Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
120° (2002), Vol. XXVI, fasc. 1, pagg. 1-15

SILVANA MARCHI (*)

L^p Regularity of the Derivative in the Second Commutator Direction for Nonlinear Elliptic Equations on the Heisenberg Group (**)

ABSTRACT. — We study the local L^p regularity of the first derivatives, in the second commutator's direction, of local weak solutions of nonlinear elliptic equations on the Heisenberg group.

Regolarità L^p delle derivate secondo i commutatori del secondo ordine per equazioni ellittiche nonlineari sul gruppo di heisenberg

SUNTO. — Si studia la regolarità L^p locale delle derivate prime, nella direzione dei commutatori del secondo ordine, di soluzioni deboli locali di equazioni ellittiche nonlineari sul gruppo di Heisenberg.

1. - INTRODUCTION

The main purpose of this paper is to study the local L^p regularity, $p > 1$, of the first derivatives, in the second commutator's direction, of local weak solutions of nonlinear elliptic equations on the Heisenberg group

$$div_{IH} A(x, Xu) = 0 \tag{1}$$

whose protype is

$$div_{IH}[(1 + |Xu|^2)^{(p-2)/2} Xu] = 0 .$$

We establish the result for $1 + \frac{1}{\sqrt{5}} < p < 1 + \sqrt{5}$.

(*) Indirizzo dell'Autore: Dipartimento di Matematica, Università di Parma, via D'Azeglio, 85 - 43100 Parma, Italia.
(**) Memoria presentata il 22 gennaio 2002 dal Socio Marco Biroli.

To be more precise, we need to introduce some notation. Let $X_1, \ldots, X_{2n}$ be the generators of the Lie algebra with their commutators up to the second order:

$$X_i = \partial_{x_i} + 2x_{i+n}\partial_t$$

$$X_{i+n} = \partial_{x_{i+n}} - 2x_i\partial_t$$

$$T = -4\partial_t$$

for $i = 1, \ldots, n$, where $[X_i, X_{i+n}] = -[X_{i+n}, X_i] = T$ for $i = 1, \ldots, n$, and $[X_i, X_k] = [X_k, X_i] = 0$ in any other case.

If $\Omega \subset lH^n$ is an open set, $k \in lN$ and $1 \leqslant q < +\infty$, we denote by $S^{k,q}(lH^n)$ the Sobolev space of $L^q(lH^n)$ functions whose horizontal derivatives (the ones along $X_1, \ldots, X_{2n}$) of order less then or equal to k are in $L^q(lH^n)$ and by $S^{k,q}_{\text{loc}}(\Omega)$ the space of functions f such that $\varphi f \in S^{k,q}(lH^n)$ for any $\varphi \in C_0^\infty(\Omega)$.

Let $p > 1$. Let $A : lH^n \times lR^{2n} \to lR^{2n}$ be a differentiable vector-valued function of scalar components A^k, $k = 1, \ldots, 2n$. We assume the existence of some constants c_1, c_2, $c_3 > 0$ such that, for any $(x, \xi) \in lH^n \times lR^{2n}$

$$(a)\quad A^k_{\xi_j}(x, \xi)\,\eta_k\eta_j \geqslant c_1 V(\xi)^{p-2}\,|\eta|^2, \qquad \eta \in lR^{2n}.$$

(Here and in the following repeated indices k or j denote summation from 1 to $2n$) and, for every $k, j = 1, \ldots, 2n$ and $l = 1, \ldots, 2n+1$

$$(b)\quad |A^k_{\xi_j}(x, \xi)| \leqslant c_2 V(\xi)^{p-2}$$

$$(c)\quad |A^k_{x_l}(x, \xi)| \leqslant c_3 V(\xi)^{p-1}$$

where $V(\xi) := (1 + |\xi|^2)^{1/2}$.

Let $Xu = (X_1 u, \ldots, X_{2n} u)$. By a local weak solution of (1) we mean a function $u \in S^{1,p}_{\text{loc}}(\Omega)$ such that

$$\int_\Omega A^k(x, Xu)\, X_k\varphi\, dx = 0 \tag{2}$$

for all $\varphi \in S^{1,p}(\Omega)$ with $supp\,\varphi \subset \Omega$.

