Von Koch curves (see [12]) $K_1$, $K_2$ and $K_3$ as shown in Figure 1.a. We assume that the junction points $A_1$, $A_3$ and $A_5$ are the vertices of a regular triangle with unit side length, i.e. $|A_1 - A_3| = |A_1 - A_5| = |A_3 - A_5| = 1$. Obviously, $F$ can also be seen as the union of the three other standard von Koch curves $K_4$, $K_5$ and $K_6$ (with junction points $A_2$, $A_4$ and $A_6$), as shown in Figure 1.b. From now on we assume that a clockwise orientation is given on $F$.

The Hausdorff dimension of the von Koch snowflake is given by $d_f = \frac{\ln 4}{\ln 3}$. This fractal is no longer self-similar (and hence, not nested).

One can define, in a natural way, a finite Borel measure $\mu_F$ supported on $F$ by

$$\mu_F := \mu_1 + \mu_2 + \mu_3,$$

where $\mu_i$ denotes the normalized $d_f$-dimensional Hausdorff measure, restricted
Fig. 1. – a: first decomposition; b: second decomposition.

to $K_i$, $i = 1, 2, 3$, it also holds that $\mu_F = \mu_4 + \mu_5 + \mu_6$, where $\mu_i$ is the normalized $d_i$-dimensional Hausdorff measure, restricted to $K_i$, $i = 4, 5, 6$.

$K_1$ is the uniquely determined self–similar set with respect to four suitable contractions $\psi_1^{(1)}, \ldots, \psi_4^{(1)}$, with the same ratio $1/3$ (see Section 3.2 in [13]). We approximate $K_1$ by a sequence of finite sets of points. Let $V_0^{(1)} := \{A_1, A_3\}$, $V_4^{(1)} := \psi_1^{(1)} \circ \cdots \circ \psi_4^{(1)}(V_0^{(1)})$ and

$$\text{(1.2)} \quad V_n^{(1)} := \bigcup_{j_1, \ldots, j_n = 1}^4 V_{j_1, \ldots, j_n}^{(1)}.$$

We set $V_4^{(1)} := \bigcup_{n \geq 0} V_n^{(1)}$. It holds that $K_1 = \overline{V_4^{(1)}}$. Let $K_1^{(0)}$ denote the unit segment whose endpoints are $A_1$ and $A_3$ and $K_{11}^{(1)} := \psi_1^{(1)} \circ \cdots \circ \psi_4^{(1)}(K_1^{(0)})$.

For $n > 0$, we denote

$$\text{(1.3)} \quad F_n^{(1)} = \{\psi_1^{(1)} \circ \cdots \circ \psi_{4n}^{(1)}(K_1^{(0)}), j_1, \ldots, j_n = 1, \ldots, 4\}.$$

We set $K_4^{(1)} = \bigcup_{j = 1}^4 \psi_1^{(1)}(K_1^{(0)})$, $K_4^{(n+1)} = \bigcup_{M \in F_n^{(1)}} \bigcup_{j = 1}^4 \psi_1^{(1)}(M)$, here $M$ denotes a segment of the $n+1$-th generation; $K_4^{(n+1)}$ the polygonal curve and $V_n^{(1)}$ the set of its vertices.

In a similar way, we approximate the von Koch curves $K_2, \ldots, K_6$ by the sequences $(V_n^{(2)})_{n \geq 0}, \ldots, (V_n^{(6)})_{n \geq 0}$, and denote their limits by $V_{\infty}^{(2)}, \ldots, V_{\infty}^{(6)}$, and the corresponding polygonal curves by $K_2^{(n+1)}, \ldots, K_6^{(n+1)}$.

In order to approximate $F$, we define the increasing sequence of finite sets of points $\mathcal{V}_n := \bigcup_{i = 1}^3 V_{n}^{(i)}$ and $\mathcal{V}_n := \bigcup_{n \geq 1} \mathcal{V}_n$. It holds that $\mathcal{V}_n =$
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\[ F_{n+1} = \bigcup_{i=1}^{5} K_i^{(n+1)} = \bigcup_{i=4}^{6} K_i^{(n+1)} \]

the closed polygonal curve approximating \( F \) at the \( (n+1) \)-th step.

The measure \( \mu_F \) has the property that there exist two positive constants \( c_1, c_2 \)

\[ c_1 r^d \leq \mu_F(B(P, r) \cap F) \leq c_2 r^d, \quad \forall P \in F \]

where \( d = d_f = \frac{\log 4}{\log 3} \) and where \( B(P, r) \) denotes the Euclidean ball in \( \mathbb{R}^2 \). As \( \mu_F \) is supported on \( F \), it is not ambiguous to write in (1.5) \( \mu_F(B(P, r)) \) in place of \( \mu_F(B(P, r) \cap F) \). In the terminology of the following section we say that \( F \) is a \( d \)-set with \( d = d_f \).

**Remark 1.1:** The von Koch snowflake can be also regarded as a fractal manifold (see [13] Section 3.2).

Let \( Q \) denote a bounded open set in \( \mathbb{R}^3 \); in our basic model \( Q \) denotes the parallelepiped \( Q = (-1, 1)^2 \times (0, 1) \) and \( S \) denotes a «cylindrical» layer in \( Q \) of the type \( S = F \times I \), where \( I = [0, 1] \) and \( F \) is the von Koch snowflake. We assume that \( S \) is located in a median position inside \( Q \) and divides \( Q \) in two sub-domains \( Q_1 \) and \( Q_2 \) (see Figure 2).

We give a point \( P \in S \) the cartesian coordinates \( P = (x, y) \), where \( x = (x_1, x_2) \) are the coordinates of the orthogonal projection of \( P \) on the plane containing \( F \) and \( y \) is the coordinate of the orthogonal projection of \( P \) on the \( y \)-line containing the interval \( I: P = (x, y) \in S, x = (x_1, x_2) \in F, \ y \in I \).

One can define in a natural way, a finite Borel measure \( m \) supported on \( S \) as the

Fig. 2. – Two different viewpoints of the domain \( Q \) and the layer \( S \).
product measure

\[ dm = d\mu_y \times dy \]

where \( dy \) denotes the one-dimensional Lebesgue measure on \( I \). The measure \( m \) has the property that there exist two positive constants \( c_1, c_2 \)

\[ c_1 r^d \leq m(B(P, r) \cap S) \leq c_2 r^d, \quad \forall P \in S \]

where \( d = d_f + 1 = \frac{\log 12}{\log 3} \) and where \( B(P, r) \) denotes the Euclidean ball in \( \mathbb{R}^3 \). As \( m \) is supported on \( S \), it is not ambiguous to write in (1.7) \( m(B(P, r), OS) \).

