Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
121° (2003), Vol. XXVII, fasc. 1, pagg. 145-158

M. BAHAJ - I. CHERTI (*)

Fractional Powers of Operators and Almost Periodic Solutions for Semilinear Equations (**)

ABSTRACT. — We establish the existence of bounded, periodic, almost periodic and asymptotically almost periodic classical solutions of semilinear equations. The key of our approach is the employment of the theory of fractional powers of operators.

Potenze frazionarie di operatori e soluzioni quasi periodiche per equazioni semilineari

SUNTO. — Per equazioni semilineari si stabiliscono risultati di esistenza di soluzioni classiche limitate che siano periodiche, quasi periodiche o asintoticamente quasi periodiche. Lo strumento chiave è costituito dalla teoria delle potenze frazionarie di operatori.

1. - INTRODUCTION

Periodic and almost periodic solutions of semilinear equations are of great interest both in applications and theory. Various methods were developed to ensure the existence of periodic and almost periodic solutions, see for instance [2, 18, 19, 20, 21, 22].

Consider the evolution equation

$$x'(t) = A(t)\,x(t) + f(t). \tag{1}$$

Equations (1.0) with almost periodic $A(t)$ are treated in [4, 9, 11].

(*) Indirizzo degli Autori: Université Hassan 1, Faculté des Sciences & Techniques, Département de Mathématiques & Informatique, Settat - Maroc. E-mail: bahaj@caramail.com; iliascherti@yahoo.fr

(**) Memoria presentata l'11 novembre 2003 da Giorgio Letta, uno dei XL.

It is shown in [10] that U (the evolution family solving the homogeneous problem) has an exponential dichotomy with an almost periodic Green's function if and only if there is a unique almost periodic mild solution of (1.0).

Recently, L. Maniar and R. Schnaubelt in [13] proved the (asymptotic) almost periodicity of the bounded solution to the parabolic evolution equation (1.0) on $\mathbb{R}$ (on $\mathbb{R}_+$) assuming that the linear operators $A(t)$ satisfy the «Acquistapace-Terreni» conditions, that the evolution family generated by $A(.)$ has an exponential dichotomy, and $R(\omega, A(t))$ and f are (asymptotically) almost periodic. The method used is similar to Henry's approach in [9, § 7.6] who derived the almost periodicity of Green's function corresponding to U.

In [3] Ballotti, Goldstein and Parrott gave necessary and sufficient conditions for the existence of almost periodic solutions of the associated homogeneous equation of (1.0):

$$x'(t) = A(t)\, x(t)$$

where $A(t)$ is the generator of a $\mathcal{C}_0$ semigroup on a Banach space.

These authors used the mean ergodic theorem.

In this paper we consider the semilinear equation

$$x'(t) + Ax(t) = f(t, x(t)) \tag{1.1}$$

where $-A$ is the infinitesimal generator of an analytic $\mathcal{C}_0$ semigroup $S(t)$ verifying the exponential stability and investigate inheritance of asymptotic almost periodicity or uniform almost periodicity from f to a (classical) solution of (1.1).

In [17], Prüss proved the existence of periodic mild solution of (1.1) under some conditions of compactness.

In [2], we proved that if $-A$ is the infinitesimal generator of a compact semigroup $\{S(t)\}_{t \geq 0}$ satisfying $|S(t)|_{\mathcal{L}(X)} \leq e^{\beta t}$ $(\beta < 0)$, then equation (1.1) has at least one almost periodic mild solution.

In the case when f is uniformly Lipschitz continuous with a Lipschitz constant small enough, existence and uniqueness of an almost periodic mild solution of (1.1) were proved in [19].

We will be interested in imposing further conditions on f so that the almost periodic mild solution becomes a almost periodic (classical) solution and thus proving, under these conditions, the existence of almost periodic solution of (1.1).

Throughout this work, we denote by X a real or complex Banach space endowed with a norm $|.|$ and $\mathcal{L}(X)$ stands for the Banach algebra of bounded linear operators defined on X with a norm $|.|_{\mathcal{L}(X)}$. If A is a linear operator, we denote by $D(A)$, resp. $\mathcal{R}(A)$, the domain, resp. the range of A, and by $\varrho(A)$ the resolvent set of A.

If $-A$ is the infinitesimal generator of an analytic semigroup in a Banach space and $0 \in \varrho(A)$, for any $\alpha > 0$ we define the fractional power $A^{-\alpha}$ by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t)\, dt.$$

Let $0 < \alpha \leq 1$, set

$$A^{\alpha} = (A^{-\alpha})^{-1}.$$

Note that A^{α} is closed linear operator whose domain $D(A^{\alpha}) \supset D(A)$ is dense in X. The closedness of A^{α} implies that $D(A^{\alpha})$ endowed with the graph norm of A^{α}:

$$|x|_{D(A)} = |x| + |A^{\alpha} x|, \quad x \in D(A^{\alpha})$$

is a Banach space. Since $0 \in \varrho(A)$, A^{α} is invertible, its graph norm is equivalent to the norm $|x|_{\alpha} = |A^{\alpha} x|$. Thus $D(A^{\alpha})$ equipped with the norm $|.|_{\alpha}$ is a Banach space which we denote X_{α}.

