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RAFFAELA CAPITANELLI (*)

Homogeneous p-Lagrangians and Self-Similarity (**)

ABSTRACT. — We discuss self-similarity in connection with homogeneous p-Lagrangians and the associated nonlinear energy forms.

p-Lagrangiane omogenee e auto-similarità

SUNTO. — Si studia l'auto-similarità in connessione con p-Lagrangiane omogenee e le corrispondenti forme di energia.

1. - INTRODUCTION

The aim of this paper is to discuss self-similarity in connection with homogeneous p-Lagrangians and the associated nonlinear energy forms (for a study of self-similarity in the special context of quadratic energy functional, see [16]).

We are motivated by the recent interest in the study of various non Euclidean structures that are invariant under suitable self-similarities of the structures themselves (see [9] and references therein); moreover, nonlinear energy forms have been recently constructed on these structures, in particular, on the Koch curve type fractals in [4] and on the Sierpinski type fractals in [10].

By using the approach of *variational metrics* developed by Mosco in [15], [17], [18], we introduce suitable quasi metrics of variational nature: in this way, the *variational fractal* gives a *metric fractal* (see [19]) and we can apply the functional inequalities developed in the framework of the theory of p-Lagrangians on homogeneous spaces (see [13] and [5]).

(*) Indirizzo dell'Autore: Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università degli Studi di Roma «La Sapienza», Via A. Scarpa 16, 00161 Roma, Italy. E-mail: raffaela.capitanelli@uniroma1.it
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More precisely, the plan of the paper is the following.

In the second section, we consider *variational fractals*, that is, self-similar fractals possessing non trivial self-similar Lagrangians. In particular, we recall the definition and some properties of self-similar fractals (according to Hutchinson's theory [11]) and of homogeneous p-Lagrangians (firstly introduced in the paper of Malý and Mosco [13]).

In the third section, we introduce a suitable quasi-distance on the variational fractal by defining a new metric d on the fractal such that d^p has the same scaling as the p-Lagrangian. In this setting, the fractal with this metric can be viewed as a homogeneous space (see Theorem 3.2). Moreover, by assuming that a global Poincaré inequality holds, we prove a family of scaled Poincaré inequalities on the homogeneous balls (see Theorem 3.6). These inequalities are the starting point of the variational theory for measure-valued Lagrangians in homogeneous spaces developed in [13] and [5].

In section 4, we study the relation between Lagrangian and the corresponding energy form. In particular, we obtain a representation formula for the homogeneous p-Lagrangians (see Theorem 4.1). Moreover, we prove that if the total energy is self-similar then the Lagrangian inherits the same invariance property (the converse being obvious) (see Theorem 4.2).

In the last section, we describe a basic example. In particular, we reformulate a result of [4] in terms of the theory of p-Lagrangians on homogeneous spaces. In [4], we examined the functions of finite nonlinear energy on the Koch curve, that is, the functions that belong to the domain of the nonlinear form. These functions, by direct calculations, are shown to be Hölder continuous, with Euclidean Hölder exponent $\beta_e = \dfrac{p-1}{p} \log_3 4$. Now, using the intrinsic approach, this property can be compared with the Morrey embedding proved in [5]: as the homogeneous dimension $\nu = 1 < p$, the functions of finite energy are Hölder continuous with respect to the intrinsic metric d, with Hölder exponent $\beta = 1 - \dfrac{\nu}{p}$.

2. - Variational fractals

Throughout this paper, we shall use the following notation: $\mathbb{R}^D$ is the D-dimensional Euclidean space, $D \geqslant 1$,

$$d_e(x, y) \equiv |x - y| = \left(\sum_{h=1}^{D} |x_h - y_h|^2 \right)^{\frac{1}{2}}$$

the Euclidean distance, $B_e(x, r) := \{y \in \mathbb{R}^D : |x - y| < r\}$, $x \in \mathbb{R}^D$, $r > 0$, are the Euclidean balls (denoted also by $B_{e,r}$), $\mathrm{diam}_e A$ the Euclidean diameter of a subset $A \subset \mathbb{R}^D$.

We suppose that $\Psi = \{\psi_1, \dots, \psi_N\}$ is a given set of contractive similitudes $\psi_i \colon \mathbb{R}^D \to \mathbb{R}^D$, with contraction factors $\alpha_i^{-1} < 1$, that is,

$$|\psi_i(x) - \psi_i(y)| = \alpha_i^{-1} |x - y|$$

for every $x, y \in \mathbb{R}^D$, $i = 1, \dots, N$. In [11], it is proved that there exists a unique closed bounded set K, which is *invariant* under $\Psi = \{\psi_1, \dots, \psi_N\}$, that is,

$$K = \bigcup_{i=1}^{N} \psi_i(K). \tag{2.1}$$

The invariant set K of a given family $\Psi = \{\psi_1, \dots, \psi_N\}$ will be called a *self-similar fractal*. The real number d_f, uniquely determined by the relation

$$\sum_{i=1}^{N} \alpha_i^{-d_f} = 1 ,$$

is the *similarity dimension* of K.

Let us choose N constants $r_i \in (0, 1)$, with $\sum_{i=1}^{N} r_i = 1$. Then, there exists a unique Borel regular measure μ in $\mathbb{R}^D$, with $\operatorname{supp} \mu = K$ and unit total mass, which is *invariant* with respect to the given $\Psi = \{\psi_1, \dots, \psi_N\}$ and $\{r_1, \dots, r_N\}$, that is, μ satisfies

$$\mu = \sum_{i=1}^{N} r_i \psi_{i\#} \mu \tag{2.2}$$

where $\psi_{i\#}\mu(\cdot) := \mu(\psi_i^{-1}(\cdot))$, $i = 1, \dots, N$ with $\operatorname{supp} \psi_{i\#}\mu = \psi_i(\operatorname{supp} \mu)$ (see [11]). The relation (2.2) can be equivalently written as

$$\int_K \varphi \, d\mu = \sum_{i=1}^{N} r_i \int_K \varphi \circ \psi_i \, d\mu \tag{2.3}$$

for every $\varphi \in C(K)$ (where $C(K)$ is the space of continuous functions on K).

In the following, the measure obtained by the special choice $r_i := \alpha_i^{-d_f}$ will be simply called the *invariant measure* of K: it only depends on the given family $\Psi = \{\psi_1, \dots, \psi_N\}$.

More specific metric informations on K and μ are available when the family $\Psi = \{\psi_1, \dots, \psi_N\}$ satisfies the following *open set condition*: there exists a bounded open set $U \subset \mathbb{R}^D$, such that

$$\bigcup_{i=1}^{N} \psi_i(U) \subset U, \quad \text{with } \psi_i(U) \cap \psi_j(U) = \emptyset \text{ if } i \neq j . \tag{2.4}$$

In fact, under this assumption, the following important metric properties hold, (see [11]): the similarity dimension d_f equals the Hausdorff dimension of K and $0 < H^{d_f}(K) < \infty$, where H^{d_f} denotes the d_f-dimensional Hausdorff measure in $\mathbb{R}^D$.

The invariant measure μ coincides with the restriction to K of the d_f-dimensional

Hausdorff measure of $\mathbb{R}^D$, $H^{d_f} \lfloor K$, normalized:

$$\mu = (H^{d_f}(K))^{-1} H^{d_f} \lfloor K;$$

d_f is also called the *fractal dimension* of K. In the special case $\alpha_1, \ldots, \alpha_N = \alpha > 1$, we have

$$d_f = \frac{\ln N}{\ln \alpha}. \tag{2.5}$$

We will use the notations $\psi_{i_1 \ldots i_n} := \psi_{i_1} \circ \psi_{i_2} \circ \ldots \circ \psi_{i_n}, A_{i_1 \ldots i_n} := \psi_{i_1 \ldots i_n}(A)$ for arbitrary n-tuples of indices $i_1, \ldots, i_n \in \{1, \ldots, N\}$ and arbitrary $A \subset K$.