We give a point \( P \in S \) the Cartesian coordinates \( P = (x, y) \), where \( x = (x_1, x_2) \) are the coordinates of the orthogonal projection of \( P \) on the plane containing \( F_h \) and \( y \) is the coordinate of the orthogonal projection \( P \) on the \( y \)-line containing the interval \( I \).

2. - Functional Spaces and Traces

2.1. - Sobolev spaces

Let \( Q \) be a polyhedral domain: just to fix the ideas, the parallelepiped as in the previous section. For every integer \( b \geq 1 \) let \( S_b \) be the prefractal surface approximating the Koch-type surface \( S \) and let us denote every affine «face» of \( S_b \) by \( S_b(j) \). \( S_b \) divides \( Q \) into two subsets \( Q_{1b} \) and \( Q_{2b} \).

By \( L^2(\cdot) \) we denote the Lebesgue space with respect to the Lebesgue measure on subsets of \( \mathbb{R}^3 \), which will be left to the context whenever that does not create ambiguity. Let \( T \) be a closed set of \( \mathbb{R}^3 \), by \( C(T) \) we denote the space of continuous functions on \( T \), by \( C_0(T) \) we denote the space of continuous functions vanishing on \( \partial T \). Let \( \mathcal{S} \) be an open set of \( \mathbb{R}^3 \), by \( H^r(\mathcal{S}) \), where \( r \in \mathbb{R}^+ \) we denote the usual (possibly fractional) Sobolev spaces (see Necas [41]); \( H^0(\mathcal{S}) \) is the closure of \( C(\mathcal{S}) \), (the smooth functions with compact support on \( \mathcal{S} \)), with respect to the \( \| \cdot \|_r \)-norm. By \( H^2_{loc}(\mathcal{S}) \) we denote the space of functions \( u \in H^2(\partial \mathcal{S}) \) on every open set \( \partial \subset \mathcal{S} \). In the following, we will make use of trace spaces on boundaries of polyhedral domains of \( \mathbb{R}^3 \).

By \( H^1(S_b) \), \( 0 < r \leq 1 \) we denote the Sobolev space on \( S_b \), defined by local Lipschitz charts as in Necas [41]. By \( H^1(S_b) \) we denote the closure in \( H^1(S_b) \) of the set

\[ \{ v \vert_{\partial Q^b}; v \in C^\infty(Q^b); v \text{ vanishes in a neighborhood of } S_b \} \]

It is to be pointed out that the Sobolev space \( H^1(S_b) \) (defined in [41])
coincides, with equivalent norms, with the trace space defined in Buffa e Ciarlet in [7] (see also [6] for the case of polygonal boundaries).

When \( r > 1 \) the trace spaces on non smooth boundaries can be defined in different ways; we now recall two trace theorems, specialized to our case, which we will use, referring to [16], [17] and [10] for a more general discussion.

For \( f \in H^s(G) \), we put

\[
\gamma_0 f(P) = \lim_{r \to 0} \frac{1}{|B(P, r) \cap \tilde{G}|} \int_{B(P, r) \cap \tilde{G}} f(Q) \, dQ
\]

at every point \( P \in \mathbb{T} \) where the limit exists. It is known that the limit (2.1) exists at quasi every \( P \) with respect to the \((s, 2)\)-capacity [1].

We now recall the results of Theorem 3.1 in [19] specialized to our case, referring to [17] and [10] for a more general discussion.

**Proposition 2.1:** Let \( Q_i \), \( Q_i^j \), \( Q_i^\delta \) and let \( \Gamma \) denote \( S_h \), \( \partial Q_i^\delta \), \( \partial Q_i^j \), \( \partial Q_i \). Let \( 1/2 < s < 3/2 \). Then \( H^{s - 1/2} (\Gamma) \), is the trace space to \( \Gamma \) of \( H^s(G) \) in the following sense:

1. \( \gamma_0 \) is a continuous and linear operator from \( H^s(G) \) to \( H^{s - 1/2} (\Gamma) \),

2. there is a continuous linear operator \( \text{Ext} \) from \( H^{s - 1/2} (\Gamma) \) to \( H^s(G) \), such that \( \gamma_0 \circ \text{Ext} \) is the identity operator in \( H^{s - 1/2} (\Gamma) \).

In particular we use the Lions-Magenes space \( H_{0,0}^{1/2} (S_h) \) defined as

\[
H_{0,0}^{1/2} (S_h) = \{ \theta \in L^2(S_h) : \exists \nu \in H^1_0(Q) : \gamma_0 \nu = \theta \text{ on } S_h \},
\]
equipped with the quotient norm

\[
\|\theta\|_{H_{0,0}^{1/2} (S_h)} = \inf \{ \|\nu\|_{H^1_0(Q)} : \nu \in H^1_0(Q), \gamma_0 \nu = \theta \text{ on } S_h \}.
\]

We note that \( H_{0,0}^{1/2} (S_h) \) is the subspace of the functions \( \theta \in H^1(S_h) \) for which the trivial extension \( \bar{\theta} \), \( \bar{\theta} = \theta \) on \( S_h \) and \( \bar{\theta} = 0 \) on \( \partial Q \setminus S_h \) belongs to the space \( H^1(\partial Q) \); (see [7]).

Finally \( (H_{0,0}^{1/2} (S_h))^\prime \) denotes the dual space of \( H_{0,0}^{1/2} (S_h) \).

For the present application, we will make use also of the Sobolev trace space \( H^2(S_h) \); in the author’s opinion, the definition of this Sobolev space (on polyhedral boundaries), which best fits to this problem, is that given in Section 2 of [7].

We set

\[
H^2(S_h) = \{ \varphi \in H^1(S_h) : \varphi|_{S_h^\delta} \in H^2(S_h^\delta), \forall S_h^\delta \in S_h \}.
\]

**Proposition 2.2:** \( H^2(S_h) \), is the trace space to \( Q_i^\delta \) of \( H^2(Q_i^\delta) \) in the following sense:

(i) \( \gamma_0 \) is a continuous and linear operator from \( H^2(Q_i) \) to \( H^2(S_h) \),

(ii) there is a continuous linear operator \( \text{Ext} \) from \( H^2(S_h) \) to \( H^2(Q_i) \), such that \( \gamma_0 \circ \text{Ext} \) is the identity operator in \( H^2(S_h) \).

(see Theorem 2.4 in [7] and Theorem 1 in [17]).

In the sequel we denote by the symbol \( f|_{S_h} \) the trace \( g_0 f \) to \( S_h \).

2.2. - Besov spaces

**Definition 2.3:** Let \( T \subset \mathbb{R}^D \) be a closed non-empty subset. It is a \( d \)-set \((0 < d \leq D)\) if there exists a Borel measure \( \mu \) with \( \text{supp} \mu = T \) such that for some constants \( c_1 = c_1(T) > 0 \) and \( c_2 = c_2(T) > 0 \)

\[
c_1 r^d \leq \mu(B(P, r)) \leq c_2 r^d \quad (P \in T, 0 < r \leq 1).
\]

Such a \( \mu \) is called a \( d \)-measure on \( T \).