(Fore more details, we refer the reader to [8,16]).

Our main assumptions in this work will be:

1. The operator $-A$ is the infinitesimal of an analytic semigroup $\{S(t)\}_{t \geq 0}$ satisfying $|S(t)|_{\mathcal{L}(X)} \leq M \exp(-\delta t)$, $\forall t > 0$, $(\delta > 0)$.

2. $(A1)$ The function $f : \mathbb{R} \times X_{\alpha} \to X$ verifies the assumption:

$$|f(t_1, x_1) - f(t_2, x_2)| \leq L(R)(|t_1 - t_2|^{\theta} + |x_1 - x_2|_{\alpha})$$

if $x_1, x_2 \in X_{\alpha}$, $|x_1|_{\alpha}$, $|x_2|_{\alpha} \leq R$, and $t_1, t_2 \in \mathbb{R}$ where $L : [0, +\infty) \to [0, +\infty)$ is a continuous nondecreasing function such that $L(0) = 0$,

$(A2)$ The function $f(t, 0)$ vanishes for all $t \in \mathbb{R}$.

We shall prove that under these assumptions, then if is uniformly asymptotically almost periodic (resp. uniformly almost periodic), equation (1.1) has a unique asymptotically almost periodic (resp. almost periodic) solution.

The proof makes use of the theory of fractional powers of operators.

More precisely, our main effort is to establish that the map

$$(Tx)(t) = \int_{-\infty}^{t} A^{\alpha} S(t-\sigma)\, f(\sigma, A^{-\alpha} x(\sigma))\, d\sigma$$

is a strict contraction.

Our work is organized as follows. The second section is devoted to a review of some results almost periodic functions with values in a Banach space. In section 3 we state and prove our main results. The last section is devoted to giving some examples illustrating the abstract results.

2. - Almost periodic functions in Banach spaces

The theory of almost periodic functions with values in a Banach space was developed by H. Bohr, S. Bochner, J. von Neumann, and others; cf., e.g., [1,5].

We mention several known results which will be used in this work.

We let $C_b(\mathbb{R}, X)$ denote the usual Banach space of bounded continuous functions from $\mathbb{R}$ into X under the supremum norm $|.|_\infty$. Further, given a function $f : \mathbb{R} \to X$ and $\omega \in \mathbb{R}$, the ω-translate f_ω of f is defined by $f_\omega(t) = f(t+\omega)$, $t \in \mathbb{R}$, and $H(f) = \{f_\omega \colon \omega \in \mathbb{R}\}$ will denote the set of all translates of f.

Definition 2.1 (Bochner's characterization of almost periodicity): *A function $f \in C_b(\mathbb{R}, X)$ is said to be almost periodic if and only if $H(f)$ is relatively compact in $C_b(\mathbb{R}, X)$.*

Of course, almost periodic functions can as well be characterized in terms of relatively dense sets in $\mathbb{R}$ of τ-almost periods.

Definition 2.2: *A function $f : \mathbb{R} \to X$ is called almost periodic if*

i) *f is continuous, and*

ii) *for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$, such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that*:

$$|f(t+\tau) - f(t)| < \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

An almost periodic function is bounded, uniformly continuous on $\mathbb{R}$ and has a relatively compact range in X.

Next. Y denotes a Banach space and Ω is an open subset of Y.

Definition 2.3: *A continuous function $f : \mathbb{R} \times \Omega \to X$ is called uniformly almost periodic if for every $\varepsilon > 0$ and every compact set $K \subset \Omega$ there exists a relatively dense set P_ε in $\mathbb{R}$ such that $|f(t+\tau, x) - f(t, x)| \leqslant \varepsilon$ for all $t \in \mathbb{R}$, $\tau \in P_\varepsilon$ and all $x \in K$.*

The essential result is the following ([18], theorem I.2.7)

Lemma 2.1: *Let $f : \mathbb{R} \times \Omega \to X$ be uniformly almost periodic and $y : \mathbb{R} \to \Omega$ be an almost periodic function such that $\overline{\mathcal{R}(y)} \subset \Omega$, then the function $t \to f(t, y(t))$ also is almost periodic.*

The space of almost periodic functions with values in X will be denoted $AP(X)$.

The concept of asymptotic almost periodicity was introduced by M. Fréchet in [6, 7].