We call $K_{i_1 \ldots i_n} = \psi_{i_1 \ldots i_n}(K)$, $n \geq 1$ $i_1, \ldots, i_n \in \{1, \ldots, N\}$, an *n-complex*. We have

$$K = \bigcup_{i_1, \ldots, i_n = 1}^{N} K_{i_1 \ldots i_n},$$

and

$$\mu(K) = \sum_{i_1, \ldots, i_n = 1}^{N} \mu(K_{i_1 \ldots i_n}).$$

We say that two complexes are contiguous if their intersection is not empty.

The diameter of $K_{i_1 \ldots i_n}$ satisfies

$$\mathrm{diam}_e K_{i_1 \ldots i_n} = \alpha_{i_1}^{-1} \ldots \alpha_{i_n}^{-1} \mathrm{diam}_e K. \tag{2.6}$$

We say that $K_{i_1 \ldots i_n}$ is *of size* R with $0 < R < \mathrm{diam}_e K$ if

$$\alpha_1^{-1} R \leq \mathrm{diam}_e K_{i_1 \ldots i_n} < R\,,$$

(we are assuming that $\alpha_1 = \max\{\alpha_1, \ldots, \alpha_N\}$).

By G_R we denote the set

$$G_R := \{K_{i_1 \ldots i_n} \text{ of size } R\}.$$

Note that $G_R = \{K\}$ if $R = \mathrm{diam}_e K$.

We recall that a self-similar fractal enjoys the following *finite-overlapping property* ([11], Theorem 5.3; [17], Theorem 2.1). This property says that, if we intersect the fractal K with a Euclidean ball of radius R, then the intersection $K \cap B_{e,R}$ is covered by at most M n-complexes $K_{i_1 \ldots i_n}$ of size R, where M is independent of the scale R. More precisely, the following theorem holds.

THEOREM 2.1: *Let K be a self-similar fractal satisfying* (2.1) *and* (2.4). *Let*

$$M = \left(1 + 2\,\frac{c_2}{\mathrm{diam}_e K}\right)^D \left(\alpha_1^{-1}\,\frac{c_1}{\mathrm{diam}_e K}\right)^{-D},$$

where c_1 is the radius of a Euclidean ball contained in U and c_2 is the radius of a Euclidean ball containing U. Then for every x and $0 < R \leqslant \mathrm{diam}_e K$ the family

$$G_{x,R} := \{K_{i_1 \dots i_n} \colon K_{i_1 \dots i_n} \in G_R \quad K_{i_1 \dots i_n} \bigcap B_e(x, R) \neq \emptyset\},$$

contains at most M distinct complexes and

$$K \cap B_e(x, R) \subset \bigcup_{G_{x,R}} K_{i_1 \dots i_n}.$$

We define the boundary Γ of K as

$$\Gamma = \bigcup_{i \neq j} \psi_i^{-1}(K_i \cap K_j).$$

We have that Γ is a compact subset of $K \cap \partial U$ and $\mu(\Gamma) = 0$ (see [17], Theorem 2.3).

In the following, we shall assume that for every $n \geqslant 1$ and every for $i_1, \dots, i_n \neq j_1, \dots, j_n$ we have

$$K_{i_1 \dots i_n} \cap K_{j_1 \dots j_n} = \Gamma_{i_1 \dots i_n} \cap \Gamma_{j_1 \dots j_n} \tag{2.7}$$

The notion of measure-valued Lagrangians has been introduced in [13] and later developed by Biroli and Vernole in [2] and in [3]. We now give the definition of homogeneous p-Lagrangians which best fits in our context in an easier form than that given in [2]: in particular, we do not require the absolute continuity of the Lagrangian with respect to the volume measure and the completion of the domain.

Let now X be a locally compact Hausdorff topological space and μ a bounded Radon measure on X with $\mathrm{supp}\, \mu = X$. Let $\widetilde{\mathcal{L}}^{(p)}$ be a Radon measure valued nonnegative map defined on a dense subalgebra $\mathcal{C}^{(p)}$ of the space $C_b(X)$ of bounded continuous functions on X. We make the following assumptions on $\widetilde{\mathcal{L}}^{(p)}$, $(p > 1)$:

i) $\widetilde{\mathcal{L}}^{(p)}$ is positive semidefinite and convex in the space $\mathcal{M}$ of Radon measure.

ii) $\widetilde{\mathcal{L}}^{(p)}$ is homogeneous of degree p.

iii) $\widetilde{\mathcal{L}}^{(p)}$ is such that

$$\|u\| = \left(\int_X |u|^p \, d\mu + \int_X d\widetilde{\mathcal{L}}^{(p)}(u) \right)^{\frac{1}{p}} \tag{2.8}$$

is a norm in $\mathcal{C}^{(p)}$.

iv) Strong locality: if $u - v =$ constant on $\mathrm{supp}\, \varphi$, then

$$\int_X \varphi(x) \, d\widetilde{\mathcal{L}}^{(p)}(u) = \int_X \varphi(x) \, d\widetilde{\mathcal{L}}^{(p)}(v)$$

for any $\varphi \in C(X)$, $u, v \in \mathcal{C}^{(p)}$.

v) for every $u, v \in \mathcal{C}^{(p)}$ there exists in the weakly* topology of $\mathcal{M}$ the following limit:

$$\lim_{t \to 0} \frac{\widetilde{\mathcal{L}}^{(p)}(u+tv) - \widetilde{\mathcal{L}}^{(p)}(u)}{t} = \langle \partial \widetilde{\mathcal{L}}^{(p)}(u), v \rangle.$$

We define $\mathcal{L}^{(p)} : \mathcal{C}^{(p)} \times \mathcal{C}^{(p)} \to \mathcal{M}$ as

$$\mathcal{L}^{(p)}(u, v) = \langle \partial \widetilde{\mathcal{L}}^{(p)}(u), v \rangle. \tag{2.9}$$

vi) The chain rules: if $u, v \in \mathcal{C}^{(p)}$ and $g \in C^1(\mathbb{R})$, with g' bounded on $\mathbb{R}$, then

$$g(u) : x \to g(u(x))$$

belongs to $\mathcal{C}^{(p)}$,

$$\mathcal{L}^{(p)}(g(u), v) = |g'(u)|^{p-2} g'(u) \mathcal{L}^{(p)}(u, v),$$

$$\mathcal{L}^{(p)}(v, g(u)) = g'(u)\, \mathcal{L}^{(p)}(v, u).$$

DEFINITION 2.2: *The measure* $\mathcal{L}^{(p)}(u, v)$ *in* (2.9) *satisfying the previous assumptions i),…,vi) will be called homogeneous p-Lagrangian.*

From the definition of $\mathcal{L}^{(p)}(u, v)$, we get the following properties (see [2]).

