Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
120° (2002), Vol. XXVI, fasc. 1, pagg. 167-181

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Fixed Points and Periodic Points of Semiflows of Holomorphic Maps of the Unit Ball of a Hilbert Space (**)

SUMMARY. — Let ϕ be a continuous semiflow of holomorphic maps of the open unit ball D of a complex Banach space. Any fixed point of ϕ can obviously be viewed as a periodic point of ϕ with any arbitrary period. Under which conditions on the geometry of D and on the behaviour of ϕ does the existence of *proper* (*i.e.*, non-fixed) periodic points of ϕ imply the existence of fixed points? The answer to this question turns out to be affirmative in the case in which D is the open unit ball of a complex Hilbert space and the elements of ϕ are holomorphic isometries for the hyperbolic metric of D. In this case, the periodicity of ϕ coupled with the differentiability of all orbits implies furthermore that the dimension of D is finite. The case in which ϕ has neither periodic nor fixed points in D provides some information on the behaviour of the continuous extension of ϕ to the boundary of D. As a consequence, the hypothesis concerning the existence of periodic points of the flow ϕ can be replaced by the weaker condition whereby the orbit by ϕ of some point is relatively compact in D.

Punti periodici e punti fissi di semiflussi continui di applicazioni olomorfe del disco unità di uno spazio di Hilbert

SUNTO. — Sia ϕ un semiflusso continuo di applicazioni olomorfe del disco unità aperto D di uno spazio di Banach complesso. Un punto fisso di ϕ può essere considerato un punto periodico di ϕ con periodo arbitrario. Sotto quali condizioni su D e su ϕ, la presenza di punti periodici di ϕ in senso stretto implica l'esistenza di punti fissi? Questo problema ha una soluzione positiva nel caso in cui D è il disco unità aperto di uno spazio di Hilbert, e gli elementi di ϕ sono isometrie olomorfe per la metrica iperbolica di D. In questo caso, la periodicità di ϕ, insieme alla differenziabilità di tutte le orbite, impone inoltre che la dimensione di D sia finita. Infine, l'assenza di punti periodici e di punti fissi fornisce indicazioni sul comportamento di ϕ sulla frontiera di D. Questi risultati consentono di rimpiazzare l'esistenza di punti periodici per il flusso ϕ con la condizione più debole secondo cui l'orbita di un punto è relativamente compatta in D.

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(**) Memoria presentata il 30 dicembre 2002 da Edoardo Vesentini, uno dei XL.

Let D be a bounded domain in a complex Banach space, endowed with a metric invariant under the action of the group $\mathrm{Aut}(D)$ of all holomorphic automorphisms of D. Let $\phi : \mathbb{R}_+ \times D \to D$ be a continuous semiflow of holomorphic isometries. Fixed points and periodic points play a crucial role in the dynamics of ϕ. The question: under which conditions does the existence of a periodic orbit Γ imply that ϕ itself is periodic (and therefore is the restriction to $\mathbb{R}_+$ of a continuous flow $\mathbb{R} \times D \to D$ of holomorphic automorphisms of D) was investigated in [11] in the case in which D is the open unit ball of a J^*-algebra $\mathcal{A}$. One of the main conclusions of [11] was that ϕ is periodic when Γ spans a dense affine subspace of $\mathcal{A}$: a result that was made more precise in the cases in which the J^*-algebra $\mathcal{A}$ is a Cartan factor of type one or of type four. The most relevant example of a Cartan factor of type one is represented by any complex Hilbert space. It was considered in [11] and [8] and will be investigated more systematically in the present paper proving, among other things, that the existence of a periodic orbit of ϕ implies that the set $\mathrm{Fix}(\phi)$ of all fixed points of ϕ is non-empty. On the opposite extreme, the absence of periodic points implies the existence of fixed points of the (unique) continuous extension $\widehat{\phi}$ of ϕ to the closure $\overline{D}$ of D. It turns out that, if ϕ is a flow, $\mathrm{Card\,Fix}\,(\widehat{\phi}) = 2$, and $\widehat{\phi}$ has no (non-fixed) periodic points.

1. Let D be the open unit ball of a complex Hilbert space $\mathcal{H}$ ($\dim_{\mathbb{C}} \mathcal{H} > 1$), and let

$$J = \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathbb{C}).$$

For $x \in \mathcal{H}$, $\lambda \in \mathbb{C}$, let

$$z = \begin{pmatrix} x \\ \lambda \end{pmatrix} \in \mathcal{H} \oplus \mathbb{C}\,. \tag{1}$$

Then, any $y \in \mathcal{H}$ is contained in D (respectively, in $\overline{D}$) if, and only if, there are $x \in \mathcal{H}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ such that

$$y = \frac{1}{\lambda} x$$

and $(Jz|z) < 0$ (respectively, $(Jz|z) \leqslant 0$).

Let $A \in \mathcal{L}(\mathcal{H} \oplus \mathbb{C})$ be such that

$$A^* J A = J, \tag{2}$$

where A^* is the adjoint of A.