**Proposition 2.4:** The set \( F \) is a \( d \)-set with \( d = k_f \). The measure \( \mu_F \) is a \( d \)-measure.

The layer \( S \) is a \( d \)-set with \( d = k_l + 1 \). The measure \( m \) is a \( d \)-measure.

See [13] and [39].

Throughout the paper \( c \) will denote different constants.

We now come to the definition of the Besov spaces. Actually there are many equivalent definitions of these spaces see for instance [43] and [24]. We recall here the one which best fits our aims and we will restrict ourselves to the case \( \alpha \) positive and non-integer, \( p = q = 2 \); the general setting being much more involved see [24].

Let \( T \) be a \( d \)-set in \( \mathbb{R}^D \).

Let \( \alpha > 0 \) non integer, \( k = \lceil \alpha \rceil \) the integer part of \( \alpha \), \( j \) a \( D \)-dimensional multi-index of length \( |j| \leq k \). If \( f \) and \( \{ f^{(j)} \} \) are functions defined \( \mu \)-a.e. on \( T \), we set

\[
R_j(P, P') = f^{(j)}(P) - \sum_{|r| \leq k} \frac{f^{(j+r)}(P)}{r!} (P - P')^r,
\]

where \( f^{(0)} = f \) and \( l \) denotes a \( D \)-dimensional multi-index. We now define the Besov space \( B^{2,2}_a(T) \equiv B^{2,2}_a(T, \mu) \).

**Definition 2.5:** We say that \( f \in B^{2,2}_a(T) \) if there exists a family \( \{ f^{(j)} \} \) with \( |j| \leq k \), as above, such that \( f^{(j)} \in L^2(T, \mu) \) and \( \| \{ a_n \} \|_b < \infty \) where \( a_n \) is the smallest number such that

\[
\left( 3^n \int_{|P - P'| < 3^{-n}} |R_j(P, P')|^2 d\mu(P) \right)^{1/2} \leq 3^{-\alpha(|j| - 1)} a_n.
\]

The norm of \( f \) in \( B^{2,2}_a(T) \) is

\[
\| f \|_{B^{2,2}_a(T)} = \| f \|_{L^2, \mu} + \| \{ a_n \} \|_b.
\]
The family \( \{ f^{(j)} \} \) in the previous definition is uniquely determined by \( f \), as shown in [24], for \( d \)-sets with \( d > D - 1 \).

Let us note that for \( 0 < \alpha < 1 \) the norm \( \| f \|_{B_{2,2}^{s}(T)} \) can be written as

\[
\| f \|_{B_{2,2}^{s}(T)} = \left( \sum_{j=0}^{\infty} 2^{(3d + 2\alpha)j} \int_{|P - P'| < 2^{-j}} |f(P) - f(P')|^2 d\mu(P) d\mu(P') \right)^{1/2}.
\]

**Proposition 2.6:** Let \( T \) be a \( d \)-set, \( T \subset Q \). Let \( s > \frac{3 - d}{2} \), \( \left( s - \frac{3 - d}{2} \right) \notin \mathbb{N} \), then \( B_{2,2}^{s,2,\alpha}(T) \) is the trace space to \( T \) of \( H^{s}(Q) \) in the following sense:

(i) \( \gamma_{0} \) is a continuous linear operator from \( H^{s}(Q) \) to \( B_{2,2}^{s,2,\alpha}(T) \),

(ii) there is a continuous linear operator \( \text{Ext} \) from \( B_{2,2}^{s,2,\alpha}(T) \) to \( H^{s}(Q) \) such that \( \gamma_{0} \circ \text{Ext} \) is the identity operator in \( B_{2,2}^{s,2,\alpha}(T) \).

For the proof we refer to Theorem 1 of Chapter VII in [24], see also [43].

From Proposition 2.6 it follows that when \( T = S \) and \( s = 1 \) the trace space of \( H^{1}(Q) \) is \( B_{2,2}^{s/2}(S) \).

Let \( \beta = \frac{dj}{2} \). The space \( B_{2,2}^{s,2,\alpha}(S) \) is a subspace of \( B_{2,2}^{s}(S) \), more precisely

\[
B_{2,2}^{s,2,\alpha}(S) = \left\{ u \in L^{2}(S, m) \mid \text{there exists } w \in H^{1}(Q) \text{ such that } \gamma_{0} w = u \text{ on } S \right\},
\]
equipped with the norm

\[
\| u \|_{B_{2,2}^{s,2,\alpha}(S)} = \inf \left\{ \| u \|_{H^{1}(Q)} : w \in H^{1}(Q), \gamma_{0} w = u, \text{ on } S \right\}.
\]

In the sequel we denote by the symbol \( f_{S} \) the trace \( \gamma_{0} f \) to \( S \).

In the following, we also make use of the dual of Besov spaces on \( S \). These spaces as shown in [25] coincide with a subspace of Schwartz distributions \( D'(\mathbb{R}^{3}) \), which are supported in \( S \). They are built by means of atomic decomposition. Actually, Jonsson and Wallin [25] proved this result in the general framework of \( d \)-sets. Here we do not give a detailed description of the duals of Besov spaces on \( d \)-sets and we refer to [25] for a complete discussion.

### 3. Variational Formulation for the Fractal Problem

**3.1. The energy forms**

In Definition 4.5 of [13], a Lagrangian measure \( E_{F} \) on \( F \) and the corresponding energy form \( \mathcal{E}_{F} \) as

\[
\mathcal{E}_{F}(u, v) = \int_{F} dE_{F}(u, v)
\]

with domain \( D(F) \) have been introduced. The domain \( D(F) \) – which is a Hilbert space
with norm \( \|u\|_{L^2(F, \mu_F)} + \varepsilon_F(u, u)^{\frac{1}{2}} \) – has been characterized in terms of the domains of the energy forms on \( K_i \) (see [13] Theorem 4.6).

In the following we will omit the subscript \( F \), the Lagrangian measure will be simply denoted by \( \mathcal{L}(u, v) \) and we will set \( \mathcal{L}[u] = \mathcal{L}(u, u) \).

We define the energy forms \( E_\delta \) on the fractal layer \( S = F \times I \) by setting

\[
E_\delta[u] = \sigma^1 \int \mathcal{L}_\delta[u](dx) \ dy + \sigma^2 \int |D_y u|^2 d\mu_F(dx)
\]

where \( \sigma^1 \) and \( \sigma^2 \) are positive constants. Here \( \mathcal{L}_\delta(\cdot, \cdot)(dx) \) denotes the measure-valued Lagrangian (of the energy form \( E \) of \( F \) with domain \( D(F) \)) now acting on \( u(x, y) \) and \( v(x, y) \) as function of \( x \in F \) for a.e. \( y \in I \); \( \mu_F(dx) \) is the Hausdorff measure acting on each section \( F \) of \( S \) for a.e. \( y \in I \) with \( \frac{\log 4}{\log 3} \), \( D_y(\cdot) \) denotes the derivative in the \( y \) direction.