DEFINITION 2.4: *A function* $f : \mathbb{R}^+ \to X$ *is called asymptotic almost periodic* (a.a.p) *if*

i) f *is continuous, and*

ii) *for each* $\varepsilon > 0$, $\exists T(\varepsilon) \geq 0$ *and* $l(\varepsilon) > 0$, *such that every interval* I *of length* $l(\varepsilon)$ *contains a number* τ *with the property that*:

$$\sup_{t \geq T(\varepsilon)} |f(t+\tau) - f(t)| < \varepsilon .$$

An asymptotically almost periodic function is bounded, uniformly continuous on $\mathbb{R}^+$ and the set $\{f(t),\ t \in \mathbb{R}^+\}$ is relatively compact in X.

DEFINITION 2.5: *A continuous function* $f : \mathbb{R}^+ \times \Omega \to X$ *is called uniformly a.a.p (abbreviated, u.a.a.p.) if, for every* $\varepsilon > 0$ *and every compact set* $K \subset \Omega$, *there exists a relatively dense set* P_ε *in* $\mathbb{R}^+$ *and* $T(\varepsilon) > 0$ *such that*

$$|f(t+\tau, x) - f(t, x)| < \varepsilon \text{ for all } t \geq T(\varepsilon),\ \tau \in P_\varepsilon, \text{ and all } x \in K.$$

LEMMA 2.2: *Let* $f : \mathbb{R}^+ \times \Omega \to X$ *be u.a.a.p. and* $y : \mathbb{R}^+ \to \Omega$ *be an a.a.p. function such that* $\overline{\mathcal{R}(y)} \subset \Omega$, *then the function* $t \to f(t, y(t))$ *also is a.a.p.*

The space of a.a.p. functions with values in X will be denoted $AAP(X)$.

3. - MAIN RESULTS

The following Proposition presents a simplified version of the results in [12] where the reader is referred to [9, 12] for more details on the subject. Here, a direct approach by means of fractional powers was proposed.

PROPOSITION 3.1: *Assume* (A1) *and* (A2) *hold, then Equation* (1.1) *has a unique bounded solution.*

PROOF: Let R be chosen so that

$$L(R) < \delta^{1-\alpha} \sin \pi\alpha \Gamma(\alpha)\, M_\alpha^{-1}$$

$(0 < \alpha < 1)$ and set $D = \{x \in \mathcal{C}_b(\mathbb{R}; X) : |x|_\infty \leq R\}$.

On D we define a mapping T by

$$(Tx)(t) = \int_{-\infty}^{t} A^\alpha S(t-\sigma)\, f(\sigma, A^{-\alpha} x(\sigma))\, d\sigma \tag{3.1}$$

We have to show that T maps D into itself.

$$|(Tx)(t)| \leqslant \int_{-\infty}^{t} |A^{\alpha} S(t-\sigma)[f(\sigma, A^{-\alpha} x(\sigma)) - f(\sigma, 0)]| d\sigma$$

$$\leqslant M_{\alpha} RL(R) \int_{0}^{+\infty} \sigma^{-\alpha} \exp(-\delta\sigma)\, d\sigma = M_{\alpha} RL(R)\, \delta^{\alpha-1} \Gamma(1-\alpha)$$

where $\Gamma(.)$ is the classical gamma function.

We use the well known identity

$$\Gamma(\alpha)\, \Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha} \qquad \text{for } 0 < \alpha < 1.$$

Then a simple computation yields that $T: D \to D$ is a contraction.

Therefore there exists $x \in D$ such that

$$x(t) = \int_{-\infty}^{t} A^{\alpha} S(t-\sigma)\, f(\sigma, A^{-\alpha} x(\sigma))\, d\sigma \tag{3.2}$$

Next we want to show that $t \to f(t, A^{-\alpha} x(t))$ is Hölder continuous on $\mathbb{R}$. To this end we show first that the solution $x(t)$ of (3.2) is Hölder continuous on $\mathbb{R}$.

We note that for every β satisfying $0 < \beta < 1 - \alpha$ we have:

$$|(S(h) - I)\, A^{\alpha} S(t-\sigma)| \leqslant C_{\beta} h^{\beta}\, |A^{\alpha+\beta} S(t-\sigma)| \tag{3.3}$$

$$|x(t+h) - x(t)| \leqslant \left| \int_{-\infty}^{t} (S(h) - I)\, A^{\alpha} S(t-\sigma)\, f(\sigma, A^{-\alpha} x(\sigma))\, d\sigma \right| + \tag{3.4}$$

$$+ \left| \int_{t}^{t+h} A^{\alpha} S(t+h-\sigma)\, f(\sigma, A^{-\alpha} x(\sigma))\, d\sigma \right|.$$

We estimate each of the terms of (3.4) separately.