We can now state the main results of this paper. In the following Ω' will be an arbitrary open bounded subset of Ω such that $\Omega' \subset\subset \Omega$. From now on we will denote C any positive constant which may depend only on Ω' and the structural constants $p, c_1, c_2, c_3, \ldots$, not necessarily the same at each occurrance.

THEOREM 1.1: *Let* $1 + \frac{1}{\sqrt{5}} < p < 1 + \sqrt{5}$ *and let* $u \in S^{1,p}_{\text{loc}}(\Omega)$ *be a local weak solution of* (1). *Let* $B(3R)$ *be an arbitrary gauge ball of radius* $3R$ *such that* $B(3R) \subset \Omega'$ *and* *let g be a cut-off function between B(R) and B(2R). Then* $Tu \in L^p_{\mathrm{loc}}(\Omega')$ *and*

$$\int_{\Omega'} |T(g^4 u)|^p \, dx \leq C R^{-4p} \int_{\Omega'} (V^p + |u|^p) \, dx \tag{3}$$

where $V = V(Xu)$.

THEOREM 1.2: *Let* $2 \leq p < 1 + \sqrt{5}$ *and let* u, g *and* $B(R)$ *be as in Theorem* 1.1. *Then* $Tu \in S^{1,2}_{\mathrm{loc}}(\Omega')$ *and*

$$\int_{\Omega'} g^4 V^{p-2} |XTu|^2 \, dx \leq C R^{-2} \int_{\Omega'} (V^p + |u|^p) \, dx \tag{4}$$

where $V = V(Xu)$.

If $2 \leq p < 1 + \sqrt{5}$ Theorems 1.1 and 1.2 can be easily extended to the case $A^k = A^k(x, u, \xi)$, for $k = 1, \ldots, 2n$, on condition that (a), (b) and (c) hold and, on addition, A^k is differentiable with respect to u, with

$$(d) \quad |A^k_u(x, u, \xi)| \leq c_4 V(\xi)^{p-2}$$

for any $(x, u, \xi) \in \mathbb{H}^n \times \mathbb{R} \times \mathbb{R}^{2n}$, where c_4 is a suitable positive constant.

Any new technique is required in this case because every adding term coming from the differentiation of A^k with respect to u results by (d) of the same type of some other appearing yet in the proofs (see the following formula (20)).

The extension of Theorem 1.2 to the case $p < 2$ is more tricky. This will be the subject of a next paper.

If $2 \leq p < 1 + \sqrt{5}$ we stated Theorem 1.1 for the simpler case $A_\varepsilon = (\varepsilon + |Xu|^{p-2}) Xu$, for $\varepsilon > 0$, in a previous paper [8] where we applied it to prove the $C^{1,\alpha}_{\mathrm{loc}}$ regularity for the solutions relative to $A = |Xu|^{p-2} Xu$.

Although if that is not the main goal of this paper, it should be noted that, thanks to Theorem 1.1, the cited result could be improved to obtain, for the same range of p, the regularity $C^{1,\alpha}_{\mathrm{loc}}$ for the solutions of

$$div_{\mathbb{H}} A(x, u, Xu) = 0$$

satisfying $(a), \ldots, (d)$ with $V(\xi) = |\xi|$. The method, yet applied in [8] and essentially due to Di Benedetto [3], is suitable for all the equations which can be approximated by a sequence of «non degenerate» ones $div_{\mathbb{H}} A_\varepsilon(x, u, Xu) = 0$ satisfying $(a), \ldots, (d)$, with $V(\xi)^2 = \varepsilon + |\xi|^2$, for arbitrary $\varepsilon > 0$, whose solutions we denote by u_ε. The first step of the method is proving $u_\varepsilon \in S^{2,2}_{\mathrm{loc}}(\Omega)$.

We avoid these computations as they require only few and obvious adaptations from [3], [8].

In virtue of this result it is possible to differentiate the approximating equations in order to obtain $u_\varepsilon \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, «uniformly on ε». A limit argument could give then the result for $\varepsilon = 0$.