The form \( E_\delta \) is defined for \( u \in D(S) \) where \( D(S) \) is the closure in the intrinsic norm

\[
\|u\|_{D(S)} = (E_\delta[u] + \|u\|_{L^2(S, \mu_F)}^2)^{\frac{1}{2}}
\]

of the set

\[
C_0(S) \cap L^2(0, 1; D(F)) \cap H^1_0(0, 1; L^2(F))
\]

where \( L^2(F) = L^2(F, \mu_F(dx)) \).

In the following we shall also use the form \( E_\delta(u, v) \) which is obtained from \( E_\delta[u] \) by the polarization identity:

\[
E_\delta(u, v) = \frac{1}{2} \{ E_\delta[u + v] - E_\delta[u] - E_\delta[v] \}, \quad u, v \in D(S).
\]

**Proposition 3.1**: The space \( D(S) \) is continuously embedded in \( B_{a_1}^{2,2}(S), a < 1 \).

**Proof**: According to [36] Section 2.2, we now introduce the following spaces:

\[
W(0, 1) = \{ z: S \rightarrow \mathbb{R} : z \in L^2(0, 1; D(F)) \cap H^1(0, 1; L^2(F)) \}
\]

equipped with the norm

\[
\|u\|_{W(0, 1)} = (\|u\|_{L^2(0, 1; D(F))}^2 + \|D_y u\|_{L^2(0, 1; L^2(F))}^2)^{\frac{1}{2}}.
\]

Obviously \( D(S) \subset W(0, 1) \).

From theorem 3.1 in [31] we deduce that \( D(F) \) is embedded in \( B_{a_1}^{2,2}(F) \). Thus, in the notations of [36], we introduce the Hilbert space

\[
B_{a_1}^{2,2}(S) = \{ L^2(0, 1; B_{a_1}^{2,2}(F)) \cap H^1(0, 1; L^2(F)) \}
\]
with norm

\[(3.9) \quad \left( \int_0^1 \|u\|_{B^\alpha_{2,2,2}((F))}^2 \, dy + \|u\|_{H^1(0,1,L^2(F))}^2 \right)^{\frac{1}{2}}.\]

From [36] page 8 it follows that the space \(W(0, 1) \subset B^\alpha_{2,2,2}(S)\), for every \(\varepsilon > 0\). From the embedding Theorem 1 of [24] we deduce that the space \(B^\alpha_{2,2,2}(S)\) is continuously embedded in \(B^\alpha_{2,2}(S)\), \(\alpha < 1\).

It can be proved, as in Proposition 3.1 of [39], that

**Proposition 3.2:** In the previous notations and assumptions the form \(E_S\) with domain \(D(S)\) is a regular Dirichlet form in \(L^2(S, m)\) and the space \(D(S)\) is a Hilbert space under the intrinsic norm (3.3).

We now define the Laplace operator on \(S\). As \((E_S, D(S))\) is a closed, bilinear form on \(L^2(S, m)\), there exists (see Chap. 6, Theorem 2.1 in [18]) a unique self-adjoint, non positive operator \(\Delta_S\) on \(L^2(S, m)\) – with domain \(\partial(D(S)) \subset D(S)\) dense in \(L^2(S, m)\) – such that

\[(3.10) \quad E_S(u, v) = -\int_S (\Delta_S u) v \, dm, \quad u \in \partial(D(S)), v \in D(S).\]

Let \((D(S))'\) denote the dual of the space \(D(S)\). We now introduce the Laplace operator on the fractal \(S\) as a variational operator from \(D(S)\) by

\[(3.11) \quad E_S(z, w) = -\langle \Delta_S z, w \rangle_{(D(S))', D(S)}\]

for \(z \in D(S)\) and for all \(w \in D(S)\) where \(\langle \cdot, \cdot \rangle_{(D(S))', D(S)}\) is the duality pairing between \((D(S))'\) and \(D(S)\). We use the same symbol \(\Delta_S\) to define the Laplace operator both as a self-adjoint operator in (3.10) and as a variational operator in (3.11). It will be clear from the context to which case we refer.

Consider now the space of functions \(u : Q \to \mathbb{R}\)

\[(3.12) \quad V(Q, S) = \{ u \in H^1_0(Q) : u|_S \in D(S) \}.\]

**Proposition 3.3:** The space \(V(Q, S) = \{ u \in H^1_0(Q) : u|_S \in D(S) \}\) is non trivial.

**Proof:** We will prove that non trivial functions in \(D(S)\) have a suitable extension in \(H^1_0(Q)\).

Let \(I_1\) denote the interval \(\left[\frac{1}{4}, \frac{3}{4}\right]\), and \(I_2\) the interval \(\left[\frac{3}{8}, \frac{5}{8}\right]\), \(S_1 = F \times I_1\) and \(S_2 = F \times I_2\). Let \(\phi\) be the capacity potential of \(S_2\) with respect to \(S_1\) (for its existence see Theorem 2.1.5 in [15]) the function \(\tilde{\phi} = \phi\) on \(S_1\) and \(\tilde{\phi} = 0\) on \(S\setminus S_1\) belongs to \(D(S)\), its compact support is contained on \(S_1\). From Proposition 3.1 and Proposition 2.6 Ext \(\tilde{\phi} \in H^1_0(Q)\), then \(w = \eta \ \text{Ext} \ \tilde{\phi} \in H^1_0(Q)\), where \(\eta\) is a suitable cut-off function.
We now introduce the energy form

\[ E[u] = \int_Q |Du|^2 dQ + c_0 E_S[u|_S] \]  

(3.13)

defined on the domain \( V(Q, S) \).
As \( c_0 \) is not relevant for our purposes we set \( c_0 = 1 \).
In the following, by \( E(u, v) \), we will denote the corresponding bilinear form

\[ E(u, v) = \int_Q Du \cdot Dv dQ + E_S(u|_S, v|_S) \]  

(3.14)

defined on \( V(Q, S) \times V(Q, S) \).
As in Theorem 3.2 of [39], it can be proved

**Proposition 3.4:** The form \( E[\cdot] \) defined in (3.13) is a regular Dirichlet form in \( L^2(Q) \) and the space \( V(Q, S) \) is a Hilbert space equipped with the scalar product

\[ (u, v)_V = \int_Q Du \cdot Dv dQ + \int_S u|_S v|_S d\mu \]  

(3.15)

where \( E_S(u|_S, v|_S) \) is the Dirichlet form defined in (3.5).