Using [16, p. 74, theorem 6.13 c] and (3.3) we have

$$\left| \int_{-\infty}^{t} (S(h) - I)\, A^{\alpha} S(t-\sigma)\, f(\sigma, A^{-\alpha} x(\sigma)) - f(\sigma, 0)\, d\sigma \right| \leqslant$$

$$\leqslant M_{\alpha+\beta} RL(R)\, C_{\beta} h^{\beta} \int_{-\infty}^{t} (t-\sigma)^{-(\alpha+\beta)} \exp(-\delta(t-\sigma))\, d\sigma.$$

We also have, by $(A1),(A2)$ and [16, p. 74, theorem 6.13 c], that:

$$\left| \int_t^{t+h} A^\alpha S(t+h-\sigma)\ f(\sigma, A^{-\alpha}\varphi(\sigma))\ d\sigma \right| \leq M_\alpha RL(R) \int_t^{t+h} (t+h-\sigma)^{-\alpha}\, d\sigma$$

$$\leq M_\alpha RL(R)\ \frac{h^{1-\alpha}}{1-\alpha}\ .$$

Combining (3.4) with these estimates it follows that there is a constant C such that

$$|x(t+h) - x(t)| \leq Ch^\beta$$

and therefore x is Hölder continuous on $\mathbb{R}$.

Finally, we show now that $t \to f(t, A^{-\alpha}x(t))$ is Hölder continuous on $\mathbb{R}$. So, in view of $(A1)$, we have:

$$|f(t, A^{-\alpha}x(t)) - f(s, A^{-\alpha}x(s))| \leq L(R)(|t-s|^\theta + |x(t) - x(s)|).$$

Therefore $t \to f(t, A^{-\alpha}(t))$ is Hölder continuous on $\mathbb{R}$.

Let x be the solution of (3.2) and consider the equation

$$\frac{dy(t)}{dt} + Ay(t) = f(t, A^{-\alpha}x(t)) \tag{3.5}$$

This equation has a unique bounded solution y given by

$$y(t) = \int_{-\infty}^{t} S(t-\sigma)\ f(\sigma, A^{-\alpha}x(\sigma))\ d\sigma. \tag{3.6}$$

Moreover, we have $y(t) \in D(A)$ for all $t \in \mathbb{R}$ and a fortiori $y(t) \in D(A^\alpha)$. Operating on both sides of (3.6) with A^α we have

$$\begin{aligned} A^\alpha y(t) &= \int_{-\infty}^{t} A^\alpha S(t-\sigma)\ f(\sigma, A^{-\alpha}x(\sigma))\ d\sigma \\ &= x(t) \end{aligned} \tag{3.7}$$

From (3.5) and (3.7) we easily see that $y(t) = A^{-\alpha}x(t)$ is a solution of (1.1). On the other hand if $x, y \in \mathcal{C}_b(\mathbb{R}; X)$ are bounded solutions of (1.1), let R such that $|x|_\infty$, $|y|_\infty \leq R$.

$$x(t) - y(t) = \int_{-\infty}^{t} A^\alpha S(t-\sigma)[f(\sigma, A^{-\alpha}x(\sigma)) - f(\sigma, A^{-\alpha}y(\sigma))]\ d\sigma.$$

$$|x(t) - y(t)| \leq L(R)\ M_\alpha \int_{-\infty}^{t} (t-\sigma)^{-\alpha} \exp(-\delta(t-\sigma))\, |x(\sigma) - y(\sigma)|\, d\sigma,$$

then the Gronwall-Bellman lemma implies that $x(t) = y(t)$.

REMARK 3.1: **1.** Observe that the proof of proposition 3.1 ensures the existence of closed, convex bounded subset of $\mathcal{C}_b(\mathbb{R}; X)$ invariant for T, which will be used in the sequel.

2. Existence and boundedness results in Hölder spaces obtained without making use the above assumptions, but using techniques of interpolation spaces and interpolatory estimates, may be found in [12].

More precisely, in her book [12, chap. 7] A. Lunardi treated equations of the type (1.1), where $A : D(A) \to X$ is a linear sectorial operator, f is a continuous function defined in $[0, T] \times X_\alpha$, $X_0 = X$, and for $0 < \alpha < 1$, X_α is any Banach space continuously embedded in X and such that: $D_A(\alpha, 1) \subset X_\alpha \subset D_A(\alpha, \infty)$ and the part of A in X_α is sectorial in X_α. Under several regularity assumptions on f, the author gives some sufficient conditions for the existence of bounded solutions.

It has been shown in [15] that if there is a bounded solution of a periodic ordinary differential equation and the solutions can be continued for all futures times then the O.D.E. has a periodic solution. In [21], a similar result was extended to semilinear equations. By using the Massera approach, a similar result is valid in this case.