PROPOSITION 2.3: *i) If* $u \in \mathcal{C}^{(p)}$ *and* $g \in C^1(\mathbb{R})$, *with* g' *bounded on* $\mathbb{R}$, *then* $g(u) : x \to g(u(x))$ *belongs to* $\mathcal{C}^{(p)}$ *and*

$$\mathcal{L}^{(p)}(g(u), g(u)) = |g'(u)|^p \mathcal{L}^{(p)}(u, u).$$

ii) For every $u \in \mathcal{C}^{(p)}$,

$$\mathcal{L}^{(p)}(u, u) = p \widetilde{\mathcal{L}}^{(p)}(u).$$

iii) Leibniz rule on the second argument: for any $u, v, w \in \mathcal{C}^{(p)}$,

$$\mathcal{L}^{(p)}(u, vw) = v \mathcal{L}^{(p)}(u, w) + w \mathcal{L}^{(p)}(u, v).$$

We conclude by giving the definition of *variational fractal.*

DEFINITION 2.4: *A variational fractal* is a triple $K \equiv (K, \mu, \mathcal{L}^{(p)})$ *where*

– *K is the invariant set of a given family* $\Psi = \{\psi_1, \dots, \psi_N\}$ *satisfying* (2.1), (2.4) *and* (2.7);

– μ *is the invariant measure* (2.2) *on* K;

– $\mathcal{L}^{(p)}$ *is a nonlinear p-homogeneous Lagrangian with domain* $\mathcal{C}^{(p)}$ *in* $L^p(K, \mu)$ *in* *the sense of Definition* 2.2 *such that, for every* $u \in \mathcal{C}^{(p)}$ *and for every* $\varphi \in C(K)$, *we have*

$$\int_K \varphi d\mathcal{L}^{(p)}[u] = \sum_{i=1}^{N} \varrho_i^{(p)} \int_K \varphi \circ \psi_i d\mathcal{L}^{(p)}[u \circ \psi_i]$$

with the real constants $\varrho_i^{(p)} > 0$, $i = 1, \ldots, N$, *satisfy* $\varrho_i^{(p)} = \mu(K_i)^{\sigma}$, $i = 1, \ldots, N$, *for some real constant* $\sigma < 1$, *independent of* $i = 1, \ldots, N$.

3. - Metric fractals

Given a variational fractal $K \equiv (K, \mu, \mathcal{L}^{(p)})$, we consider quasi-distances d on K with Euclidean scaling

$$d(x, y) = |x - y|^{\delta}, \quad x, y \in K$$

indexed by a real parameter $\delta > 0$.

The quasi-balls associated with d will be denoted by $B(x, r)$, that is, $B(x, r) :=$ $:= \{y \in K : d(x, y) < r\}$, $x \in K$, $r > 0$. For every $x \in K$ and every $r > 0$ we have $B(x, r) = B_e(x, r^{\frac{1}{\delta}}) \cap K$. For every A, the diameter of A with respect to the quasi metric d will be denoted by

$$\operatorname{diam} A = (\operatorname{diam}_e A)^{\delta}. \tag{3.1}$$

We choose d by requiring d^p to obey on K the same scaling as $\mathcal{L}^{(p)}$ itself:

$$d^p(x, y) = \sum_{i=1}^{N} \varrho_i^{(p)} d^p(\psi_i(x), \psi_i(y)),$$

for every $x, y \in K$.

Lemma 3.1: *Let K be a variational fractal, with given structural constants N, $\alpha_1, \ldots, \alpha_N$ and σ. Then, there exists one and only one constant $\delta > 0$, such that the following identities hold:*

$$d(x, y) = |x - y|^{\delta}$$

$$d^p(x, y) = \sum_{i=1}^{N} \varrho_i^{(p)} d^p(\psi_i(x), \psi_i(y)),$$

for every $x, y \in K$.

Such a δ is uniquely determined by the identity

$$\sum_{i=1}^{N} \varrho_i^{(p)} \alpha_i^{-p\delta} = 1$$

and is given by

$$\delta = d_f(1-\sigma)/p .$$

PROOF: By replacing $d(x, y) = |x-y|^{\delta}$ in the scaling identity for d^p, we obtain

$$|x-y|^{p\delta} = \sum_{i=1}^{N} \varrho_i^{(p)} \, |\psi_i(x) - \psi_i(y)|^{p\delta} = \sum_{i=1}^{N} \varrho_i^{(p)} \alpha_i^{-p\delta} |x-y|^{p\delta},$$

which gives

$$\sum_{i=1}^{N} \varrho_i^{(p)} \alpha_i^{-p\delta} = 1 .$$

Taking into account the expression of the scaling factors $\varrho_i^{(p)}$ in (2.10), we have

$$\sum_{i=1}^{N} \varrho_i^{(p)} \alpha_i^{-p\delta} = \sum_{i=1}^{N} \mu(K_i)^{\sigma} \alpha_i^{-p\delta} = \sum_{i=1}^{N} \alpha_i^{-d_f\sigma - p\delta};$$

by the definition of d_f as similarity dimension, we have

$$d_f\sigma + p\delta = d_f. \qquad \blacksquare$$

We note that in the special case $\alpha_i = \alpha > 1$ and $\varrho_i^{(p)} = \varrho^{(p)}$ for every $i \in \{1, \dots, N\}$, we have

$$\delta = \frac{\ln_{\alpha} N\varrho^{(p)}}{p} . \tag{3.2}$$

When endowed with this quasi-metric, the fractal K becomes a space of homogeneous type of dimension $\nu = \dfrac{d_f}{\delta}$: in fact, the following theorem holds (see [17], Theorem 3.1).

THEOREM 3.2: *Let $K \equiv (K, \mu, \mathcal{L}^{(p)})$ be a variational fractal endowed with its intrinsic metric.*

Then K is a homogeneneous space of dimension

$$\nu = \frac{d_f}{\delta} : \tag{3.3}$$

for every $x \in K$ and for every $0 < r \leqslant R \leqslant \operatorname{diam} K = (\operatorname{diam}_e K)^{\delta}$ we have

$$M^{-1} \alpha_1^{-d_f} \mu(B(x, R)) \left(\frac{r}{R} \right)^{\nu} \leqslant \mu(B(x, r)) \leqslant M \alpha_1^{d_f} \mu(B(x, R)) \left(\frac{r}{R} \right)^{\nu}$$

where

$$M = \left(1 + 2\,\frac{c_2}{\mathrm{diam}_e K}\right)^{D} \left(\alpha_1^{-1}\,\frac{c_1}{\mathrm{diam}_e K}\right)^{-D},$$

c_1 is the radius of a Euclidean ball contained in U and c_2 is the radius of a Euclidean ball containing U. Moreover,

$$\sigma = (\nu - p)/\nu\,. \tag{3.4}$$

We will call d the *(intrinsic) homogeneous metric*, ν the *(intrinsic) homogeneous dimension* of $K \equiv (K, \mu, \mathcal{L}^{(p)})$.

We now show that the scaling laws for the Lagrangian can be stated more precisely in the intrinsic metric of K.

THEOREM 3.3: *Let $K \equiv (K, \mu, \mathcal{L}^{(p)})$ be a variational fractal endowed with its intrinsic metric. Then, for every $n \geq 1$, we have*

$$\int_K \varphi\, d\mathcal{L}^{(p)}[u] = \sum_{i_1, \dots, i_n = 1}^{N} (\mathrm{diam}\, K_{i_1 \dots i_n}/\mathrm{diam}\, K)^{\nu - p} \int_K \varphi \circ \psi_{i_1 \dots i_n}\, d\mathcal{L}^{(p)}[u \circ \psi_{i_1 \dots i_n}]\,, \tag{3.5}$$

for every $u \in \mathcal{C}^{(p)}$ and for every $\varphi \in C(K)$.