We denote by \( \|u\|_{V(Q, S)} \) the norm in \( V(Q, S) \) (associated with (3.15)) and we note that from the trace theorem (see Proposition 2.6) there exists a positive constant \( c \), such that

\[ \|u\|_{L^2(S, m)} \leq c \|u\|_{V(Q, S)} \]  

(3.16)

thus an equivalent norm in \( V(Q, S) \) is

\[ \|u\|_{V(Q, S)} = \left( E_S[u|_S] + \int_Q |Du|^2 dQ \right)^{1/2}. \]  

(3.17)

**Proposition 3.5:** Given \( f \in L^2(Q) \), there exists a unique \( u \in V(Q, S) \) such that

\[ E(u, v) = \int_Q fv dQ \]  

(3.18)

for every \( v \in V(Q, S) \).
Moreover, \( u \) is obtained by

\[ \min_{v \in V(Q, S)} \left\{ \frac{1}{2} E[v] - \int_Q fv dQ \right\}. \]  

(3.19)

**Proof:** The thesis follows by applying Lax-Milgram theorem to the bilinear form \( E(u, v) \).
3.2. - The strong formulation of the transmission problem on the fractal layer $S$

We consider the problem (P) formally stated as

\[
\begin{cases}
-\Delta u = f & \text{in } Q', \ i = 1, 2 \quad (i) \\
-\Delta_S u = \left[ \frac{\partial u}{\partial n} \right] & \text{on } S, \quad (ii) \\
u = 0 & \text{on } \partial Q, \quad (iii) \\
u_1 = u^2 & \text{on } S, \quad (iv) \\
u = 0 & \text{on } \partial S \quad (v),
\end{cases}
\] (3.20)

where $u_i$ is the restriction to $Q'$, $\frac{\partial u_i}{\partial n_i}$, $i = 1, 2$ is the outward «normal derivative», to be defined in a suitable sense, $\left[ \frac{\partial u}{\partial n} \right] = \frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2}$ is the jump of the normal derivative and $\Delta_S$ is the fractal Laplacian defined in Section 3.1.

We now prove that the variational solution of (3.18) satisfies problem (P) in a suitable strong sense. We state some preliminary results.

**Proposition 3.6:** The space $D(S)$ is embedded in $B^{b}_{\beta,0} (S)$, $\beta = \frac{d_f}{2}$.

**Proof:** The proof can be achieved by making use of a general extension theorem, proved by Jonsson in [22], for Besov spaces defined on general closed sets which are not possibly $d$-sets such as the set $T = \partial Q \cup S$. More precisely, for any given function $z \in D(S)$ consider the function $\tilde{z}$ defined as $\tilde{z} = z$ on $S$ and $\tilde{z} = 0$ on $\partial Q \setminus S$. In order to extend $\tilde{z}$ to a function $w \in H^1_0 (Q)$ we apply Theorem 1 in [22]. This theorem provides, in particular, a continuous linear operator $\text{Ext}$, $\tilde{z} \mapsto w$, $\text{Ext} : \tilde{B}^{b}_{\alpha,2} (S \cup \partial Q) \rightarrow H^1 (\mathbb{R}^3)$, where $\tilde{B}^{b}_{\alpha,2} (\cdot)$ is the Besov space defined in [22] page 356. We note that $z$ belongs to $B^{\alpha}_{\alpha,2} (S)$, $\alpha < 1$ (see Proposition 3.1) and $\tilde{z}$ belongs to $\tilde{B}^{b}_{\beta,2} (S \cup \partial Q)$, $\beta < \frac{2 - d_f}{2}$, hence in particular to $\tilde{B}^{b}_{\beta,2} (S \cup \partial Q)$. Therefore $w = \text{Ext} \tilde{z} |_Q$ belongs to $H^1_0 (Q)$ and $w |_S = z$ on $S$. 

**Theorem 3.7:** Let $u$ be the variational solution for problem (3.18) then we have that

\[
u' \in H^2_{\text{vis}} (Q')
\]
(3.21)

\[
\frac{\partial u'}{\partial n_i} \in (B^{b}_{\beta,0} (S))', \quad \beta = \frac{d_f}{2}, \quad i = 1, 2
\]
(3.22)
and the transmission condition \( jj \) holds in \((D(S))'\) that is

\[
\langle A_S u | S, z \rangle_{(D(S))', D(S)} = \left( \frac{\partial u}{\partial n} \right)_{S} \quad \text{for every } z \in D(S).
\]

where \((B_{\beta,0}^2(S))'\) is the dual of \(B_{\beta,0}^2(S)\) defined in (2.4); \((D(S))'\) is the dual of \(D(S)\), \(A_S\) is the variational operator from \(D(S) \to (D(S))'\) defined in (3.11) and \(\langle \cdot, \cdot \rangle_{(D(S))', D(S)}\) is the duality pairing between \((D(S))'\) and \(D(S)\).

**Proof:** We recall that by \( u_i \) we denote the restriction to \( \Omega_i \) of the solution \( u \in V(Q, S) \) of (3.18). We choose in (3.18) \( v \in \mathcal{O}(Q') \) and we obtain

\[
\begin{align*}
\int_{Q'} D u_i' D \phi_i dQ &= \int_{Q'} f \phi_i dQ \\
\text{for every } \phi_i \in \mathcal{O}(Q') &\quad (i = 1, 2).
\end{align*}
\]

From the density of \( L^2(Q') \) in \( D(Q') \) and from the fact that \( f \in L^2(Q') \) we deduce that

\[
\begin{align*}
-\Delta u_1 &= f \quad \text{in } L^2(Q') \\
-\Delta u_2 &= f \quad \text{in } L^2(Q')
\end{align*}
\]

This gives that \( u \in V(Q') = \{ u \in H^1(Q) | \Delta u \in L^2(Q') \} \) where the Laplacian is intended in the distributional sense. The classical theory on local regularity results (see [5]) gives also that \( u \in H^2_{00}(Q') \). Moreover, proceeding by duality (see Theorem 4.15 in [30] and [3] Appendix 4) we prove that the normal derivative \( \frac{\partial u_i}{\partial n_i} \) is in the dual \((B_{\beta,0}^2(S))'\) of the space \(B_{\beta,0}^2(S)\), where \( \beta = \frac{d_f}{2} \) and

\[
\langle \frac{\partial u_i}{\partial n_i}, v \rangle_{(B_{\beta,0}^2(S))', B_{\beta,0}^2(S)} = \int_{Q'} D u_i' D v dQ + \int_{Q'} v \Delta u_i' dQ, \quad \text{for every } v \in H^1_0(Q).
\]