Setting

$$A = \begin{pmatrix} A_{11} & A_{12} \\ (\bullet|A_{21}) & A_{22} \end{pmatrix},$$

with $A_{11} \in \mathcal{L}(\mathcal{H})$, $A_{12}, A_{21} \in \mathcal{H})$ and $A_{22} \in \mathbb{C}$, (2) is equivalent to the following equations:

(3) $$|A_{22}|^2 = 1 + \|A_{12}\|^2,$$

(4) $$A_{21} = \frac{1}{A_{22}} A_{11}^* A_{12},$$

(5) $$A_{11}^* A_{11} = I_{\mathcal{H}} + \frac{1}{|A_{22}|^2} (\bullet|A_{11}^* A_{12})_{\mathcal{H}} A_{11}^* A_{12}.$$

The operator A is invertible in $\mathcal{L}(\mathcal{H} \oplus \mathbb{C})$ if, and only if, A_{11} is invertible in $\mathcal{L}(\mathcal{H})$.

According to [3] (see also [8]), if $y \in D$,

(6) $$(y|A_{21})_{\mathcal{H}} + A_{22} \neq 0,$$

and the function $\widetilde{A}$ defined by

(7) $$\widetilde{A} : D \ni y \mapsto \frac{1}{(y|A_{21})_{\mathcal{H}} + A_{22}} (A_{11} y + A_{12})$$

is a holomorphic isometry for the hyperbolic metric (Kobayashi-Carathéodory metric) of B. It turns out, [8], that $\widetilde{A}$ is a holomorphic automorphism of D if, and only if, A is invertible in $\mathcal{L}(\mathcal{H} \oplus \mathbb{C})$.

It is shown in [3] that, if F is a holomorphic isometry of D, there exists $A \in \mathcal{L}(\mathcal{H} \oplus \mathbb{C})$ satisfying (2) such that $F = \widetilde{A}$.

The Fréchet differential $d\widetilde{A}(x) \in \mathcal{L}(\mathcal{H})$ of $\widetilde{A}$ at a point $x \in D$, whose action on $v \in \mathcal{H}$ is expressed by

$$d\widetilde{A}(x)\, v = \frac{1}{(x|A_{21})_{\mathcal{H}} + A_{22}} \left[A_{11} v - \frac{1}{\overline{A_{22}}} (A_{11} v|A_{12})_{\mathcal{H}} \widetilde{A}(x) \right],$$

is an isometry for the hyperbolic metric, in the sense that, if $|v|_x$ is the lenght of v at the point $x \in D$, then

(8) $$|d\widetilde{A}(x)\, v|_{\widetilde{A}(x)} = |v|_x.$$

The hyperbolic metric has the following expression, which is computed in ([3], pp. 153-154). For $x \in D$, let $v_1, v_2 \rightarrow \{v_1, v_2\}_x$ be the continuous, sesquilinear, positive definite, hermitian form defined by

$$\{v_1, v_2\}_x = (1-\|x\|^2_{\mathcal{H}})^{-2}((v_1\,|\,x)_{\mathcal{H}}(x\,|\,v_2)_{\mathcal{H}} + (1-\|x\|^2_{\mathcal{H}})(v_1\,|\,v_2)_{\mathcal{H}}).$$

Then

$$|v|^2_x = \{v, v\}_x.$$

As a consequence, the polarization formula

$$\{v_1, v_2\}_x = \frac{1}{4}[\,|v_1+v_2|^2_x - |v_1-v_2|^2_x + i(|v_1+iv_2|^2_x - |v_1-iv_2|^2_x)]\,,$$

together with (8), yields the following lemma.

LEMMA 1: *If F is a holomorphic isometry of the hyperbolic metric of D, then*

$$\{dF(x)\,v_1, dF(x)\,v_2\}_{F(x)} = \{v_1, v_2\}_x$$

for all $x \in D$ and all $v_1, v_2 \in \mathcal{H}$.

In other words, holomorphic isometries are conformal maps of the hyperbolic metric of D.

In the following, A will be replaced by the value $T(t)$ of a strongly continuous semigroup $T:\mathbb{R}_+ \to \mathcal{L}(\mathcal{H}\oplus\mathbb{C})$ such that

$$T(t)^* J T(t) = J \tag{9}$$

for all $t \in \mathbb{R}_+$.

Setting

$$T(t) = \begin{pmatrix} T_{11}(t) & T_{12}(t) \\ (\bullet\,|\,T_{21}(t)) & T_{22}(t) \end{pmatrix},$$

with $T_{11}(t) \in \mathcal{L}(\mathcal{H})$, $T_{12}(t)$, $T_{21}(t) \in \mathcal{H})$ and $T_{22}(t) \in \mathbb{C}$, the above considerations still hold when $A_{11}, A_{12}, A_{21}, A_{22}$ and $\widetilde{A}$ are replaced by $T_{11}(t)$, $T_{12}(t)$, $T_{21}(t)$, $T_{22}(t)$ and $\widetilde{T(t)}$, which is now expressed by

$$\widetilde{T(t)}: D \ni y \to \frac{1}{(y\,|\,T_{21}(t))_{\mathcal{H}} + T_{22}(t)}\,(T_{11}(t)\,y + T_{12}(t))\,. \tag{10}$$

As a consequence, T defines a continuous semiflow

$$\phi : t \mapsto \phi_t = \widetilde{T(t)} \tag{11}$$

of holomorphic isometries of the open unit ball B of $\mathcal{H}$.