PROOF: By iterating (2.10) along a finite sequences of indices $i_1 \dots i_n \in \{1, \dots, N\}$, $n \geq 1$,

$$\int_K \varphi\, d\mathcal{L}^{(p)}[u] = \sum_{i_1, \dots, i_n = 1}^{N} \varrho_{i_1}^{(p)} \dots \varrho_{i_n}^{(p)} \int_K \varphi \circ \psi_{i_1 \dots i_n}\, d\mathcal{L}^{(p)}[u \circ \psi_{i_1 \dots i_n}]\,.$$

As $\varrho_i^{(p)} = \mu(K_i)^{\sigma}$, $i = 1, \dots, N$, we have, for some real number $\sigma < 1$ independent of i

$$\varrho_{i_1}^{(p)} \dots \varrho_{i_n}^{(p)} = \mu(K_{i_1})^{\sigma} \dots \mu(K_{i_n})^{\sigma} = \alpha_{i_1}^{-d_f \sigma} \dots \alpha_{i_n}^{-d_f \sigma},$$

hence, for (2.6) and (3.1),

$$\varrho_{i_1}^{(p)} \dots \varrho_{i_n}^{(p)} = (\mathrm{diam}\, K_{i_1 \dots i_n}/\mathrm{diam}\, K)^{d_f \sigma/\delta}.$$

By (3.3) and (3.4), this gives

$$\varrho_{i_1}^{(p)} \dots \varrho_{i_n}^{(p)} = (\mathrm{diam}\, K_{i_1 \dots i_n}/\mathrm{diam}\, K)^{\nu - p}. \qquad \blacksquare$$

We also obtain the following «change of variable formula» (for $p = 2$, see [17], Theorem 4.5).

THEOREM 3.4: *Let $K \equiv (K, \mu, \mathcal{L}^{(p)})$ be a variational fractal endowed with its intrinsic metric; let Γ be the boundary of K. Then, for every $n \geq 1$ and for every*

$i_1, \dots, i_n \in \{1, \dots, N\}$, *we have*

$$\int_{K_{i_1 \dots i_n} - \Gamma_{i_1 \dots i_n}} \varphi \, d\mathcal{L}^{(p)}[u] \lfloor (K_{i_1 \dots i_n} - \Gamma_{i_1 \dots i_n}) = \tag{3.6}$$

$$= (\operatorname{diam} K_{i_1 \dots i_n} / \operatorname{diam} K)^{\nu - p} \int_{K - \Gamma} \varphi \circ \psi_{i_1 \dots i_n} \, d\mathcal{L}^{(p)}[u \circ \psi_{i_1 \dots i_n}] \lfloor (K - \Gamma)$$

for every $\varphi \in C(K)$ *with* $\operatorname{supp} \varphi \subset K_{i_1 \dots i_n} - \Gamma_{i_1 \dots i_n}$.

PROOF: Let $i_1, \dots, i_n = 1 \in \{1, \dots, N\}$ be fixed and let $\varphi \in C(K)$ be such that $\operatorname{supp} \varphi \subset K_{i_1 \dots i_n} - \Gamma_{i_1 \dots i_n}$. Since $K_{i_1 \dots i_n} - \Gamma_{i_1 \dots i_n}$ is open in K, the restriction of the Lagrangian to $K_{i_1 \dots i_n} - \Gamma_{i_1 \dots i_n}$ depends only on the restriction of the function u to $K_{i_1 \dots i_n} - \Gamma_{i_1 \dots i_n}$. Therefore, for $\varphi \in C(K)$ with $\operatorname{supp} \varphi \subset K_{i_1 \dots i_n} - \Gamma_{i_1 \dots i_n}$, we have

$$\int_K \varphi \, d\mathcal{L}^{(p)}[u] = \int_{K_{i_1 \dots i_n} - \Gamma_{i_1 \dots i_n}} \varphi \, d\mathcal{L}^{(p)}[u] \lfloor (K_{i_1 \dots i_n} - \Gamma_{i_1 \dots i_n}). \tag{3.7}$$

On the other hand, let us remark that for every $j_1, \dots, j_n \in \{1, \dots, N\}$ with $j_1, \dots, j_n \neq i_1, \dots, i_n$, we have $(K_{i_1 \dots i_n} - \Gamma_{i_1 \dots i_n}) \cap K_{j_1 \dots j_n} = \emptyset$. Therefore, $\varphi \circ \psi_{j_1 \dots j_n} \equiv 0$ on K, whenever $j_1, \dots, j_n \neq i_1, \dots, i_n$. Moreover $\operatorname{supp} \varphi \circ \psi_{i_1 \dots i_n} \subset K - \Gamma$.

Thus

$$\sum_{j_1, \dots, j_n = 1}^{N} (\operatorname{diam} K_{j_1 \dots j_n} / \operatorname{diam} K)^{\nu - p} \int_K \varphi \circ \psi_{j_1 \dots j_n} \, d\mathcal{L}^{(p)}[u \circ \psi_{j_1 \dots j_n}] = \tag{3.8}$$

$$= (\operatorname{diam} K_{i_1 \dots i_n} / \operatorname{diam} K)^{\nu - p} \int_K \varphi \circ \psi_{i_1 \dots i_n} \, d\mathcal{L}^{(p)}[u \circ \psi_{i_1 \dots i_n}] =$$

$$= (\operatorname{diam} K_{i_1 \dots i_n} / \operatorname{diam} K)^{\nu - p} \int_{K - \Gamma} \varphi \circ \psi_{i_1 \dots i_n} \, d\mathcal{L}^{(p)}[u \circ \psi_{i_1 \dots i_n}] \lfloor (K - \Gamma).$$

In order to get (3.6) it suffices now to replace both (3.7) and (3.8) into (3.5) of Theorem 3.3. ■

COROLLARY 3.5: *Under the assumptions of the previous theorem, we have*

$$\int_{K_{i_1 \dots i_n} - \Gamma_{i_1 \dots i_n}} d\mathcal{L}^{(p)}[u] \lfloor (K_{i_1 \dots i_n} - \Gamma_{i_1 \dots i_n}) = \tag{3.9}$$

$$= (\operatorname{diam} K_{i_1 \dots i_n} / \operatorname{diam} K)^{\nu - p} \int_{K - \Gamma} d\mathcal{L}^{(p)}[u \circ \psi_{i_1 \dots i_n}] \lfloor (K - \Gamma).$$

PROOF: Since the Lagrangian is, in particular, a regular measure, from (3.6) we obtain (3.9). ■

We now prove that a family of *scaled Poincaré inequalities* on the homogeneous balls holds. In particular, we show that if the structure enjoys a self-similar invariance then a much simpler starting point can be given to the whole theory: this is the following Poincaré inequality

$$\int_K |u-u(z)|^p\,d\mu \leq c_P \int_{K-\Gamma} d\mathcal{L}^{(p)}[u] \tag{3.10}$$

for every $u \in \mathcal{C}^{(p)}$ and every $z \in \Gamma$.

In fact, the following theorem holds (for p=2, see [17], Theorem 5.1).

THEOREM 3.6: *Let $K \equiv (K, \mu, \mathcal{L}^{(p)})$ be a variational fractal satisfying* (3.10). *Then, there exist two constants $C > 0$ and $q \geq 1$, such that the following inequalities hold*

$$\int_{B(x,\,r)} |u-u_{B(x,\,r)}|^p\,d\mu \leq C(r/\operatorname{diam} K)^p \int_{B(x,\,qr)} d\mathcal{L}^{(p)}[u] \tag{3.11}$$

for every $u \in \mathcal{C}^{(p)}$, where $B(x, r)$ are the balls of the intrinsic metric d, $0 < r \leq \operatorname{diam} K$ and $q = 2^\delta$.

Before proving the theorem, we need some preliminary results.

LEMMA 3.7: *For every $n \geq 1$, for every $i_1, \dots, i_n \in \{1, \dots, N\}$, for every $\zeta \in \Gamma$, we have*

$$\int_{K_{i_1\dots i_n}} |u-u\circ\psi_{i_1\dots i_n}(\zeta)|^p\,d\mu \leq c_P(\operatorname{diam} K_{i_1\dots i_n}/\operatorname{diam} K)^p \int_{K_{i_1\dots i_n}} d\mathcal{L}^{(p)}[u] \tag{3.12}$$

for every $u \in \mathcal{C}^{(p)}$.