COROLLARY 3.1: *Suppose that the hypotheses of proposition* 3.1 *hold and that* $f(t, x)$ *is* ω*-periodic at* t, *then there is an* ω*-periodic classical solution of* (1.1).

PROOF: Let x be the bounded solution of (3.2) and let T the map given by (3.1). Defining $y(t) = x(t+\omega)$ $t \in \mathbb{R}$, it is easy to see that

$$\begin{aligned} y(t) &= x(t+\omega) \\ &= \int_{-\infty}^{t+\omega} A^\alpha S(t+\omega-\sigma)\, f(\sigma, A^{-\alpha} x(\sigma))\, d\sigma \\ &= \int_{-\infty}^{t} A^\alpha S(t-\sigma)\, f(\sigma+\omega, A^{-\alpha} x(\sigma+\omega))\, d\sigma \\ &= \int_{-\infty}^{t} A^\alpha S(t-\sigma)\, f(\sigma, A^{-\alpha} y(\sigma))\, d\sigma \\ &= (Ty)(t). \end{aligned}$$

Therefore, from the uniqueness of the fixed point of T, it follows that $y = x$, which implies that x is ω-periodic.

THEOREM 3.1: *If the hypotheses of proposition* 3.1 *hold and f is uniformly asymptotically almost periodic, then there is a unique asymptotically almost periodic solution of equation* (1.1).

PROOF: We consider the set $\widetilde{D} = D \cap AAP$. It is clear that $\widetilde{D}$ is a closed subset of $\mathcal{C}_b([0, +\infty); X)$.

Let $x \in \widetilde{D}$, we show first that the function

$$(Tx)(t) = \int_{-\infty}^{t} A^{\alpha} S(t-\sigma)\, f(\sigma, A^{-\alpha} x(\sigma))\, d\sigma$$

when restricted to $\mathbb{R}^+$ is asymptotic almost periodic.

Note that the function $t \to f(t, A^{-\alpha} x(t))$ is a.a.p (see lemma 2.2).

Given $\varepsilon > 0$, consider numbers $T(\varepsilon)$ and $L(\varepsilon)$ corresponding to $t \to \to f(t, A^{-\alpha} x(t))$.

In any interval $[a, a+L] \subset \mathbb{R}^+$, take a τ such that

$$|f(t+\tau, A^{-\alpha} x(t+\tau)) - f(t, A^{-\alpha} x(t))| < \varepsilon \quad \text{for all } t \geq T(\varepsilon).$$

Then we have that

$$\begin{aligned}
&|(Tx)(t+\tau) - (Tx)(t)| = \\
&= \left| \int_{-\infty}^{t+\tau} A^{\alpha} S(t+\tau-\sigma)\, f(\sigma, A^{-\alpha} x(\sigma))\, d\sigma - \int_{-\infty}^{t} A^{\alpha} S(t-\sigma)\, f(\sigma, A^{-\alpha} x(\sigma))\, d\sigma \right| \leq \\
&\leq \int_{-\infty}^{t} |A^{\alpha} S(t-\sigma)[f(\sigma+\tau, A^{-\alpha} x(\sigma+\tau)) - f(\sigma, A^{-\alpha} x(\sigma))]|\, d\sigma \leq \\
&\leq 2RL(R) \int_{-\infty}^{T(\varepsilon)} |A^{\alpha} S(t-\sigma)|_{\mathcal{L}(X)}\, d\sigma + \\
&+ \int_{T(\varepsilon)}^{t} |A^{\alpha} S(t-\sigma)|_{\mathcal{L}(X)}\, |[f(\sigma+\tau, A^{-\alpha} x(\sigma+\tau)) - f(\sigma, A^{-\alpha} x(\sigma))]|\, d\sigma \leq \\
&\leq 2RL(R) \int_{t-T(\varepsilon)}^{+\infty} \sigma^{-\alpha} e^{-\delta\sigma}\, d\sigma + \varepsilon \int_{0}^{t-T(\varepsilon)} \sigma^{-\alpha} e^{-\delta\sigma}\, d\sigma \leq \\
&\qquad\qquad \leq 2RL(R) \int_{t-T(\varepsilon)}^{+\infty} \sigma^{-\alpha} e^{-\delta\sigma}\, d\sigma + \varepsilon \delta^{\alpha} \Gamma(1-\alpha).
\end{aligned}$$

Take now any $\varepsilon' > 0$, then choose ε such that $\varepsilon \delta^{\alpha} \Gamma(1-\alpha) < \dfrac{\varepsilon'}{2}$ and then $T_1(\varepsilon') > T(\varepsilon)$ such that, for $t > T_1$,

$$2RL(R) \int_{t-T(\varepsilon)}^{+\infty} \sigma^{-\alpha} e^{-\delta\sigma}\, d\sigma < \frac{\varepsilon'}{2},$$

therefore $|(Tx)(t+\tau) - (Tx)(t)| < \varepsilon'$ if $t > T_1(\varepsilon')$, which shows that the function Tx also is a.a.p. We proceed as in the proof of proposition 3.1, we can establish that the map $T : \widetilde{D} \to \widetilde{D}$ is a contraction, which implies the existence of an asymptotic almost periodic solution.