We observe that Theorem 1.1 intervenes in the first step: to prove $u_\varepsilon \in S^{2,2}_{loc}(\Omega)$ we need appropriate test functions which require $Tu_\varepsilon \in L^p_{loc}(\Omega)$ (see [8], Theorem 4.1).

These results could be used to prove regularity results for obstacle problems (see for example [10], [9]).

We would point out that, if $p = 2$, L. Capogna [1] established $Tu \in S^{1,2}_{loc}(\Omega)$ directly for $A = A(x, \xi)$, satisfying (*a*), (*b*), (*c*) with $V(\xi) = |\xi|$, and, as a by-product, he proved $u \in C^{1,\alpha}_{loc}(\Omega)$.

The proof of Theorem 1.1 is characterized by repeated recourses to Lemma 3.4 (see the following) and imbeddings between functional spaces. These impose some restrictions on p which generate the bounds to its range.

2. - Basic knowledge

The Heisenberg group lH^n is the Lie group whose underlying manifold is lR^{2n+1} with the following group law: for all $x = (x', t) = (x_1, \ldots, x_{2n}, t)$, $y = (y', s) = = (y_1, \ldots, y_{2n}, s)$

$$x \circ y = (x' + y', t + s + 2[x', y'])$$

where $[x', y'] := \sum_{i=1}^{n} (y_i x_{i+n} - x_i y_{i+n})$.

lH^n is a homogeneous group, that is a group with dilations, defined as

$$\delta_\lambda(x', t) = (\lambda x', \lambda^2 t)$$

where the direction t plays a particular role (the space is non-isotropic) corresponding to the definition of the group action.

A norm for lH^n which is homogeneous of degree 1 with respect to the dilations is

$$|x|^4 = |(x', t)|^4 = |x'|^4 + t^2$$

for any $x = (x', t) \in lH^n$ and the associated distance is

$$d(x, y) := |y^{-1} \circ x|$$

for $x, y \in lH^n$ where $y^{-1} = -y$.

$B(x, r)$ will denote the homogeneous ball centered in $x \in lH^n$ with radius $r > 0$.

The existence of cut-off functions in the Heisenberg group follows by standard methods wheanever one observes that the horizontal gradient of the gauge distance has lenght less or equal than one (this is a trivial computation from the definition).

For every function ψ defined on lH^n, both left and right translations are defined on lH^n as

$$L_y \psi(x) = \psi(y \circ x)$$

$$R_y \psi(x) = \psi(x \circ y).$$

The Lebesgue measure is invariant with respect to the translations of the group, though the shape of the ball changes if one shifts its center, and it is proportional to the Q-th power of the radius, where $Q = 2n + 2$ is the homogeneous dimension of lH^n, that is $|B(x, r)| \simeq r^Q |B(0, 1)|$.

An operator N on lH^n is left-invariant if $L_y(N\psi) = N(L_y\psi)$, and similarly for right-invariance.

The Lie algebra $\mathcal{L}(X)$ of left-invariant vector fields corresponding to lH^n is generated by X_i, X_{i+n} for $i = 1, \dots, n$ and their second commutator T. For this reason the vector fields X_i, $i = 1, \dots, 2n$ satisfy the Hörmander condition of order 2 [2].

The vector fields X_i don't commute with right translations. In particular we cannot interchange them with difference quotiens operators

$$D_h\psi(x) = \frac{\psi(x \circ h) - \psi(x)}{|h|}, \qquad D_{-h}\psi(x) = \frac{\psi(x \circ h^{-1}) - \psi(x)}{-|h|}$$

for $x, h \in lH$.

For any $i = 1, \dots, 2n$ and for any $s > 0$, let h_s^i, $(h_s^i)^{-1}$ be the elements of the group whose j-th component is s, or resp. $-s$, if $j = i$ and 0 otherwise. We have

$$X_i D_{\pm h_s^i} = D_{\pm h_s^i} X_i$$

for every $i = 1, \dots, 2n$, but $X_k D_{\pm h_s^i} \neq D_{\pm h_s^i} X_k$ if $k \neq i$.