PROOF: By (3.10) we have

$$\int_{K_{i_1\dots i_n}} |u-u\circ\psi_{i_1\dots i_n}(\zeta)|^p\,d\mu = (\operatorname{Lip}\psi_{i_1\dots i_n})^{d_f}\int_K |u\circ\psi_{i_1\dots i_n}-u\circ\psi_{i_1\dots i_n}(\zeta)|^p\,d\mu \leq$$

$$\leq c_P\,\alpha_{i_1}^{-d_f}\dots\alpha_{i_n}^{-d_f}\int_{K-\Gamma} d\mathcal{L}^{(p)}[u\circ\psi_{i_1\dots i_n}] = c_P(\operatorname{diam} K_{i_1\dots i_n}/\operatorname{diam} K)^\nu \int_{K-\Gamma} d\mathcal{L}^{(p)}[u\circ\psi_{i_1\dots i_n}];$$

moreover, by Corollary 3.5,

$$\int_{K-\Gamma} d\mathcal{L}^{(p)}[u\circ\psi_{i_1\dots i_n}] = (\operatorname{diam} K_{i_1\dots i_n}/\operatorname{diam} K)^{p-\nu} \int_{K_{i_1\dots i_n}-\Gamma_{i_1\dots i_n}} d\mathcal{L}^{(p)}[u] \leq \tag{3.13}$$

$$\leq (\operatorname{diam} K_{i_1\dots i_n}/\operatorname{diam} K)^{p-\nu} \int_{K_{i_1\dots i_n}} d\mathcal{L}^{(p)}[u]$$

and so (3.12) follows. ∎

Lemma 3.8: *Let $K_{i_1 \dots i_n}$, $K_{j_1 \dots j_n}$, $n \geq 1$, $i_1, \dots, i_n \neq j_1, \dots, j_n \in \{1, \dots, N\}$ be two contiguous complexes. Let $Q = K_{i_1 \dots i_n} \cup K_{j_1 \dots j_n}$. Then, there exists a constant C such that for every $u \in \mathcal{C}^{(p)}$*

$$\int_Q |u - u_Q|^p \, d\mu \leq C \left\{ (\operatorname{diam} K_{i_1 \dots i_n} / \operatorname{diam} K)^p \int_{K_{i_1 \dots i_n}} d\mathcal{L}^{(p)}[u] + \right. \tag{3.14}$$

$$\left. + (\operatorname{diam} K_{j_1 \dots j_n} / \operatorname{diam} K)^p \int_{K_{j_1 \dots j_n}} d\mathcal{L}^{(p)}[u] \right\}.$$

Proof: As $K_{i_1 \dots i_n} \cap K_{j_1 \dots j_n} \neq \emptyset$, there exists $\xi \in K_{i_1 \dots i_n} \cap K_{j_1 \dots j_n} = \Gamma_{i_1 \dots i_n} \cap \Gamma_{j_1 \dots j_n}$ and $\xi = \psi_{i_1 \dots i_n}(\zeta_1) = \psi_{j_1 \dots j_n}(\zeta_2)$ with $\zeta_1, \zeta_2 \in \Gamma$.

We have, by Lemma 3.7

$$\int_{K_{i_1 \dots i_n}} |u - u(\xi)|^p \, d\mu = \int_{K_{i_1 \dots i_n}} |u - u \circ \psi_{i_1 \dots i_n}(\zeta_1)|^p \leq$$

$$\leq c_P (\operatorname{diam} K_{i_1 \dots i_n} / \operatorname{diam} K)^p \int_{K_{i_1 \dots i_n}} d\mathcal{L}^{(p)}[u].$$

In a similar way, we obtain

$$\int_{K_{j_1 \dots j_n}} |u - u(\xi)|^p \, d\mu = \int_{K_{j_1 \dots j_m}} |u - u \circ \psi_{j_1 \dots j_n}(\zeta_2)|^p \leq$$

$$\leq c_P (\operatorname{diam} K_{j_1 \dots j_n} / \operatorname{diam} K)^p \int_{K_{j_1 \dots j_n}} d\mathcal{L}^{(p)}[u].$$

Then as

$$\int_Q |u - u_Q|^p \, d\mu \leq 2^p \int_Q |u - u(\xi)|^p \, d\mu = 2^p \left(\int_{K_{i_1 \dots i_n}} |u - u(\xi)|^p \, d\mu + \int_{K_{j_1 \dots j_n}} |u - u(\xi)|^p \, d\mu \right),$$

we conclude the proof. ■

The following lemma allows us to extend Poincaré inequality across two contiguous sets that overlap on a set of positive measure.

Lemma 3.9: *Let Q_1, Q_2 be two subsets of K such that $\mu(Q_1 \cap Q_2) > 0$. Then,*

$$\int_{Q_1 \cup Q_2} |u - u_{Q_1 \cup Q_2}|^p \, d\mu \leq 2^{p+1} \left(\frac{\mu(Q_1 \cup Q_2)}{\mu(Q_1 \cap Q_2)} \right)^p \max_{i = 1, 2} \int_{Q_i} |u - u_{Q_i}|^p \, d\mu. \tag{3.15}$$

PROOF: We have that

$$
\int_{Q_1 \cup Q_2} |u(x) - u_{Q_1 \cup Q_2}|^p \, d\mu(x) \leqslant \tag{3.16}
$$

$$
\begin{aligned}
&\leqslant 2^p \int_{Q_1 \cup Q_2} |u(x) - u_{Q_1 \cap Q_2}|^p \, d\mu(x) = \\
&= 2^p \left(\frac{1}{\mu(Q_1 \cap Q_2)} \right)^p \int_{Q_1 \cup Q_2} \left| \int_{Q_1 \cap Q_2} (u(x) - u(y)) \, d\mu(y) \right|^p d\mu(x) \leqslant \\
&\leqslant 2^p \left(\frac{1}{\mu(Q_1 \cap Q_2)} \right)^p \int_{Q_1} \left| \int_{Q_1 \cap Q_2} (u(x) - u(y)) \, d\mu(y) \right|^p d\mu(x) + \\
&+ 2^p \left(\frac{1}{\mu(Q_1 \cap Q_2)} \right)^p \int_{Q_2} \left| \int_{Q_1 \cap Q_2} (u(x) - u(y)) \, d\mu(y) \right|^p d\mu(x) \leqslant \\
&\leqslant 2^p \left(\frac{1}{\mu(Q_1 \cap Q_2)} \right)^p \int_{Q_1} \left| \int_{Q_1} (u(x) - u(y)) \, d\mu(y) \right|^p d\mu(x) + \\
&+ 2^p \left(\frac{1}{\mu(Q_1 \cap Q_2)} \right)^p \int_{Q_2} \left| \int_{Q_2} (u(x) - u(y)) \, d\mu(y) \right|^p d\mu(x) = \\
&= 2^p \left(\frac{\mu(Q_1)}{\mu(Q_1 \cap Q_2)} \right)^p \int_{Q_1} \left| u(x) - u_{Q_1} \right|^p d\mu(x) + \\
&+ 2^p \left(\frac{\mu(Q_2)}{\mu(Q_1 \cap Q_2)} \right)^p \int_{Q_2} |u(x) - u_{Q_2}|^p \, d\mu(x) \leqslant \\
&\leqslant 2^{p+1} \left(\frac{\mu(Q_1 \cup Q_2)}{\mu(Q_1 \cap Q_2)} \right)^p \max_{i=1,2} \int_{Q_i} |u(x) - u_{Q_i}|^p \, d\mu(x) . \qquad \blacksquare
\end{aligned}
$$

By iterating Lemma 3.9, we get the following lemma.