Now we give sufficient conditions for the existence of almost periodic solutions.

THEOREM 3.2: *If the hypotheses of proposition* 3.1 *hold and f is uniformly almost periodic, then there is a unique almost periodic solution of equation* (1.1).

PROOF: We show first that maps $AP(X)$ into itself.

If $x \in AP(X)$, it follows from lemma 2.2 that $t \to f(t, A^{-\alpha} x(t))$ is almost periodic. Hence, for each $\varepsilon > 0$ there exists a set P_ε relatively dense in $\mathbb{R}$ such that

$$|f(t+\tau, A^{-\alpha} x(t+\tau)) - f(t, A^{-\alpha} x(t))| \leq \varepsilon \quad \text{for all } \tau \in \mathbb{R} \text{ and } \tau \in P_\varepsilon .$$

Therefore, the map T defined by (3.1) satisfies

$$|Tx(t+\tau) - Tx(t)| =$$

$$= \left| \int_{-\infty}^{t+\tau} A^\alpha S(t+\tau-\sigma)\, f(\sigma, A^{-\alpha} x(\sigma))\, d\sigma - \int_{-\infty}^{t} A^\alpha S(t-\sigma)\, f(\sigma, A^{-\alpha} x(\sigma))\, d\sigma \right| =$$

$$= \left| \int_{-\infty}^{t} A^\alpha S(t-\sigma)\, f(\sigma+\tau, A^{-\alpha} x(\sigma+\tau))\, d\sigma - \int_{-\infty}^{t} A^\alpha S(t-\sigma)\, f(\sigma, A^{-\alpha} x(\sigma))\, d\sigma \right| \leq$$

$$\leq \int_{-\infty}^{t} |A^\alpha S(t-\sigma)|_{\mathcal{L}(X)}\, |f(\sigma+\tau, A^{-\alpha} x(\sigma+\tau)) - f(\sigma, A^{-\alpha} x(\sigma))|\, d\sigma$$

$$\leq \varepsilon M_\alpha \int_{-\infty}^{t} (t-\sigma)^{-\alpha} \exp(-\delta(t-\sigma))\, d\sigma .$$

Which shows that the function Tx also is almost periodic and that $T: AP(X) \to \to AP(X)$. We proceed as in the proof of proposition (3.1). In this case we can see that the map $T: AP(X) \to AP(X)$ is a contraction, which implies the existence of an almost periodic solution.

4. - EXAMPLE

1) Consider the following partial differential equation

$$\begin{cases} \dfrac{\partial u(\xi, t)}{\partial t} = d\, \dfrac{\partial^2 u(\xi, t)}{\partial \xi^2} + h(t)\, g\left(\dfrac{\partial u(\xi, t)}{\partial \xi} \right) \\ u(0, t) = u(1, t) = 0 \end{cases}$$

for all $(\xi, t) \in [0, 1] \times \mathbb{R}$ where d is positive constant).

To represent this problem in a abstract frame, we take $X = L^2((0, 1); \mathbb{R})$ and $x(t) := u(., t)$. We define an operator A by the operator

(4.1) $$Au = -du'' \quad \text{for } u \in D(A) = \left\{u \in H_0^1((0, 1); \mathbb{R});\ u'' \in X\right\}.$$

The operator $-A$ is self-adjoint, with a compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$ on $L^2((0, 1); \mathbb{R})$. Furthermore, A has a discrete spectrum with simple eigenvalues $n^2\pi^2 d$, $n \in \mathbb{N}$. The set of normalized eigenvectors is complete in X, which shows that $|S(t)| \leq e^{-d\pi^2 t}$ ([14]). We take $\alpha = \dfrac{1}{2}$, that is $X_{1/2} = (D(A^{1/2}), |.|_{1/2})$.