It turns out, [8], that ϕ is the restriction to $\mathbb{R}_+$ of a continuous flow of holomorphic automorphisms of D if, and only if, T is the restriction to $\mathbb{R}_+$ of a group

$\mathbb{R} \to \mathcal{L}(\mathcal{H} \oplus \mathbb{C})$ (that will be denoted by the same symbol T) for which (9) holds whenever $t \in \mathbb{R}$.

According to Theorem VI.4.5 of [8], for any $t \geqslant 0$ there is a neighbourhood W of $\overline{D}$ in $\mathcal{H}$ such that (6) holds for all $y \in W$, and consequently ϕ_t, is the restriction to D of a holomorphic map, still expressed by (10). Denoting by $\widehat{\phi_t}$ the restriction of this map to $\overline{D}$, then $\widehat{\phi_t}(\overline{D}) \subset \overline{D}$ and $\widehat{\phi_{t_1+t_2}} = \widehat{\phi_{t_1}} \circ \widehat{\phi_{t_2}}$ for all t_1, t_2 in $\mathbb{R}_+$. Since $\widehat{\phi_t}$ is continuous for the weak topology on $\overline{D}$, which is weakly compact in $\mathcal{H}$, the Schauder-Tychonoff theorem implies that

$$\operatorname{Fix}\left(\widehat{\phi_t}\right) \neq \emptyset \tag{12}$$

for all $t \in \mathbb{R}_+$.

Let $y = \dfrac{1}{\lambda} x \in \overline{D}$, with $\lambda \neq 0$, and let z be given by (1).

By (10), $\widehat{\phi_\tau}(y) = y$ for some $\tau > 0$ if, and only if, there is some $\zeta \in \mathbb{C}$ such that

$$T(\tau)\, z = e^{\zeta\tau} z\,. \tag{13}$$

Thus, looking for the fixed points of $\widehat{\phi_\tau}$ is the same as looking for the eigenvectors z of $T(\tau)$ with non-vanishing eigenvalues, and such that $(Jz|z) \leqslant 0$. This search involves the spectral structure of the infinitesimal generator of the semigroup T.

2. If

$$X : \mathcal{D}(X) \subset \mathcal{H} \oplus \mathbb{C} \to \mathcal{H} \oplus \mathbb{C}$$

is the infinitesimal generator of the semigroup T, X is closed, JX is skew-symmetric, and, [8], there is a dense linear subspace $\mathcal{D} \subset \mathcal{H}$ such that

$$\mathcal{D}(X) = \mathcal{D} \oplus \mathbb{C}\,.$$

Thus, [8], X is represented by the matrix

$$X = \begin{pmatrix} X_{11} & X_{12} \\ (\bullet | X_{12})_{\mathcal{H}} & iX_{22} \end{pmatrix}, \tag{14}$$

where $X_{22} \in \mathbb{R}$, $X_{12} \in \mathcal{H}$, and X_{11} is a skew-symmetric, closed, linear operator in $\mathcal{H}$ with domain $\mathcal{D}(X_{11}) = \mathcal{D}$, whose resolvent set $r(X_{11}) \supset \{\zeta \in \mathbb{C} : \Re\zeta > 0\}$.

Vice versa, if X_{11}, X_{12}, X_{21} and X_{22} satisfy these conditions, the operator X represented by (14) is the infinitesimal generator of a strongly continuous semigroup $T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{H} \oplus \mathbb{C})$ for which (9) holds for all $t \in \mathbb{R}_+$.

The semiflow ϕ is the restriction to $\mathbb{R}_+$ of a continuous flow of holomorphic automorphisms of D if, and only if, X_{11} is skew-self adjoint.

If (13) holds, there is a sequence $\{z_\nu\}$ in $\mathcal{D}(X)$ for which

$$(15) \qquad \begin{cases} z \quad = \sum z_\nu \\ Xz_\nu = \left(\zeta + \dfrac{2\pi i}{\tau} n_\nu\right) z_\nu \qquad (n_\nu \in \mathbb{Z}) \\ \nu' \neq \nu'' \Rightarrow n_{\nu'} \neq n_{\nu''} . \end{cases}$$

If $z', z'' \in (\mathcal{K} \oplus \mathbb{C}) \setminus \{0\}$ are two eigenvectors of X with eigenvalues ζ' and ζ'', and if $\zeta' + \overline{\zeta''} \neq 0$, then

$$\begin{aligned} (Jz' \,|\, z'')_{\mathcal{K}\oplus\mathbb{C}} &= \frac{1}{\zeta' + \overline{\zeta''}} (\zeta' + \overline{\zeta''})(Jz' \,|\, z'')_{\mathcal{K}\oplus\mathbb{C}} \\ &= \frac{1}{\zeta' + \overline{\zeta''}} [(JXz' \,|\, z'')_{\mathcal{K}\oplus\mathbb{C}} + (Jz' \,|\, Xz'')_{\mathcal{K}\oplus\mathbb{C}}] \\ &= \frac{1}{\zeta' + \overline{\zeta''}} ((JX + X^* J)\, z' \,|\, z'')_{\mathcal{K}\oplus\mathbb{C}} = 0 \end{aligned}$$

because JX is skew-symmetric. Hence,

$$(16) \qquad \zeta' + \overline{\zeta''} \neq 0 \;\Rightarrow\; (Jz' \,|\, z'')_{\mathcal{K}\oplus\mathbb{C}} = 0 \,.$$