LEMMA 3.10: *Let* $Q_1, \dots, Q_m$ *be* $m \geqslant 2$ *subsets of* K *such that* $\mu(Q_s \cap Q_{s+1}) > 0$ *for every* $s = 1, \dots, m-1$. *Let* $Q = Q_1 \cup \dots \cup Q_m$. *Then,*

$$(3.17) \qquad \int_Q |u - u_Q|^p \, d\mu \leqslant$$

$$\leqslant \left[2^{p+1} \left(\frac{\mu(Q)}{\min\limits_{s=1,\dots,m-1} \mu(Q_s \cap Q_{s+1})} \right)^p \right]^{m-1} \max_{s=1,\dots,m} \int_{Q_s} |u - u_{Q_s}|^p \, d\mu \, .$$

From Lemma 3.10 we get the following lemma.

LEMMA 3.11: *Let* $K_{i_1^s \dots i_{n_s}^s}$, $s = 1, \dots, L$, $L \geqslant 3$ *be given* n_s*-complexes,* $n_s \geqslant 1$, $i_1^s, \dots, i_{n_s}^s \in \{1, \dots, N\}$. *Let* $Q = \bigcup\limits_{s=1}^{L} K_{i_1^s \dots i_{n_s}^s}$ *and for each* $s = 1, \dots, L-1$, *let* $Q_s = K_{i_1^s \dots i_{n_s}^s} \cup K_{i_1^{s+1} \dots i_{n_{s+1}}^{s+1}}$. *Then*

$$(3.18) \qquad \int_Q |u - u_Q|^p \, d\mu \leqslant$$

$$\leqslant \left[2^{p+1} \left(\frac{\mu(Q)}{\min\limits_{s=1,\dots,L-2} \mu(K_{i_1^{s+1} \dots i_{n_{s+1}}^{s+1}})} \right)^p \right]^{L-2} \max_{s=1,\dots,L-1} \int_{Q_s} |u - u_{Q_s}|^p \, d\mu \, .$$

We now combine the previous result with Lemma 3.8.

LEMMA 3.12: *Let* $K_{i_1^s \dots i_{n_s}^s}$, $s = 1, \dots, L$, $L \geqslant 3$ *be given* n_s*-complexes,* $n_s \geqslant 1$, $i_1^s, \dots, i_{n_s}^s \in \{1, \dots, N\}$ *for every* $s = 1, \dots, L$, $(i_1^s, \dots, i_{n_s}^s) \neq (i_1^{s+1}, \dots, i_{n_{s+1}}^{s+1})$ *for every* $s = 1, \dots, L-1$. *Then, there exists a constant* C *such that, if* $Q = \bigcup\limits_{s=1}^{L} K_{i_1^s \dots i_{n_s}^s}$, *for every* $u \in \mathcal{C}^{(p)}$ *we have*

$$(3.19) \qquad \int_Q |u - u_Q|^p \, d\mu \leqslant C \left[2^{p+1} \left(\frac{\mu(Q)}{\min\limits_{s=2,\dots,L-1} \mu(K_{i_1^s \dots i_{n_s}^s})} \right)^p \right]^{L-2} \cdot$$

$$\cdot \max_{s=1,\dots,L} (\operatorname{diam} K_{i_1^s \dots i_{n_s}^s} / \operatorname{diam} K)^p \int_{K_{i_1^s \dots i_{n_s}^s}} d\mathcal{L}^{(p)}[u] \, .$$

PROOF: By the previous lemma, the inequality 3.18 holds with $Q_s = K_{i_1^s \dots i_{n_s}^s} \cup K_{i_1^{s+1} \dots i_{n_{s+1}}^{s+1}}$ for every $s = 1, \dots, L-1$. Moreover, by Lemma 3.8, for each $s = 1, \dots, L-1$ we have

$$\int_{Q_s} |u - u_{Q_s}|^p \, d\mu \leqslant C \max_{l=s,\,s+1} (\operatorname{diam} K_{i_1^l \dots i_{n_l}^l} / \operatorname{diam} K)^p \int_{K_{i_1^l \dots i_{n_l}^l}} d\mathcal{L}^{(p)}[u] \, .$$

By taking this inequality into account, we get from (3.18)

$$\int_Q |u - u_Q|^p \, d\mu \leqslant \left[2^{p+1} \left(\frac{\mu(Q)}{\min\limits_{s=1,\dots,L-2} \mu(K_{i_1^{s+1} \dots i_{n_{s+1}}^{s+1}})} \right)^p \right]^{L-2} \cdot$$

$$\cdot \max_{s=1,\dots,L-1} \int_{Q_s} |u - u_{Q_s}|^p \, d\mu \leqslant C \left[2^{p+1} \left(\frac{\mu(Q)}{\min\limits_{s=1,\dots,L-2} \mu(K_{i_1^{s+1} \dots i_{n_{s+1}}^{s+1}})} \right)^p \right]^{L-2} \cdot$$

$$\cdot \max_{s=1,\dots,L-1} \max_{l=s,s+1} (\operatorname{diam} K_{i_1^l \dots i_n^l} / \operatorname{diam} K)^p \int_{K_{i_1^l \dots i_n^l}} d\mathcal{L}^{(p)}[u] \leqslant$$

$$\leqslant C \left[2^{p+1} \left(\frac{\mu(Q)}{\min\limits_{s=2,\dots,L-1} \mu(K_{i_1^s \dots i_{n_s}^s})} \right)^p \right]^{L-2} \cdot$$

$$\cdot \max_{s=1,\dots,L} (\operatorname{diam} K_{i_1^s \dots i_{n_s}^s} / \operatorname{diam} K))^p \int_{K_{i_1^s \dots i_{n_s}^s}} d\mathcal{L}^{(p)}[u]$$

and this proves the lemma. ■

We now prove Theorem 3.6.

Proof of Theorem 3.6: Let $x \in K$, $0 < r \leqslant \operatorname{diam} K$. By Theorem 2.1 we have

$$B(x, r) = K \cap B_e(x, r^{1/\delta}) \subset \bigcup_{G_{x,R}} K_{i_1 \dots i_n}$$

where the family $G_{x,R}$, with $R = r^{1/\delta}$, contains at most M elements. It is not restrictive to assume, up to renumbering, that the sets $K_{i_1^s \dots i_{n_s}^s}$ in $G_{x,R}$ are such that any two successive $K_{i_1^s \dots i_{n_s}^s}$, $K_{i_1^{s+1} \dots i_{n_{s+1}}^{s+1}}$ are contiguous complexes and $Q = \bigcup_{s=1}^{L} K_{i_1^s \dots i_{n_s}^s}$ with $L \leqslant M$.

By Lemma 3.12, we have for $L \leqslant M$,

$$\int_Q |u - u_Q|^p \, d\mu \leqslant C \left[\left(\frac{\mu(Q)}{\min\limits_{s=1,\dots,L} \mu(K_{i_1^s \dots i_{n_s}^s})} \right)^p \right]^{L-2} \cdot \tag{3.20}$$

$$\cdot \max_{s=1,\dots,L} (\operatorname{diam} K_{i_1^s \dots i_{n_s}^s} / \operatorname{diam} K)^p \int_{K_{i_1^s \dots i_{n_s}^s}} d\mathcal{L}^{(p)}[u] \, .$$

We now recall that for every $s = 1, \dots, L$ we have

$$\alpha_1^{-1} r^{1/\delta} < \operatorname{diam}_e K_{i_1^s \dots i_{n_s}^s} \leqslant r^{1/\delta} \qquad K_{i_1^s \dots i_{n_s}^s} \cap B_e(x, r^{1/\delta}) \neq \emptyset \, .$$