From Proposition 3.6 and proceeding as in Section 6 of [30], it can be proved that the transmission condition

\[
A_S(u|S) = \left[ \frac{\partial u}{\partial n} \right] \quad \text{on } S
\]

holds in \((D(S))'\) that is

\[
\langle A_S(u|S), z \rangle_{(D(S))', D(S)} = \left( \frac{\partial u}{\partial n} \right)_{S} \quad \text{for every } z \in D(S).
\]

As a consequence of Theorem 3.7, the (variational) solution of problem (3.18) is
the solution of problem \((P)\) which can be rigorously stated as follows

\[
\begin{aligned}
-\Delta u^i &= f & \text{in } L^2(Q^i), \; i = 1, 2 \\
-\Delta_s u &= \left[ \frac{\partial u}{\partial n} \right] & \text{on } (D(S))^i \\
u^i &= 0 & \text{on } H^\frac{1}{2}(\partial Q) \\
u^2 &= u^2 & \text{on } B^2_{\frac{1}{2}}(S) \\
\end{aligned}
\]

\(\text{(P)}\)

Remark 3.8: Actually from Proposition 3.1 one deduces that equalities \(\nu^2\) and \(\nu\) respectively hold in \(B^2_{\frac{1}{2}}(S)\) and in \(B^2_{\frac{1}{2}}(\mathcal{S})\) with \(\alpha < 1\).

4. - Variational formulation for the pre-fractal layer problem

4.1. - The energy forms

By \(Q\) we denote the parallelepiped as defined in Section 2 and by \(S_b\) we denote the pre-fractal layer of the type \(S_b = F_b \times I, \; b = 1, 2, \ldots, \; F_b\) is the piecewise linear pre-fractal approximation of \(F\) at the step \(b\) (see Section 1). \(S_b\) divides \(Q\) in two subdomains \(Q_b^i, \; i = 1, 2\).

We give a point \(P \in S_b\) the Cartesian coordinates \(P = (x, y), \; x = (x_1, x_2)\) are the coordinates of the orthogonal projection of \(P\) on the plane containing \(F_b\) and \(y\) is the coordinate of the orthogonal projection \(P\) on the \(y\)-line containing the interval \(I\).

We first construct the energy forms \(E_{\Sigma}\) on the pre-fractal layers \(S_b = F_b \times I, \; b \in I\).

By \(l\) we denote the natural arc-length coordinate on each edge of \(F_b\) and we introduce the coordinates \(x_1 = x_1(l), \; x_2 = x_2(l), \; y = y\) on every affine «face» \(S_b^j\) of \(S_b\). By \(d\sigma\) we denote the one-dimensional measure given by the arc-length \(l\) and by \(d\sigma\) the surface measure on each face \(S_b^j\) of \(S_b\), that is \(d\sigma = dl^y\). We define \(E_{\Sigma}\) by setting

\[
E_{\Sigma}[u] = \sum_{S_b^j} \left( \sigma_1^1 \left| D_{\ell} u \right|^2 + \sigma_2^2 \left| D_{\ell} u \right|^2 \right) d\sigma
\]

(4.1)

where \(\sigma_1^1, \sigma_2^2\) are positive constants and \(u \in H^1(S_b), \; \) the Sobolev space of functions on the piece-wise affine set \(S_b\) (see Section 2.1). By Fubini theorem, we can write this functional in the form

\[
E_{\Sigma}[u] = \sigma_1^1 \int_{F_b} \left( \int_{S_b} \left| D_{\ell} u \right|^2 d\ell \right) dy + \sigma_2^2 \int_{F_b} \left( \int_{S_b} \left| D_{\ell} u \right|^2 dy \right) d\ell.
\]

(4.2)

We denote the corresponding bilinear form by \(E_{\Sigma}(u, v)\).
Consider now the space of functions $u : Q \to \mathbb{R}$

$$V(Q, S_b) = \{ u \in H^1_0(Q) : u|_{S_b} \in H^1_0(S_b) \},$$

it is not trivial as it contains $\partial(Q)$.

Consider now the energy form

$$E^{(b)}[u] = \int_Q |D u|^2 \, d Q + E_{S_b}[u|_{S_b}]$$

defined on the domain $V(Q, S_b)$.

By $E^{(b)}(u, v)$ we will denote the corresponding bilinear form

$$E^{(b)}(u, v) = \int_Q D u \, D v \, d Q + E_{S_b}(u|_{S_b}, v|_{S_b})$$

defined on $V(Q, S_b) \times V(Q, S_b)$.

**Theorem 4.1:** The form $E^{(b)}[u]$, defined in (4.4), with domain $V(Q, S_b)$ is a regular Dirichlet form in $L^2(Q)$ and the space $V(Q, S_b)$ is a Hilbert equipped with the scalar product

$$(u, v)_{V(Q, S_b)} = \int_Q D u \, D v \, d Q + E_{S_b}(u|_{S_b}, v|_{S_b}) + \int_{S_b} u|_{S_b} \, v|_{S_b} \, d \sigma.$$

**Proof:** The completeness follows from the completeness of $H^1_0(Q)$ and $H^1_0(S_b)$ and from Proposition 2.1 (with $s = 1$, $g = Q$ and $\Gamma = S_b$). The regularity of the form follows from Proposition 2.2 and from the density of $\partial(Q)$ in $V(Q, S_b)$ (see also Proposition 4.1 in [33] where the two-dimensional case is studied).

We denote by $\| u \|_{V(Q, S_b)}$ the corresponding energy norm in $V(Q, S_b)$. By proceeding as in Section 4.1 one can prove, via the trace theorem that there exists a positive constant $c$

$$\| u \|_{L^2(S_b)} \leq c \| u \|_{H^1_0(Q)},$$

thus an equivalent norm in $V(Q, S_b)$ is

$$\| u \|_{V(Q, S_b)} = \int_Q D u \, D v \, d Q + E_{S_b}[u|_{S_b}].$$

**Proposition 4.2:** Given $f \in L^2(Q)$, for every $b \in \mathbb{N}$, there exists a unique $u_b \in V(Q, S_b)$ such that

$$E^{(b)}(u_b, v) = \int_Q f v \, d Q$$

for every $v \in V(Q, S_b)$. 
Moreover, \( u_h \) is obtained as the minimizer of the variational problem

\[
E^{(b)}[u_h] = \min_{u \in \mathcal{V}(Q, S_b)} \left\{ \frac{1}{2} E^{(b)}[u] - \int_{\partial Q} f u \, dQ \right\}.
\]

**Proof:** The thesis follows by applying Lax-Milgram theorem to the form \( E^{(b)}(u, v) \). \( \blacksquare \)

### 4.2. The strong formulation of the transmission problem on the pre-fractal layer \( S_b \)

We consider now the problems \((P_b)\), formally stated as:

\[
(P_b) \quad \begin{cases} 
- \Delta u = f & \text{in } Q^i_1, \ i = 1, 2 \ j \\
- \Delta_{S_b} u = \left[ \frac{\partial u}{\partial n} \right] & \text{on } S_b \ jj \\
u = 0 & \text{on } \partial Q \ jjj \\
u^1 = u^2 & \text{on } S_b \ jv \\
u = 0 & \text{on } \partial S_b \ v
\end{cases}
\]

where \( u^i = u|_{Q^i_1} \), \( \Delta_{S_b} \) is the «piecewise» tangential Laplacian on \( S_b \) associated to the Dirichlet form \( E_{S_b} \), \( \left[ \frac{\partial u}{\partial n} \right] = \frac{\partial u^1}{\partial n_1} + \frac{\partial u^2}{\partial n_2} \) is the jump of the normal derivatives of \( u \) across \( S_b \), \( n_i, i = 1, 2 \) being the outward normal to \( Q^i_1 \).