As a consequence, (15) implies that, if (13) holds, then

$$(17) \qquad (Jz|z)_{\mathcal{K}\oplus\mathbb{C}} = \sum (Jz_\nu \,|\, z_\nu)_{\mathcal{K}\oplus\mathbb{C}} .$$

If $x \in \overline{D}$ is fixed by $\widehat{\phi_\tau}$ for some $\tau > 0$, then, for

$$(18) \qquad z = \begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathcal{K} \oplus \mathbb{C} \,,$$

$(Jz|z)_{\mathcal{K}\oplus\mathbb{C}} \leqslant 0$, being $(Jz|z)_{\mathcal{K}\oplus\mathbb{C}} < 0$ if, and only if, $y \in D$. In this latter case, (16) and (17) imply that

$$(Jz_\nu \,|\, z_\nu)_{\mathcal{K}\oplus\mathbb{C}} \leqslant 0$$

for some ν, and therefore

$$T(t)\, z_\nu = \mathrm{e}^{\frac{2\pi}{\tau} n_\nu i t} z_\nu$$

for all t. That proves

THEOREM 1: *If the semiflow ϕ has a periodic point* [1], *then* $\operatorname{Fix}(\phi) \neq \emptyset$.

By (12), for every $\tau \geqslant 0$ there is some $x \in \overline{D}$ for which $\phi_\tau(x) = x$. Since $(Jz|z)_{\mathcal{H}\oplus\mathbb{C}} \leqslant \leqslant 0$ for z given by (18), the same argument as before yields

$$\operatorname{Fix}(\widehat{\phi}) \neq \emptyset \, . \tag{19}$$

If $\operatorname{Fix}(\phi) = \emptyset$, there are no periodic points of ϕ, and therefore $\operatorname{Fix}(\phi_t) = \emptyset$ for every $t > 0$. Since, by Theorem VI.4.9 of [3] [2],

$$1 \leqslant \operatorname{Card}\operatorname{Fix}(\widehat{\phi_s}) \leqslant 2 \, , \tag{20}$$

for all $s \geqslant 0$ (and all $s \in \mathbb{R}$ when ϕ is a flow), and because

$$\operatorname{Fix}(\widehat{\phi}) \subset \operatorname{Fix}(\widehat{\phi_s}),$$

then $\operatorname{Fix}(\widehat{\phi})$ consists of one or two points.

3. Further information on $\operatorname{Fix}(\widehat{\phi})$, involves the point spectrum $p\sigma(X)$ of X, which, by (19), is not empty.

Suppose first that $\operatorname{Fix}(\phi) \neq \emptyset$. Since D is homogeneous, there is no restriction in assuming $0 \in \operatorname{Fix}(\phi)$.

Inspection of (10) yields then

$$T_{12}(t) = 0 \quad \forall\, t \in \mathbb{R}_+ \, , \tag{21}$$

and therefore

$$\phi_t = d\phi_t(0)_{|D},$$

where $d\phi_t(0)$ (the Fréchet differential of ϕ_t at 0) is a linear isometry [3].

That proves

PROPOSITION 1: *If the continuous semiflow ϕ has a periodic point, ϕ is conjugate, by a holomorphic automorphism of D, to the restriction to D of a strongly continuous semigroup $V : \mathbb{R}_+ \to \mathcal{L}(\mathcal{H})$ of linear isometries of the Hilbert space $\mathcal{H}$. This semigroup is the restriction to $\mathbb{R}_+$ of a strongly continuous group $U : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ if, and only if, ϕ is the restriction to $\mathbb{R}_+$ of a continuous flow of holomorphic automorphisms of D.*

By (21), $X_{12} = 0$. Since X_{11} is skew-symmetric and $r(X_{11}) \supset \{\zeta \in \mathbb{C} : \Re\zeta > 0\}$, (14) implies

(1) A point y is periodic with period $\tau > 0$ if $\phi_\tau(y) = \phi_0(y) = y$ and $\phi_t(y) \neq y$ for all $t \in (0, \tau)$.

(2) Which extends to holomorphic isometries a theorem established by T. L. Hayden and T. J. Suffridge, [5], for holomorphic automorphisms of D; see also [8] for a different proof.

(3) This follows also from the fact that H. Cartan's linearity theorem holds for all holomorphic isometries of D, [8].

LEMMA 2: *If* $\mathrm{Fix}(\phi) \neq \emptyset$, *then* $r(X) \supset \{\zeta \in \mathbb{C} : \Re\zeta > 0\}$. *If moreover* ϕ *is the restriction to* $\mathbb{R}_+$ *of a continuous flow of holomorphic automorphisms of* D, *the spectrum* $\sigma(X) \subset i\mathbb{R}$.

The infinitesimal generator of V is the closed skew-symmetric operator

$$Y = X_{11} - iX_{22} : \mathcal{D}(X_{11}) \subset \mathcal{H} \to \mathcal{H}$$

which is skew-self adjoint if, and only if, V is the restriction to $\mathbb{R}_+$ of the unitary group U. (In which case, U is generated by the skew-self adjoint operator Y.)