For any $\alpha \in (0, 1)$, $s > 0$ we will denote $h_s^* = (0, s)$ and $(h_s^*)^{-1} = (0, -s)$. For every $i = 1, \dots, 2n$ we have

$$X_i D_{\pm h^*_{s,\alpha}} = D_{\pm h^*_{s,\alpha}} X_i. \tag{5}$$

Finally we observe that

$$D_{-h^*_{s,\alpha}} D_{h^*_{s,\alpha}} \psi = \frac{\psi(x \circ h_s^*) + \psi(x \circ (h_s^*)^{-1}) - 2\psi(x)}{s^{2\alpha}} = D_{h^*_{s,\alpha}} D_{-h^*_{s,\alpha}} \psi .$$

3. - Preliminaries results

The proof of the following Lemma is an easy calculation, so we will omit it.

Lemma 3.1: *Let $p > 1$. For any function $\psi \in L^p(\Omega)$ with compact support $\omega \subset \Omega$ and for any h such that $|h| < d(\omega, \partial\Omega)$, we have*

$$\int_\Omega f D_{\pm h}\psi\, dx = -\int_\Omega \psi D_{\mp h} f\, dx$$

for any $f \in L^{p/(p-1)}_{\text{loc}}(\Omega)$.

LEMMA 3.2: *For any* $\psi \in C_0^\infty(\Omega)$ *and for any* $i = 1, \dots, 2n$

$$\lim_{s\to 0^+} D_{\pm h^*_{s,1}}\psi = \partial_t \psi\,, \qquad \lim_{s\to 0^+} D_{\pm h^i_s}\psi = X_i\psi\,.$$

PROOF: Let's observe that

$$D_{h^*_{s,1}}\psi = \int_0^1 (\partial_t\psi)(x\circ\delta_\theta h^*_s)\, d\theta\,.$$

Therefore, taking into account that $x\circ\delta_\theta h^*_s \to x$ when $s\to 0^+$, we conclude the proof. The proofs of the other limits are similar.

LEMMA 3.3 ([1], Proposition 2.3): *Let* $p>1$ *and let* $\psi\in L^p_{\mathrm{loc}}(\Omega)$ *and* $g\in C_0^\infty(\Omega)$ *with* $\omega = \mathrm{supp}\, g \subset\subset \Omega$. *Let* $i\in\{1,\dots,2n\}$. *If there are some constants* $\varepsilon_0>0$ *and* $C>0$ *such that*

$$\sup_{0<s<\varepsilon_0} \int_\omega |D_{\pm h^i_s}\psi|^p\, dx \leq C^p \tag{6}$$

then $X_i\psi\in L^p(\omega)$ *and* $\|X_i\psi\|_{L^p(\omega)}\leq C$. *Conversely, if* $X_i\psi\in L^p_{\mathrm{loc}}(\Omega)$, *then* (6) *holds for any* $\omega = \mathrm{supp}\, g\subset\subset\Omega$, $v\in C_0^\infty(\Omega)$ *and* $C = 2\|X_i\psi\|_{L^p(\omega)}$. *The same result holds if we substitute* $D_{\pm h^i_s}$ *and* X_i *by resp.* $D_{\pm h^*_{s,1}}$ *and* ∂_t.

LEMMA 3.4: *Let* $\psi\in C^\infty(\Omega)$ *and let* $g\in C_0^\infty(B(R))$, *where* $B(R)$ *is a homogeneous gauge ball such that* $B(2R)\subset\Omega$. *Then there exists a positive constant* C *such that, for any small* $\varepsilon_0>0$ *and any* $p>1$

$$\sup_{0<s<\varepsilon_0}\int_\Omega |D_{\pm h^*_{s,1/2}}(\psi g)|^p\, dx \leq \tag{7}$$

$$\leq C\sum_{i=1}^{2n}\left\{\sup_{0<s<\varepsilon_0}\int_\Omega |D_{h^i_s}(\psi g)|^p\,dx + \sup_{0<s<\varepsilon_0}\int_\Omega |D_{-h^i_s}(\psi g)|^p\,dx\right\}.$$

PROOF: We will suppose $n=1$ for sake of simplicity. We start estimating for small $s>0$ the integral $\int_\Omega |D_{-h^*_{s,1/2}}(\psi g)|^p\,dx$ by the right-hand side of (7). Further, we will estimate $\int_\Omega |D_{h^*_{s,1/2}}(\psi g)|^p\,dx$ as well.