Let \( u^i_b \) denote the restriction of the variational solution \( u_b \) to \( Q^i_1 \). By usual duality arguments (see Appendix 4 in [3]) the normal derivatives \( \frac{\partial u^i_b}{\partial n_i} \), \( i = 1, 2 \), belong to the dual space of \( H^1_{0, 0}(S_b) \) (see (2.2)).

Then, by the Green formula for Lipschitz domains, one can prove that the transmission condition \( jj \) in (4.9) can be interpreted in the sense of the dual of \( H^1_{0, 0}(S_b) \) (see Proposition 2.2 in [32]).

**Theorem 4.3:** Let \( u_b \) be the variational solution for problem (4.7) then we have that

\[
u_b \in C(\overline{Q})
\]

\[
u^1_b \in H^s_{-\epsilon}(Q^1_1), \ \nu^2_b \in H^s_{-\epsilon}(Q^2_2)
\]

\[
\frac{\partial u^i_b}{\partial n_i} \in L^2(S_b), \ i = 1, 2
\]

in particular conditions \( jjj \), \( jv \) and \( v \) are satisfied point-wise, \( j \) and \( jj \) almost every-
where and
\begin{equation}
\Delta_{S_h} = \sigma^1 \mathcal{D}^2 + \sigma^2 \mathcal{D}^2.
\end{equation}
Here $\mathcal{D}^2$ is the «piecewise» second order tangential derivative along the sides of $F_h$ and $\mathcal{D}^2$ the «usual» second order partial derivative in $y$.

In order to prove Theorem 4.3 we need some intermediate results. Consider the weak solutions $w^i_h$, $\overline{w}^i_h$ in $H^1(Q^i_h)$ of the following auxiliary problems
\begin{equation}
\begin{cases}
\Delta \overline{w}^i_h = 0 & \text{in } Q^i_h \\
\overline{w}^i_h = u_h & \text{on } \partial Q^i_h
\end{cases}
\end{equation}
\begin{equation}
\begin{cases}
-\Delta w^i_h = f & \text{in } Q^i_h \\
w^i_h = 0 & \text{on } \partial Q^i_h
\end{cases}
\end{equation}
As the link between $u_h$ and the solutions of problems (4.14) and (4.15) is
\begin{equation}
u^i_h = w^i_h + \overline{w}^i_h,
\end{equation}
then the regularity of $u^i_h$ follows from the regularity of $w^i_h$ and $\overline{w}^i_h$.

**Proposition 4.4:** In the assumptions of Theorem 4.3 and notations (4.14) we have
\begin{equation}
\frac{\partial \overline{w}^i_h}{\partial n_i} \in L^2(S^i_h)
\end{equation}
\begin{equation}
\left\| \frac{\partial \overline{w}^i_h}{\partial n_i} \right\|_{L^2(S^i_h)} \leq c(b) \| \nabla_i u_h \|_{L^2(S^i_h)}, \quad i = 1, 2.
\end{equation}

**Proof:** The proof follows from an analogous result of Jerison and Kenig (see Theorem 3 and also the proof of Theorem 2 of [21]).

Note also that the right-hand side of (4.17) can be evaluated in terms of the $L^2$-norm of $f$ in $Q$.

**Proposition 4.5:** In the notations of (4.15) we have that
\begin{equation}
w^i_h \in H^1(Q^i_h), \quad \| w^i_h \|_{H^1(Q^i_h)} \leq c(\mu, b) \| f \|_{L^2(Q^i_h)}, \quad i = 1, 2
\end{equation}
where $s_i = 2 - \mu$, with $\frac{2}{5} \leq \mu_i < 1$ and $\frac{1}{4} \leq \mu_2 < 1$; $c(\mu, b)$ is a positive constant depending on $\mu_i$ and on $h$.

**Proof:** The bounded domain $Q^i_h$ has several intersecting edges on the boundary: let $\{P^i_r, r = 1, \ldots, T\}$ be the set of the intersection points of the edges, near each intersection point $P^i_r$ the domain coincides with a cone $K^r$ cutting out a domain $\Omega^i_r$ on the sphere $S^i$. Let $r(P)$ be the distance from the point $P$.
to the set \( M \) of the edges and \( q(P) \) be the distance from the point \( P \) to the point \( P_i^t \).

From Theorem 10.2.3 in [40] we deduce the following estimates for the second derivatives of \( \mathcal{w}_i \):

\[
(4.19) \quad \sum_{|\tau|=2}^{T} \int_{Q^i_t} r^{2\mu_i} \prod_{r=1}^{T} q_t^{2(a_i^r - \mu_i)} |D^\alpha \mathcal{w}_i|^2 \, dx \, dy \leq c(\mu_i, b) \|f\|_{L^2(Q^i_t)}^2, \quad i = 1, 2
\]

the parameters \( a_i^r \) and \( \mu_i \) satisfy the following condition

\[
(4.20) \quad 0 < -\mu_i + 1 < \frac{\pi}{\theta_i}
\]

and

\[
(4.21) \quad 4(\sigma_i^r - 1)^2 < 1 + 4A_{1, r}
\]

where \( A_{1, r} = A_1(\Omega^i_t) \) is the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator in \( \Omega^i_t, \tau = 1, \ldots, T. \)

It is to be pointed out that for functions \( z \) supported in a small neighborhood of \( P_i^t \) the product \( \prod_{r=1}^{T} q_t^{2(a_i^r - \mu_i)} \) in (4.19) can be replaced by the single factor \( q_t^{2(a_i^r - \mu_i)} \). Roughly speaking the loss of the smoothness depends only on the nearest singularity of the boundary \( \partial Q^i_t \), namely either a vertex or an edge, the regularity being a local property.