By the spectral mapping theorem,

$$\mathrm{Fix}(\phi) = \mathrm{Fix}(V) = \ker Y. \tag{22}$$

Since, [3], any holomorphic automorphism of D maps onto itself the family of the intersections of D with all closed, affine subspaces of $\mathcal{H}$, (22) yields

PROPOSITION 2: *If* ϕ *is any continuous semiflow of holomorphic isometries of* D, *and* $\phi_\tau(x) = x$ *for some* $x \in D$ *and some* $\tau \geq 0$, $\mathrm{Fix}(\phi)$ *is the intersection of* D *with a closed, affine subspace of* $\mathcal{H}$.

Going back to the case in which $0 \in \mathrm{Fix}(\phi)$, let Π be the orthogonal projector of $\mathcal{H}$ onto $\ker Y$. For all $x, y \in \mathcal{H}$,

$$\begin{aligned}(V(t)\,x - x \mid \Pi y) &= (V(t)\,x \mid \Pi y) - (x \mid \Pi y)\\ &= (V(t)\,x \mid V(t)\,\Pi y) - (x \mid \Pi y)\\ &= (x \mid \Pi y) - (x \mid \Pi y) = 0,\end{aligned}$$

i.e., $\Pi V(t)\,x = \Pi x$ for all $x \in \mathcal{H}$.

Since furthermore,

$$\|V(t)\,x - \Pi x\| = \|V(t)\,x - V(t)\,\Pi x\| = \|x - \Pi x\|,$$

then

LEMMA 3: *If* $0 \in \mathrm{Fix}(\phi)$, *the orbit by* ϕ *of any* $x \in D$ *is contained in the intersection of the sphere with center* Πx *and radius* $\|x - \Pi x\|$ *with the closed affine subspace of* $\mathcal{H}$ *which contains* x *and is orthogonal to* $\ker Y$.

As a consequence, the following theorem holds.

THEOREM 2: *If the orbit* $\Gamma(x)$ *of some* $x \in D$ *by the semiflow* ϕ *spans a dense, affine subspace of* $\mathcal{H}$, *then* ϕ *fixes one point of* D *at most.*

According th Theorem 3 of [11], if furthermore x is a periodic point with period $\tau > 0$, ϕ is the restriction to $\mathbb{R}_+$ of a periodic flow, with period τ, of holomorphic automorphisms of D.

Choose now $\mathrm{Fix}(\phi) = \{0\}$. The structure of the spectrum $\sigma(Y)$ of Y is described by a theorem of H. Bart, [1], whereby

$$\sigma(Y) = p\sigma(Y) \subset i\frac{2\pi}{\tau}\mathbb{Z}$$

consists entirely of poles of the resolvent function of Y, and the corresponding eigenspaces span a dense linear subspace of $\mathcal{K}$. Since 0 is the unique fixed point of U, $\ker Y = \{0\}$. Furthermore, according to [11], all eigenspaces have complex dimension one. Finally, by Theorem 4 of [11], if the group T associated to the flow ϕ is eventually differentiable, then $\dim_{\mathbb{C}} \mathcal{K} < \infty$.

It was shown in [8] that, for every $y \in D \cap \mathcal{D}(X_{11})$, the map $t \mapsto \phi_t(y)$ is differentiable and satisfies the Riccati equation

$$\frac{d}{dt}\phi_t(y) = X_{11}\,\phi_t(y) - ((\phi_t(y)\,|X_{12})_{\mathcal{K}} + iX_{12})\,\phi_t + X_{12}. \tag{23}$$

Some of the results established so far can be rephrased in terms of this Riccati equation. For example, the following theorem holds.

THEOREM 3: *If the Riccati equation* (23) *has a periodic integral, it has also a constant integral. If the periodic integral spans a dense affine subspace of* $\mathcal{K}$, *the constant integral is unique.*

4. We consider now the case in which the affine space $H(\tau)$ spanned by the orbit of a periodic point $x \in D$ of the semiflow ϕ is not necessarily dense in $\mathcal{K}$.

Let $\tau > 0$ be the period of x. For $0 \leqslant s < t < \tau$, the intersection, R, of D with the complex affine line which is the support of the unique complex geodesic for the hyperbolic metric of D containing both $\phi_s(x)$ and $\phi_t(x)$.

Since

$$\phi_\tau(\phi_s(x)) = \phi_{\tau+s}(x) = \phi_s(\phi_\tau(x)) = \phi_s(x)$$

and

$$\phi_\tau(\phi_t(x)) = \phi_{\tau+t}(x) = \phi_t(\phi_\tau(x)) = \phi_t(x),$$

then ϕ_τ is the identity on R.

Thus, if $0 < s < t < \tau$, ϕ_τ is the identity on the intersection of D with the complex affine space determined by x, $\phi_s(x)$ and $\phi_t(x)$. A trivial inductive argument implies then that ϕ_τ is the identity on $\overline{H(\tau)} \cap D$.

Since ([3], Corollary VI.4.4, p. 176),

$$\phi_t(D \cap \overline{H(\tau)}) = D \cap \overline{H(\tau)} \qquad \forall\, t \in \mathbb{R}_+,$$

the following proposition holds.