If $x=(x_1,x_2,t)\in\Omega$ and if we put for small $s>0$

$$y := x\circ h^1_{\sqrt{s}/2}$$

$$z := y\circ h^2_{\sqrt{s}/2}$$

$$w := z\circ (h^1_{\sqrt{s}/2})^{-1}$$

then we obtain $w \circ (h^2_{\sqrt{s}/2})^{-1} = x \circ (h^*_s)^{-1}$ and, if we set $v = \psi g$, then

$$
\begin{aligned}
&|v(x \circ (h^*_s)^{-1}) - v(x)| \\
&\leq |v(w \circ (h^2_{\sqrt{s}/2})^{-1}) - v(w)| \\
&+ |v(z \circ (h^1_{\sqrt{s}/2})^{-1}) - v(z)| \\
&+ |v(y \circ h^2_{\sqrt{s}/2}) - v(y)| \\
&+ |v(x \circ h^1_{\sqrt{s}/2}) - v(x)| \,.
\end{aligned}
$$

Let's divide both sides of the latter by $\sqrt{s}/2$ and integrate with respect to $x \in \Omega$ the p-th power of them.

For s small enough we have

$$\int_{B(2R)} \left| \frac{v(y \circ h^2_{\sqrt{s}/2}) - v(y)|}{\sqrt{s}/2} \right|^p dx$$

$$\leq \int_{B(3R)} \left| \frac{v(x \circ h^2_{\sqrt{s}/2}) - v(x)}{\sqrt{s}/2} \right|^p dx$$

(the modulus of the Jacobian of the application $x \to x \circ h^1_{\sqrt{s}/2}$ is equal to 1) and the other integrals in the right-hand side can be estimated in the same way.

So the first part of the conclusion easily follows. About the second part we proceed similarly by putting

$$\bar{y} := x \circ h^2_{\sqrt{s}/2}$$

$$\bar{z} := \bar{y} \circ h^1_{\sqrt{s}/2}$$

$$\overline{w} := \bar{z} \circ (h^2_{\sqrt{s}/2})^{-1}.$$

We obtain $\overline{w} \circ (h^1_{\sqrt{s}/2})^{-1} = x \circ h^*_s$ and the same arguments of the former case give as well the result.

From Propositions 3.3 and 3.4 we easily deduce

COROLLARY 3.5: *Let the hypothesis of Proposition* 3.4 *be hold. Then there exists a constant* $C > 0$ *such that, for any small* $\varepsilon_0 > 0$ *and any* $p > 1$

$$\sup_{0 < s < \varepsilon_0} \int_\Omega |D_{\pm h^*_{s,1/2}}(\psi g)|^p \, dx \leq C \int_\Omega |X(\psi g)|^p \, dx \tag{8}$$

4. - Proof of Theorem 1.1

Let's multiply the equation (1) by the test function $\varphi = D_{-h^*_{s,1/2}}(g^4 D_{h^*_{s,1/2}} u)$, g being a cut-off function between $B(R)$ and $B(2R)$. Let's observe that $\varphi \in S_0^{1,p}(\Omega)$ in virtue of (5). On account of (5) we obtain

$$
(9) \qquad I_1 + 4I_2 := \int_{\Omega} (D_{h^*_{s,1/2}} A^k)(X_k D_{h^*_{s,1/2}} u)\, g^4\, dx + \\
+4 \int_{\Omega} (D_{h^*_{s,1/2}} A^k)(D_{h^*_{s,1/2}} u)\, g^3 X_k g\, dx = 0\,.
$$