Therefore, $K_{i_1^s \dots i_{n_s}^s} \subset K \cap B_e(x, 2r^{1/\delta}) = B(x, 2^\delta r)$ for every $s = 1, \dots, L$. It follows that

$$(3.21) \qquad \int_{B(x,r)} |u(y) - u_{B(x,r)}|^p \, d\mu(y) \le 2^p \int_{B(x,r)} |u(y) - u_Q|^p \, d\mu(y) \le$$

$$\le 2^p \int_Q |u(y) - u_Q|^p \, d\mu(y) \le C \left[\left(\frac{\mu(Q)}{\min_{s=1,\dots,L} \mu(K_{i_1^s \dots i_{n_s}^s})} \right)^p \right]^{L-2} \cdot$$

$$\cdot \max_{s=1,\dots,L} (\operatorname{diam} K_{i_1^s \dots i_{n_s}^s} / \operatorname{diam} K)^p \int_{K_{i_1^s \dots i_{n_s}^s}} d\mathcal{L}^{(p)}[u] .$$

We have

$$\mu(Q) = \sum_{s=1}^{L} \mu(K_{i_1^s \dots i_{n_s}^s}) \le L \left(\frac{r^{1/\delta}}{\operatorname{diam}_e(K)} \right)^{d_f}, \qquad \mu(K_{i_1^s \dots i_{n_s}^s}) > \alpha_1^{-d_f} \left(\frac{r^{1/\delta}}{\operatorname{diam}_e(K)} \right)^{d_f},$$

for every $s = 1, \dots, L$.

Therefore,

$$\left(\frac{\mu(Q)}{\min_{s=1,\dots,L} \mu(K_{i_1^s, \dots, i_{n_s}^s})} \right)^p \le L^p \alpha_1^{p d_f}.$$

Moreover,

$$\max_{s=1,\dots,L} (\operatorname{diam} K_{i_1^s \dots i_{n_s}^s} / \operatorname{diam} K)^p \le \left(\frac{r}{\operatorname{diam} K} \right)^p$$

and

$$\int_{K_{i_1^s \dots i_{n_s}^s}} d\mathcal{L}^{(p)}[u] \le \int_{B(x, 2^\delta r)} d\mathcal{L}^{(p)}[u]$$

for every $s = 1, \dots, L$.

Thus,

$$\int_{B(x,r)} |u - u_{B(x,r)}|^p \, d\mu \le C (L^p \alpha_1^{p d_f})^{L-2} \big(r / \operatorname{diam} K\big)^p \int_{B(x, qr)} d\mathcal{L}^{(p)}[u]$$

for every $u \in \mathcal{C}^{(p)}$. ■

REMARK 3.1: *We recall that if* (3.10) *holds, then*

$$\int_K |u-u_K|^p\,d\mu \le 2^p \int_K |u-u(z)|^p\,d\mu \le 2^p c_P \int_{K-\Gamma} d\mathcal{L}^{(p)}[u]\,,$$

where $u_K = \int_K u\,d\mu$. *We remark that if* K *is connected in capacity sense according to the Definition* 5.1 *in* [17], *the starting Poincaré inequality* (3.10) *can be replaced with by the weaker assumption*

$$\int_K |u-u_K|^p\,d\mu \le c_P \int_{K-\Gamma} d\mathcal{L}^{(p)}[u]:$$

in this case, Lemma 3.8 *still holds with suitable changes and Theorem* 3.6 *can be achieved with the same proof.*

4. - LAGRANGIANS AND ENERGIES

The Lagrangian formalism is based on the definition of a local energy $\mathcal{L}^{(p)}$, that, when integrated on a structure X, gives the total energy $E^{(p)}$ of X:

$$E^{(p)} = \int_X d\mathcal{L}^{(p)}.$$

In this section we begin by proving the following «representation formula» (for the classical case $p=2$, see [8] and [14]). In the following, we use the notations $\mathcal{L}^{(p)}[u] = \mathcal{L}^{(p)}(u,u)$.

THEOREM 4.1: *For any* $u, \varphi \in \mathcal{C}^{(p)}$, $u \ge \varepsilon > 0$, *we have*

$$\int_X \varphi d\mathcal{L}^{(p)}[u] = \frac{1}{(p-1)^2} E^{(p)}(u, u\varphi) - \frac{1}{p^{p-1}(p-1)^2} E^{(p)}(u^p, u^{p(2-p)}\varphi)\,.$$

PROOF: From the Leibniz rule on the second argument and the chain rules, we have

$$\begin{aligned}\frac{1}{(p-1)^2} E^{(p)}(u,u\varphi) &- \frac{1}{p^{p-1}(p-1)^2} E^{(p)}(u^p, u^{p(2-p)}\varphi) = \\ &= \frac{1}{(p-1)^2}\int_X d\mathcal{L}^{(p)}(u,u\varphi) - \frac{1}{p^{p-1}(p-1)^2}\int_X d\mathcal{L}^{(p)}(u^p, u^{p(2-p)}\varphi) \\ &= \frac{1}{(p-1)^2}\int_X \varphi d\mathcal{L}^{(p)}(u,u) + \frac{1}{(p-1)^2}\int_X u d\mathcal{L}^{(p)}(u,\varphi) -\end{aligned}$$

$$- \frac{1}{p^{p-1}(p-1)^2} \int_X |pu^{p-1}|^{p-2} pu^{p-1} \, d\mathcal{L}^{(p)}(u, u^{p(2-p)}\varphi) =$$

$$= \frac{1}{(p-1)^2} \int_X \varphi d\mathcal{L}^{(p)}(u, u) + \frac{1}{(p-1)^2} \int_X u d\mathcal{L}^{(p)}(u, \varphi) -$$

$$- \frac{1}{p^{p-1}(p-1)^2} \int_X |pu^{p-1}|^{p-2} pu^{p-1} u^{p(2-p)} \, d\mathcal{L}^{(p)}(u, \varphi) -$$

$$- \frac{1}{p^{p-1}(p-1)^2} \int_X |pu^{p-1}|^{p-2} pu^{p-1} p(2-p) \, u^{p(2-p)-1} \varphi d\mathcal{L}^{(p)}(u, u) =$$

$$= \int_X \varphi d\mathcal{L}^{(p)}(u, u). \qquad \blacksquare$$

The self-similar property of the total energy $E^{(p)}$ follows from the self-similar property of the relative Lagrangian trivially.

Next, we prove the converse: more precisely, we prove that if the total energy is self-similar then the Lagrangian inherits this same invariance property.

THEOREM 4.2: *Let $E^{(p)}$ be self-similar, that is, for every $u, v \in \mathcal{C}^{(p)}$ and for every $\varphi \in C(K)$,*

$$E^{(p)}(u, v) = \sum_{i=1}^{N} \varrho_i^{(p)} E^{(p)}(u \circ \psi_i, v \circ \psi_i)$$

with the real constants $\varrho_i^{(p)} > 0$, $i = 1, \dots, N$, satisfy $\varrho_i^{(p)} = \mu(K_i)^{\sigma}$, $i = 1, \dots, N$, for some real constant $\sigma < 1$, independent of $i = 1, \dots, N$.

Then, the Lagrangian $\mathcal{L}^{(p)}$ is self-similar, that is,

$$\int_K \varphi \, d\mathcal{L}^{(p)}[u] = \sum_{i=1}^{N} \varrho_i^{(p)} \int_K \varphi \circ \psi_i d\mathcal{L}^{(p)}[u \circ \psi_i], \tag{4.1}$$

for every $u \in \mathcal{C}^{(p)}$ and for every $\varphi \in C(K)$.