PROPOSITION 3: *The map* $t \mapsto \phi_{t|D \cap \overline{H(\tau)}}$ *is the restriction to* $\mathbb{R}_+$ *of a continuous flow* $\tilde{\phi} : \mathbb{R} \ni t \mapsto \tilde{\phi}_t$ *of holomorphic automorphisms of the ball* $D \cap \overline{H(\tau)}$, *which is periodic with period* τ.

By Theorem 2, $\operatorname{Card}(\operatorname{Fix}(\tilde{\phi})) = 1$, or, equivalently,

$$\operatorname{Card}(\operatorname{Fix}(\phi) \cap \overline{H(\tau)}) = 1,$$

a result more precise than Theorem 1.

Suppose now that ϕ itself is a flow, and choose $0 \in \operatorname{Fix}(\phi)$. In this case, $\overline{H(\tau)}$ is a closed linear subspace of $\mathcal{H}$ which is invariant under the action of the unitary group U, whose restriction, $U_{|\overline{H(\tau)}}$, to $\overline{H(\tau)}$ is periodic with period τ.

The intersection $\overline{H(\tau)} \cap \mathcal{D}(Y)$ is dense in $\overline{H(\tau)}$, and

$$Y_{|\overline{H(\tau)} \cap \mathcal{D}(Y)} : \overline{H(\tau)} \cap \mathcal{D}(Y) \subset \overline{H(\tau)} \to \overline{H(\tau)}$$

is the infinitesimal generator of $U_{|\overline{H(\tau)}}$.

Hence,

$$\sigma(Y_{|\overline{H(\tau)} \cap \mathcal{D}(Y)}) = p\sigma(Y_{|\overline{H(\tau)} \cap \mathcal{D}(Y)}) \subset i \frac{2\pi}{\tau} \mathbb{Z} \setminus \{0\},$$

and the eigenspaces are all one-dimensional and span a dense linear subspace of $\overline{H(\tau)}$.

Note that

$$p\sigma(Y_{|\overline{H(\tau)} \cap \mathcal{D}(Y)}) \subset p\sigma(Y). \tag{24}$$

Let now $x' \in D$ be another periodic point of the flow ϕ, whose period $\tau' > 0$ is such that $\tau'/\tau \notin \mathbb{Q}$.

Since the eigenvalues of $Y_{|\overline{H(\tau')} \cap \mathcal{D}(Y)}$ are non-vanishing integer multiples of $i \frac{2\pi}{\tau'}$, (24) implies that

$$\overline{H(\tau)} \perp \overline{H(\tau')}.$$

Summing up, the following facts have been established.

PROPOSITION 4: *Let* $0 \in \operatorname{Fix}(\phi)$, *and let* $x^1, x^2, \ldots$ *be periodic points of the flow* ϕ, *with periods* $\tau^1 > 0, \tau^2 > 0, \ldots$. *If* $\tau^1, \tau^2, \ldots$ *are linearly independent over* $\mathbb{Q}$, *the invariant spaces* $\overline{H(\tau^1)}, \overline{H(\tau^2)}, \ldots$ *are mutually orthogonal. If* ϕ *is eventually differentiable, they are all finite-dimensional.*

If the Hilbert space $\mathcal{H}$ *is separable the set* $\{\tau^1, \tau^2, \ldots\}$ *is at most countable.*

5. We consider finally the case in which the flow ϕ is such that $\operatorname{Fix}(\phi) = \emptyset$, *i.e.*, the case in which ϕ has neither periodic nor fixed points in D.

By (19), there is some $x \in \partial D$ such that

$$\widehat{\phi_t}(x) = x \qquad \forall\, t \in \mathbb{R}.$$

Hence, for z given by (18), there is some $\zeta \in \mathbb{C}$ such that

$$T(t)\, z = \mathrm{e}^{\zeta t} z\,. \tag{25}$$

By Lemma 1.4 of [8], $|\zeta| \neq 1$.

As before, there are sequences $\{z_\nu\}$ in $\mathcal{D}(X)$ and $\{n_\nu\}$ in $\mathbb{Z}$ for which (15) holds, and in particular

$$T(s)\, z_\nu = \mathrm{e}^{(\zeta + \frac{2\pi n_\nu i}{t} s)} z_\nu \tag{26}$$

for every ν and all $s \in \mathbb{R}$.

Since now $\|x\|_{\mathcal{H}} = 1$, then

$$(Jz|z)_{\mathcal{H}\oplus\mathbb{C}} = 0\,. \tag{27}$$

Because furthermore Fix $(\phi_t) = \emptyset$, then $(Jz_\nu\, |z_\nu)_{\mathcal{H}\oplus\mathbb{C}} \geqslant 0$ for all ν. Hence, this latter inequality becomes an equality for all indices ν.