Define the function $f : \mathbb{R} \times X_{1/2} \to X$, by

$$f(t, u) = h(t)\, g(u'), \quad \text{for each } t \in \mathbb{R} \text{ and all } u \in X_{1/2}$$

where $h : \mathbb{R} \to R$ is almost periodic in $\mathbb{R}$ and there exist $k_1 > 0$ and $\theta \in]0, 1[$ such that

(4.2) $$|h(t) - h(s)| \leq k_1 |t - s|^\theta, \quad \text{for all } t, s \in \mathbb{R}$$

and $g : X \to X$ is Lipschitz continuous on X. A concrete example of the function g is notably

$$g(u) = \sin(u)$$

$$g(u) = ku$$

$$g(u) = \arctan(u).$$

We give first some known results concerning the operators A defined by (4.1) and $A^{1/2}$. Let $u \in D(A)$ and $\lambda \in \mathbb{R}$, such that

$$Au = -du'' = \lambda u$$

that is,

(4.3) $$du'' + \lambda u = 0$$

We have

$$\langle Au, u\rangle = \langle \lambda u, u\rangle$$

that is

$$-\langle du'', u\rangle = d|u'|_{L^2}^2 = \lambda |u|_{L^2}^2$$

so $\lambda \in \mathbb{R}_+$.

The solutions of equation (4.3) have the form

$$u(x) = C\cos\left(\sqrt{\frac{\lambda}{d}}\,x\right) + D\sin\left(\sqrt{\frac{\lambda}{d}}\,x\right)$$

we have $u(0) = u(1) = 0$, so, $C = 0$ and $\sqrt{\dfrac{\lambda}{d}} = n\pi$ $(n \in \mathbb{N})$. Put $\lambda_n = dn^2\pi^2$. The solutions of equation (4.3) are

$$u_n(x) = D \sin(\sqrt{\lambda_n} x), \qquad n \in \mathbb{N}^*.$$

We have $\langle u_n, u_m \rangle = 0$, for $n \neq m$ and $\langle u_n, u_n \rangle = 1$. So $D = \sqrt{2}$ and

$$u_n(x) = \sqrt{2} \sin(\sqrt{\lambda_n} x)$$

$u \in D(A)$, so there exists a sequence $(\alpha_n) \in \mathbb{R}$, such that

$$u(x) = \sum_{n \in \mathbb{N}^*} \alpha_n u_n(x), \ \sum_{n \in \mathbb{N}^*} (\alpha_n)^2 < +\infty \ \text{ and } \ \sum_{n \in \mathbb{N}^*} (\lambda_n)^2 (\alpha_n)^2 < +\infty .$$

We have

$$A^{1/2} u(x) = \sum_{n \in \mathbb{N}^*} \sqrt{\lambda_n} \alpha_n u_n(x)$$

with $u \in D(A^{1/2})$, that is

$$\sum_{n \in \mathbb{N}^*} (\alpha_n)^2 < +\infty \ \text{ and } \ \sum_{n \in \mathbb{N}^*} \lambda_n (\alpha_n)^2 < +\infty .$$

We show now that f satisfies the hypothesis $(A1)$. In fact, Let $t_1, t_2 \in \mathbb{R}$ and $u_1, u_2 \in X_{1/2}$, we have

$$\begin{aligned} f(t_1, u_1) - f(t_2, u_2) &= h(t_1)\, g(u_1') - h(t_2)\, g(u_2') \\ &= [h(t_1) - h(t_2)]\, g(u_1') + h(t_2)[g(u_1') - g(u_2')]. \end{aligned}$$

So,

(4.4) $$\begin{aligned} |f(t_1, u_1) - f(t_2, u_2)|_{L^2} &\leq |h(t_1) - h(t_2)|\,|g(u_1')|_{L^2} + h(t_2)|\,|g(u_1') - g(u_2')|_{L^2} \\ &\leq |g|_\infty |h(t_1) - h(t_2)| + |g|_{\text{Lip}} |h(t_2)|\,|u_1' - u_2'|_{L^2} \end{aligned}$$

h is almost periodic, so, there exists $k_2 > 0$, such that

(4.5) $$|h(t_2)| \leq k_2 .$$

Therefore from (4,2), (4.4), (4.5), and the fact that $g(u')$ is Lipschitz on $X_{1/2}$ (see for instance [9, p. 75]) we have

$$\begin{aligned} |f(t_1, u_1) - f(t_2, u_2)|_X &\leq k_1 |g|_\infty |t_1 - t_2|^\theta + k_2 |g|_{\text{Lip}} |u_1 - u_2|_{1/2} \\ &\leq L(|t_1 - t_2|^\theta + |u_1 - u_2|_{1/2}). \end{aligned}$$

So, f satisfies the hypothesis $(A1)$, with $L = \max(k_1 |g|_\infty, k_2 |g|_{\text{Lip}})$.

2) *A semilinear evolution equation in* $\mathbb{R}^3$.

Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^3$ and consider the following semilinear equation

$$(4.6)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^{3} u \dfrac{\partial u}{\partial x_i} + f(t, x) & \text{in } \mathbb{R} \times \Omega \\ u(t, 0) = 0 & \text{in } \mathbb{R} \times \partial\Omega \end{cases}$$

Where $f\colon \mathbb{R} \times \Omega \to L^2(\Omega)$ satisfies,

$$|f(t_1, x) - f(t_2, x)|_{L^2(\Omega)} \leq C|t_1 - t_2|^\theta \qquad (0 < \theta < 1).$$

Let $X = L^2(\Omega)$ and define an operator A by

$$Au = -\Delta u, \qquad D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

The operator $-A$ self-adjoint, and is the infinitesimal generator of an analytic semigroup on $L^2(\Omega)$.