Taking into account that the dihedral angles in \( Q^i_t \) have opening equal to \( \frac{2}{3} \pi \) or \( \frac{5}{3} \pi \) it follows that \( \mu_1 \) has to be chosen greater than \( \frac{2}{5} \); on the other hand in \( Q^2_t \), the dihedral angle have opening \( \frac{\pi}{3} \) and \( \frac{4}{3} \pi \) hence \( \mu_2 \) has to be chosen greater than \( \frac{1}{4} \). As to the choice of \( \sigma_i^r \), we firstly observe that any \( \Omega^i_t, i = 1, 2 \) is contained in the «lune»

\[
\Omega = \left\{ \omega = (\omega_1, \omega_2) \in S^2; \; \omega_1 \in \{0, \pi\}, \; \omega_2 \in \left(0, \frac{5}{3} \pi\right) \right\};
\]

taking into account the monotonicity properties of the first eigenvalue of Laplace-Beltrami operator (see e.g. [4] and [2]) we conclude that

\[ A_{1, r}^i = A_1(\Omega^i_t) \geq A_1(\Omega). \]

Finally, Proposition 3.1 in [40] yields the explicit value of \( A_1(\Omega) = \)
The choice in our case will be \( \sigma_1^1 = \mu_1, \sigma_2^1 = \mu_2, \tau = 1, \ldots, T \).

Denoting by \( \delta \) the distance from the boundary, we have \( \delta^\alpha : D^\alpha w_1^i \in L^2(Q_1^i), |\alpha| = 2, \mu_1 > \frac{2}{5}, \) and

\[
\|\delta^\alpha D^\alpha w_1^i\|_{L^2(Q_1^i)} \leq c(\mu_1, b) \|f\|_{L^2(Q_1^i)},
\]

Analogously, as in \( Q_2^i \) the angles have opening equal to \( \frac{\pi}{3} \) or \( \frac{4\pi}{3} \), we prove that \( \delta^\alpha : D^\alpha w_2^i \in L^2(Q_2^i), |\alpha| = 2, \mu_2 > \frac{1}{4}, \) and

\[
\|\delta^\alpha D^\alpha w_2^i\|_{L^2(Q_2^i)} \leq c(\mu_2, b) \|f\|_{L^2(Q_2^i)},
\]

We deduce the thesis from (4.22), (4.23) and (4.24): we have \( w_1^i \in H^i(Q_1^i) \forall i < \frac{8}{5} \) and \( w_2^i \in H^i(Q_2^i) \forall i < \frac{7}{4} \). 

We are now in position to prove Theorem 4.3.

PROOF: From Proposition 4.5 we deduce that

\[
D^\alpha w_1^i \in H^{\frac{1}{2} - \epsilon}(Q_1^i), D^\alpha w_2^i \in H^{\frac{1}{2} - \epsilon}(Q_2^i), |\alpha| = 1
\]

then by trace results (see Proposition 2.1) we obtain for \( i = 1, 2 \)

\[
\frac{\partial w_1^i}{\partial n_j} \in L^2(S_0), \quad \left\| \frac{\partial w_1^i}{\partial n_j} \right\|_{L^2(S_0)} \leq c(\mu_i, b) \|f\|_{L^2(Q)}.
\]

It follows from (4.26), (4.17) and (4.16) that \( \frac{\partial u_1^i}{\partial n_j} \in L^2(S_0), i = 1, 2, \) hence the jump belongs to \( L^2(S_0) \). As \( H^{\frac{1}{2} - \epsilon}(S_0) \) is dense in \( L^2(S_0) \) (see e.g. [7]), we deduce that the transmission condition \( jf \) in (4.9) actually holds in the \( L^2 \)-sense and in particular \( A_S u_b \in L^2(S_0) \). As \( u_b \in H^1(S_0) \), from Theorem 8 in [8], we deduce that \( u_b \) is in particular in \( H^2(S_0) \).
Denote by \( \tilde{u}_h \) the trivial extension of \( u_h \) in \( \partial Q_h \)

\[
\tilde{u}_h = \begin{cases} 
    u_h & \text{on } S_h \\
    0 & \text{on } \partial Q_h \setminus S_h,
\end{cases}
\]

then \( \tilde{u}_h \mid_{\partial Q_h} \) belongs in particular to \( H^2(\partial Q_h) \) (see Proposition 2.11 in [7]). Let \( \tilde{u}_h = \tilde{E} \tilde{u}_h \) be a function in \( H^2(\partial Q_h) \) such that \( \tilde{u}_h \mid_{\partial Q_h} = \tilde{u}_h \) (see Proposition 2.2), then \( \Delta \tilde{u}_h \in L^2(\partial Q_h) \) and \( \| \Delta \tilde{u}_h \|_{L^2(\partial Q_h)} \leq c(b) \| f \|_{L^2(\partial Q)} \).

We note that the restriction \( u_h = u_h \mid_{\partial Q} \) is the weak solution in \( H^1(\partial Q) \), of the problem

\[
\begin{cases} 
    -\Delta u_h = f & \text{in } Q_h \\
    u_h = 0 & \text{on } \partial Q_h \setminus S_h \\
    u_h = u_h & \text{on } S_h.
\end{cases}
\]

Then the function \( v_h := u_h - \tilde{u}_h \) is the weak solution in \( H^1_0(Q_h) \) of the Dirichlet problem

\[
\begin{cases} 
    -\Delta v_h = f + \Delta \tilde{u}_h & \text{in } Q_h \\
    v_h = 0 & \text{on } \partial Q_h.
\end{cases}
\]

By proceeding as in the proof of Proposition 4.5 we can obtain that \( v_h \in H^s(Q_h) \) where \( s_1 < \frac{8}{5} \) and \( s_2 < \frac{7}{4} \). Finally \( u_h \) inherits the regularity of \( v_h \) as \( \tilde{u}_h \) is more regular, this yields (4.11).

We now prove (4.10), i.e. that \( u \in C(\overline{Q}) \). We note that from Morrey-Sobolev embedding it follows that \( u_h \in C(S_h) \) because \( H^2(S_h) \) is embedded in \( C(S_h) \), and that \( u_h \in C(\overline{Q_h}) \) as \( u_h \in H^{2,\gamma}(Q_h) \), \( i = 1, 2 \). We conclude the proof taking into account that \( u_h \mid_{S_h} = u_h \), \( i = 1, 2 \). \( \blacksquare \)

From Theorem 4.3 it follows that the variational solution of problem (4.7) is the solution of problem \((P_h)\) which can be rigorously stated as follows

\[
(P) \quad \begin{cases} 
    -\Delta u = f & \text{in } L^2(Q_h), \; i = 1, 2 \\
    -\Delta \delta u = \left[ \frac{\partial u}{\partial n} \right] & \text{on } L^2(S_h) \\
    u = 0 & \text{on } H^\frac{1}{2}(\partial Q) \setminus C(\partial Q) \\
    u^1 = u^2 & \text{on } H^1(S_h) \setminus C(S_h) \\
    u = 0 & \text{on } H^\frac{1}{2}(\partial S_h) \setminus C(\partial S_h) \\
\end{cases}
\]

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