Because of (20), the set of all these indices consists of at least one and at most two distinct elements. In the latter case, denoting by ν' and ν'' these two elements, and setting

$$z_{\nu'} = \begin{pmatrix} x_{\nu'} \\ 1 \end{pmatrix} \in \mathcal{H} \oplus \mathbb{C}, \qquad z_{\nu''} = \begin{pmatrix} x_{\nu''} \\ 1 \end{pmatrix} \in \mathcal{H} \oplus \mathbb{C}\,,$$

(26) shows that ϕ_t fixes every point in the (non-empty) intersection of D with the complex affine line joining $x_{\nu'}$ and $x_{\nu''}$, contradicting the hypothesis whereby Fix $(\phi) = \emptyset$. Hence, (25) holds for all $t \in \mathbb{R}$, and therefore

$$Xz = \zeta z\,. \tag{28}$$

The values of ζ satisfying (28) can be characterized among the zeros of a Weinstein-Aronszajn determinant, [6], $\omega(\zeta, X', Z)$, associated to the perturbation X of the skew-self adjoint operator

$$X' = \begin{pmatrix} X_{11} & 0 \\ 0 & iX_{22} \end{pmatrix}$$

by the degenerate operator

$$Z = \begin{pmatrix} 0 & X_{12} \\ (\bullet|X_{12})_{\mathcal{H}} & 0 \end{pmatrix}.$$

A direct computation (see [8], pp. 294-296 and 301-302) shows that, for any $\zeta \in r(X_{11}) \setminus iX_{22}$,

$$\omega(\zeta, X', Z) = 1 - \frac{((\zeta I_{\mathcal{K}} - X_{11})^{-1} X_{12} \mid X_{12})}{\lambda - iX_{22}} .$$

As was shown in [8], the function $\omega(\bullet, X', Z)$ has two zeros in $r(X_{11}) \setminus i\mathbb{R}$, which are symmetric with respect to the imaginary axis, are poles of the resolvent function of X and therefore are eigenvalues of X. They are the values of ζ which satisfy (25) and (27), yielding the set $\widehat{\phi}$.

Hence, the following theorem holds.

THEOREM 4: *If the continuous flow ϕ has neither periodic nor fixed points in D, then there are two points in ∂D which are the only fixed points of $\widehat{\phi_t}$ for any $t \in \mathbb{R}$.*

As a consequence, $\widehat{\phi}$ has no properly periodic points.

6. As an application of Theorem 4 we will now improve Theorem 1 replacing the existence of a periodic point by that of a relatively compact orbit.

As before, let ϕ be a continuous semigroup of holomorphic isometries of D, defined by a strongly continuous semigroup $T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{K} \oplus \mathbb{C})$ satisfying (9), and suppose now that there is a point $x_o \in D$ whose orbit $\Gamma(x_0)$ does not get too close to ∂D. More precisely, suppose that there is $r \in (0, 1)$ such that

$$\Gamma(x_0) \subset D_r := \{x \in \mathcal{K} : \|x\|_{\mathcal{K}} < r\} . \tag{29}$$

The invariance property of the hyperbolic metric implies that a similar fact holds then for all points of D.

A sufficient condition (also necessary if $\dim_{\mathbb{C}} < \infty$) for (29) to hold is that $\Gamma(x_0)$ be relatively compact in D.

The invariance property of the hyperbolic metric implies that a similar fact holds then for all points of D.

Since D is homogeneous, there is no restriction in assuming $x_0 = 0$.

If ϕ_t is expressed by (11) and (10),

$$\|\phi_t(0)\|_{\mathcal{K}} = \|T_{12}(t)\|_{\mathcal{K}} / |T_{22}(t)| < r$$

for all $t \geqslant 0$.

Hence, (3) - with $T_{12}(t)$ and $T_{22}(t)$ replacing A_{12} and A_{22}-yields

$$\|T_{12}(t)\|^2_{\mathcal{K}} \leqslant \frac{r^2}{1 - r^2}, \qquad |T_{22}(t)|^2 \leqslant \frac{1}{1 - r^2} .$$

Replacing A_{11} and A_{21} by $T_{11}(t)$ and $T_{21}(t)$, (4) and (5) yield then

$$\begin{aligned}\|T_{11}(t)\,x\|_{\mathcal{X}}^2 &= (T_{11}(t)^*\,T_{11}(t)\,x|x)_{\mathcal{X}}\\ &= \|x\|_{\mathcal{X}}^2 + \frac{1}{|T_{22}(t)|^2}\,|(T_{11}(t)\,x|T_{12}(t))_{\mathcal{X}}|^2\\ &\leqslant \|x\|_{\mathcal{X}}^2 + \|T_{11}(t)\,x\|_{\mathcal{X}}^2\,\frac{\|T_{12}(t)\|_{\mathcal{X}}^2}{|T_{22}(t)|^2}\\ &\leqslant \|x\|_{\mathcal{X}}^2 + r^2\,\|T_{11}(t)\,x\|_{\mathcal{X}}^2,\end{aligned}$$

whence

$$\|T_{11}(t)\,x\|_{\mathcal{X}}^2 \leqslant \frac{1}{1-r^2}\,\|x\|_{\mathcal{X}}^2 \qquad \forall\, x\in\mathcal{X}\,.$$