Denote by $g(u) = \sum_{i=1}^{3} u \dfrac{\partial u}{\partial x_i}$.

In [16, p. 240, lemma 3.4] it was established that the function g verifies the estimate, if $\gamma > \dfrac{3}{4}$ then

$$|g(u_1) - g(u_2)|_{L^2(\Omega)} \leq C(|u_1|_\gamma |u_1 - u_2|_{1/2} + |u_2|_{1/2} |u_1 - u_2|_\gamma) \quad \text{for all } u_1, u_2 \in D(A^\gamma).$$

Since $X_\gamma \subset X_{1/2}$ and the imbedding is continuous, we may conclude that the map g satisfies the hypothesis $(A1)$ in $\mathbb{R} \times X_\gamma$.

COROLLARY 4.1: *For each f, asymptotically almost periodic* (*resp. almost periodic*), *equation* (4.6) *has a unique a.a.p* (*resp. a.p*) *strong solution. This result is an immediate consequence of theorem* 3.1 *and theorem* 3.2.

REFERENCES

[1] L. AMERIO - G. PROUSE, *Almost periodic functions and functional equations*, Van Nostrand-Reinhold, New York, 1971.

[2] M. BAHAJ - E. HANEBALY, *Periodic and almost periodic solutions of semilinear equations in Banach spaces*, Rev. R. Acad. Cien. Exact. Fis. Nat., (Esp). Vol. 93, no. 2 (1999), 171-177.

[3] M. E. BALLOTTI - J. A. GOLDSTEIN - M. E. PARROTT, *Almost periodic solutions of evolution equations*, J. Math. Anal. Appl., 138 (1989), 522-536.

[4] C. J. K. BATTY - R. CHILL, *Approximation and asymptotic behaviour of evolution families*, Differential Integral Equations, **15** (2002), 477-512.

[5] S. BOCHNER - J. VON NEUMANN, *Almost periodic something is missing in a group, II*, Trans. Amer. Math. Soc., 37 (1935), 21-50.

[6] M. FRÉCHET, *Les fonctions asymptotiquement presque-périodiques continues*, C.R. Acad. Sci. Paris, 213 (1941), 520-522.

[7] M. FRÉCHET, *Les fonctions asymptotiquement presque-périodiques*, Rev. Sci., 79 (1941), 341-354.
[8] A. FRIEDMAN, *Partial differential equations Holt*, Rinehart and Winston, New York (1969).
[9] D. HENRY, *Geometric theory of semilinear parabolic equations*, Springer Lecture Notes in Mathematics, 860 (1981).
[10] W. HUTTER - F. RÄBIGER, *Spectral mapping theorems for evolution semigroups on spaces of almost periodic functions*, preprint.
[11] B. M. LEVITAN - V. V. ZHIKOV, *Almost periodic functions and differential equations*, Cambridge University Press, 1982.
[12] A. LUNARDI, *Analytic semigroups and optimal regularity in parabolic problems*, Birkhäuser, Basel 1995.
[13] L. MANIAR - R. SCHNAUBELT, *Almost periodicity of inhomogeneous parabolic evolution equations*, preprint.
[14] R. H. MARTIN, *Non linear operators and differential equations in Banach spaces*, John wiley, New York (1976).
[15] J. L. MASSERA, *The existence of periodic solutions of systems of differential equations*, Duke Math. J., **17** (1950), 457-475.
[16] A. PAZY, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New-York (1983).
[17] J. PRÜSS, *Periodic solutions of semilinear evolution equations*, Non linear Analysis, Theory, Methods & Applications, Vol. 3, no. 5 (1978), 601-612.
[18] T. YOSHIZAWA, *Stability theory and the existence of periodic solutions and almost periodic solutions*, Springer-Verlag, New York, 1975.
[19] S. ZAIDMAN, *Abstract differential equations, Pitman Publishing*, San Francisco-London-Melbourne 1979.
[20] S. ZAIDMAN, *A non linear abstract differential equation with almost-periodic solution*, Riv. Mat. Univ. Parma (4), 10 (1984), 331-336.
[21] S. ZAIDMAN, *Topics in abstract differential equations I*, Pitman Research Notes in Mathematics, 304.
[22] S. ZAIDMAN, *Topics in abstract differential equations II*, Pitman Research Notes in Mathematics, 321.

Direttore responsabile: Prof. A. BALLIO - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5 - Bologna