PROOF: We set $u^{\natural} = u - \min_K u + \varepsilon$, with $\varepsilon > 0$. By the strong locality and Theorem 4.1, we have that

$$\int_K \varphi \, d\mathcal{L}^{(p)}[u] = \int_K \varphi \, d\mathcal{L}^{(p)}[u^{\natural}] = \tag{4.2}$$

$$= \frac{1}{(p-1)^2} E^{(p)}(u^{\natural}, u^{\natural}\varphi) - \frac{1}{p^{p-1}(p-1)^2} E^{(p)}((u^{\natural})^p, (u^{\natural})^{p(2-p)}\varphi) =$$

$$= \sum_{i=1}^{N} \varrho_i^{(p)} \left\{ \frac{1}{(p-1)^2} E^{(p)}(u^\natural \circ \psi_i, (u^\natural \circ \psi_i)(\varphi \circ \psi_i)) - \right.$$

$$\left. \frac{1}{p^{p-1}(p-1)^2} E^{(p)}((u^\natural \circ \psi_i)^p, (u^\natural \circ \psi_i)^{p(2-p)}(\varphi \circ \psi_i)) \right\} =$$

$$= \sum_{i=1}^{N} \varrho_i^{(p)} \int_K \varphi \circ \psi_i \, d\mathfrak{L}^{(p)}[u^\natural \circ \psi_i] = \sum_{i=1}^{N} \varrho_i^{(p)} \int_K \varphi \circ \psi_i \, d\mathfrak{L}^{(p)}[u \circ \psi_i]. \qquad \blacksquare$$

5. - An example

A first example of nonlinear forms on fractals has been given in [4]. More precisely, self-similar energy forms $E^{(p)}$ with domains $\mathcal{C}^{(p)}$ have been constructed on the Koch curve type fractals by using suitable sequences of finite differences schemes.

We now show how we can construct the corresponding Lagrangians on these fractals. For simplicity, we consider the well known Koch curve (on generalized Koch curves, we can proceed in a analogous way just by making some small proper changes). The Koch curve K is a nested fractal (see [12]) and, in particular, it is the invariant set of a suitable family $\Psi = \{\psi_1, \ldots, \psi_N\}$, with $N = 4$, $\alpha_i = \alpha = 3$ satisfying (2.1), (2.4) and (2.7). Let $\mathcal{C}^{(p)}$ be the domain of the energy form $E^{(p)}$ defined by Theorem 3.1 in [4].

We define, for any set $A \subset K$ and $u \in \mathcal{C}^{(p)}$,

$$\widetilde{\mathfrak{L}}_n^{(p)}(u)(A) := \frac{1}{p}(4^{p-1})^n \sum_{\xi, \eta \in \Gamma} \sum_{\substack{i_1 \ldots i_n = 1 \\ \psi_{i_1 \ldots i_n}(\xi),\, \psi_{i_1 \ldots i_n}(\eta) \in A}}^{N} |u(\psi_{i_1 \ldots i_n}(\xi)) - u(\psi_{i_1 \ldots i_n}(\eta))|^p$$

and

$$\widetilde{\mathfrak{L}}^{(p)}(u)(A) := \lim_{n \to \infty} \widetilde{\mathfrak{L}}_n^{(p)}(u)(A).$$

The previous limit exists by Theorem 3.1 in [4]. Moreover, as $\widetilde{\mathfrak{L}}^{(p)}(u)$ is positive and finitely additive, that is,

$$\widetilde{\mathfrak{L}}^{(p)}(u)(A \cup B) = \widetilde{\mathfrak{L}}^{(p)}(u)(A) + \widetilde{\mathfrak{L}}^{(p)}(u)(B)$$

if $A \cap B = \emptyset$, by the Caratheodory extension theorem (see [7]), $\widetilde{\mathfrak{L}}^{(p)}(u)$ extends to a finite Borel measure that we denote again $\widetilde{\mathfrak{L}}^{(p)}(u)$.

By the definition, the assumptions i), ii), iv) are satisfied; assumption iii) follows from Proposition 4.2 and Theorems 4.1 and 4.2 in [4]. Assumption v) can be verified in the following way. First, we observe that

$$\langle \partial \widetilde{\mathcal{L}}_n^{(p)}(u), v\rangle = (4^{p-1})^n \sum_{i_1 \dots i_n = 1}^{N} \sum_{\xi, \eta \in \Gamma} |u(\psi_{i_1 \dots i_n}(\xi)) - u(\psi_{i_1 \dots i_n}(\eta))|^{p-2} \cdot$$

$$\cdot (u(\psi_{i_1 \dots i_n}(\xi)) - u(\psi_{i_1 \dots i_n}(\eta))) \cdot (v(\psi_{i_1 \dots i_n}(\xi)) - v(\psi_{i_1 \dots i_n}(\eta))) .$$

Since $\mathcal{C}^{(p)}$ is reflexive (it is a uniformly convex Banach space by Theorem 4.1 in [4]) and

$$\lim_{n \to \infty} \widetilde{\mathcal{L}}_n^{(p)}(u) = \widetilde{\mathcal{L}}^{(p)}(u) ,$$

by Theorem 3.66 in [1], we obtain that

$$\partial \widetilde{\mathcal{L}}_n^{(p)}(u) \to \partial \widetilde{\mathcal{L}}^{(p)}(u) ;$$

so, we have for $u, v \in \mathcal{C}^{(p)}$

$$\mathcal{L}^{(p)}(u, v) = \lim_{n \to \infty} (4^{p-1})^n \sum_{i_1 \dots i_n = 1}^{N} \sum_{\xi, \eta \in \Gamma} |u(\psi_{i_1 \dots i_n}(\xi)) - u(\psi_{i_1 \dots i_n}(\eta))|^{p-2} \cdot$$

$$\cdot (u(\psi_{i_1 \dots i_n}(\xi)) - u(\psi_{i_1 \dots i_n}(\eta))) \cdot (v(\psi_{i_1 \dots i_n}(\xi)) - v(\psi_{i_1 \dots i_n}(\eta))) .$$

Finally, assumption vi) follows by taking into account the previous expression of $\mathcal{L}^{(p)}$. Then, the Koch curve is a variational fractal according to Definition 2.4. Moreover, since assumption (3.10) is satisfied by Proposition 3.1 in [4], Theorem 3.6 holds and hence we obtain the scaled Poincaré inequalities (3.11).

These inequalities establish a further important connection between the homogeneous structure and the energy form. As shown in [13] and in [5], in the present general setting of measure-valued p-Lagrangians on homogeneous spaces, a whole family of important inequalities can be obtained from Theorem 3.6. In particular, when the homogeneous dimension is smaller than p, as in this case, we have the Morrey embedding.

As a consequence of this intrinsic Morrey embedding, that is, $\mathcal{C}^{(p)} \subset C^{0,\beta}$ with $\beta = 1 - \dfrac{\nu}{p}$, we obtain the Euclidean embedding $\mathcal{C}^{(p)} \subset C^{0,\beta_e}_{\mathrm{eucl}}$ where now $C^{0,\beta_e}_{\mathrm{eucl}}$ is the space of Hölder continuous functions with Hölder exponent $\beta_e = \delta\beta$ in the Euclidean metric of K: so we find the Euclidean estimates (first obtained by direct calculations in [4])

$$|u(x) - u(y)| \leqslant C|x-y|^{\beta_e} (E^{(p)}[u])^{\frac{1}{p}}$$

with

$$\beta_e = \delta\beta = \delta\left(1 - \frac{\nu}{p}\right) = \delta - \frac{d_f}{p} = \frac{\ln_\alpha N\varrho^{(p)}}{p} - \frac{\ln_\alpha N}{p} = \frac{\ln_\alpha \varrho^{(p)}}{p}\,,$$

where we have taken into account the expressions (2.5) and (3.2).

We recall that similar Euclidean estimates can be obtained by considering the identification of the domains of the nonlinear energy forms with suitable Lipschitz spaces (see [6]).

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Direttore responsabile: Prof. A. BALLIO - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5 - Bologna