For $z=\begin{pmatrix}x\\ \lambda\end{pmatrix}\in\mathcal{X}\oplus\mathbb{C}$, with $x\in\mathcal{X}$ and $\lambda\in\mathbb{C}$,

$$T(t)\,z=\begin{pmatrix}T_{11}(t)\,x+\lambda T_{12}(t)\\ (x|T_{21}(t))_{\mathcal{X}}+\lambda T_{22}(t)\end{pmatrix},$$

and

$$\begin{aligned}\|T_{11}(t)\,x+\lambda T_{12}(t)\|_{\mathcal{X}} &\leqslant \|T_{11}(t)\,x\|_{\mathcal{X}} + |\lambda|\,\|T_{12}(t)\|_{\mathcal{X}}\\ &\leqslant \frac{1}{\sqrt{1-r^2}}\,(\|x\|_{\mathcal{X}}+r|\lambda|)\\ &\leqslant \frac{1}{\sqrt{1-r^2}}\,(\|x\|_{\mathcal{X}}+|\lambda|)\,,\end{aligned}$$

$$\begin{aligned}|(x|T_{21}(t))_{\mathcal{X}}+\lambda T_{22}(t)| &\leqslant |(x|T_{21}(t))_{\mathcal{X}}| + |\lambda|\,|T_{22}(t)|\\ &= \left|\left(T_{11}(t)\,x\Big|\,\frac{1}{T_{22}(t)}\,T_{12}(t)\right)_{\mathcal{X}}\right| + |\lambda|\,|T_{22}(t)|\\ &\leqslant \frac{r}{\sqrt{1-r^2}}\,(\|x\|_{\mathcal{X}}+|\lambda|)\\ &\leqslant \frac{1}{\sqrt{1-r^2}}\,(\|x\|_{\mathcal{X}}+|\lambda|)\,.\end{aligned}$$

Hence

$$\|T(t)\,z\|^2_{\mathcal{X}\oplus\mathbb{C}} = \|T_{11}(t)\,x + \lambda T_{12}(t)\|^2_{\mathcal{X}} + |(x\,|\,T_{21}(t))_{\mathcal{X}} + \lambda T_{22}(t)\,|^2$$

$$\leqslant \frac{2}{1-r^2}\,(\|x\|^2_{\mathcal{X}} + |\lambda|^2)^2$$

$$\leqslant \frac{4}{1-r^2}\,(\|x\|^2_{\mathcal{X}} + |\lambda|^2) = \frac{4}{1-r^2}\,\|z\|^2_{\mathcal{X}\oplus\mathbb{C}},$$

and therefore

$$\|T(t)\| \leqslant \frac{2}{\sqrt{1-r^2}} \qquad \forall\, t \in \mathbb{R}_+ .$$

The fact that the semigroup T is uniformly bounded provides some information on $\sigma(X)$; namely, (see, *e.g.*, [7] and [12]): every isolated point of $\sigma(X)$ is an eigenvalue, and – denoting by $r\sigma(X)$ the residual spectrum of X – $r\sigma(X) \cap i\mathbb{R} = \emptyset$.

According to Lemma 8.2 and Propositions 8.1, 8.4 of [8], if iX_{11} is self adjoint, $\sigma(X) \setminus i\mathbb{R}$ consists of two eigenvalues at most, whereas, if iX_{11} is symmetric but not self adjoint, then $\{\zeta \in \mathbb{C} : \Re\zeta < 0\}$ minus one point at most is contained in $r\sigma(X)$.

Summing up, denoting by $c\sigma(X)$ the continuous spectrum of X, the following proposition has been established.

PROPOSITION 5: *If* (29) *holds for some* $r \in (0, 1)$, *the semigroup* T *is uniformly bounded, every isolated point of* $\sigma(X)$ *is an eigenvalue, and*

$$\sigma(X) \cap i\mathbb{R} \subset p\sigma(X) \cup c\sigma(X).$$

If ϕ *is the restriction to* $\mathbb{R}_+$ *of a continuous flow of holomorphic automorphisms of* D, *then* $r\sigma(X) = \emptyset$.

According to Propositions 8.1, 8.4 of [8], $\sigma(X) \cap \{\zeta \in \mathbb{C} : \Re\zeta > 0\}$, if not empty, consists of one eigenvalue ζ of X. Thus (25) yields

LEMMA 4: *If*

$$\sigma(X) \cap \{\zeta \in \mathbb{C} : \Re\zeta > 0\} \neq \emptyset,$$

the semigroup T *is not uniformly bounded.*

Hence, Theorem 4, implies the first part of the following theorem, which improves Theorem 1.

THEOREM 5: *Let* ϕ *be a flow of holomorphic automorphisms of* D. *Then,* $\mathrm{Fix}(\phi) \neq \emptyset$ *if, and only if, there exist* $x_0 \in D$ *and* $r \in (0, 1)$ *satisfying* (29).

The «only if» part of the theorem follows from the fact that holomorphic automorphisms are isometries for the hyperbolic distance in D.

COROLLARY 1: *If* $\Gamma(x_0)$ *is relatively compact in* D, *then* $\mathrm{Fix}(\phi) \neq \emptyset$.

Arguing as in n. 3, one can extend to any flow ϕ of holomorphic automorphisms of D satisfying (29) for $x_0 \in D$ and $r \in (0, 1)$ some of the results on the structure of $\mathrm{Fix}(\phi)$ established there in the case of periodic orbits.

In particular, the following proposition holds.

PROPOSITION 6: *If* (29) *holds,* $\mathrm{Fix}(\phi)$ *is the intersection of D with a closed affine subspace of* $\mathcal{H}$.

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Direttore responsabile: Prof. A. BALLIO - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
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