Estimate of I_1 For sake of simplicity we will denote $A_j^k := A_{\xi_j}^k$, j, $k = 1, \ldots, 2n$. Let's observe that

$$
(10) \qquad D_{h^*_{s,1/2}} A^k = \int_0^1 A_j^k(x,\, Xu + \theta s^{1/2} D_{h^*_{s,1/2}} Xu)\, d\theta\, D_{h^*_{s,1/2}} X_j u + \\
+ \int_0^1 A_{x_l}^k(x + \theta s^{1/2} D_{h^*_{s,1/2}} x,\, Xu)\, d\theta\, D_{h^*_{s,1/2}} x_l
$$

where the repetition of the index l denotes summation from 1 to $2n+1$. If we denote

$$
\alpha_{h_s^*}^{kj} := \int_0^1 A_j^k(x,\, Xu + \theta s^{1/2} D_{h^*_{s,1/2}} Xu)\, d\theta
$$

$$
\beta_{h_s^*}^{kl} := \int_0^1 A_{x_l}^k(x + \theta s^{1/2} D_{h^*_{s,1/2}} x,\, Xu)\, d\theta
$$

then from (5), (9) and (10) we obtain

$$
(11) \qquad I_1 = \int_{\Omega'} g^4 \alpha_{h_s^*}^{kj} D_{h^*_{s,1/2}} X_k u\, D_{h^*_{s,1/2}} X_j u\, dx + \\
+ \int_{\Omega'} g^4 \beta_{h_s^*}^{kl} D_{h^*_{s,1/2}} X_k u\, D_{h^*_{s,1/2}} x_l u\, dx =: J_1 + J_2 .
$$

From (a), (b), (c) and using Giusti's technique ([6], Lemma 8.3), we obtain, for $\xi \in I\!R^{2n}$ and suitable positive constants C

$$
(12) \qquad \alpha_{h_s^*}^{kj} \xi_k \xi_j \geq C \int_0^1 V^{p-2}(Xu + \theta s^{1/2} D_{h^*_{s,1/2}} Xu)\, d\theta\, \xi_k \xi_j \geq C W_{h_s^*}^{p-2} |\xi|^2
$$

$$
(13) \qquad |\alpha_{h_s^*}^{kj}| \leq C W_{h_s^*}^{p-2}, \qquad |\beta_{h_s^*}^{kl}| \leq C W_{h_s^*}^{p-1}
$$

where $W^2_{h_s^*} := 1 + |Xu(x)|^2 + |Xu(x \circ h_s^*)|^2$. On account of (12) we have

$$J_1 \geq C \int_{\Omega'} g^4 W^{p-2}_{h_s^*} |D_{h_s^*,1/2} Xu|^2 \, dx \,. \tag{14}$$

On account of the decomposition $p - 1 = \frac{p + (p-2)}{2}$ we obtain from (13) and Young's inequality

$$\begin{aligned} |J_2| &\leq C \int_{\Omega'} g^4 W^{p-1}_{h_s^*} |D_{h_s^*,1/2} Xu| \, |D_{h_s^*,1/2} x| \, dx \leq \\ &\leq \delta \int_{\Omega'} g^4 W^{p-2}_{h_s^*} |D_{h_s^*,1/2} Xu|^2 \, dx + \delta^{-1} \int_{\Omega'} g^4 W^p_{h_s^*} |D_{h_s^*,1/2} x|^2 \, dx \,. \end{aligned} \tag{15}$$

Let's observe that, for $s < R$, we have

$$\int_{B(2R)} W^p_{h_s^*} \, dx \leq \int_{B(3R)} V^p \, dx \,. \tag{16}$$

On account of (14), (15) and (16) and for small δ we obtain

$$I_1 \geq C_1 \int_{\Omega'} g^4 W^{p-2}_{h_s^*} |D_{h_s^*,1/2} Xu|^2 \, dx - C_2 \int_{\Omega'} V^p \, dx \tag{17}$$

where C_1 and C_2 are suitable positive constants.

5. - Estimate of I_2

We count again on (10), (13), (*b*) and (*c*) to obtain

$$\begin{aligned} |I_2| &= |D_{h_s^*,1/2} A^k D_{h_s^*,1/2} u g^3 X_k g \, dx| \leq \\ &\leq \left| \int_{\Omega'} \alpha^{kj}_{h_s^*} D_{h_s^*,1/2} X_j u \, D_{h_s^*,1/2} u g^3 X_k g \, dx \right| + \left| \int_{\Omega'} \beta^{kl}_{h_s^*} D_{h_s^*,1/2} x_l D_{h_s^*,1/2} u g^3 X_k g \, dx \right| \leq \\ &\leq \int_{\Omega'} W^{p-2}_{h_s^*} |D_{h_s^*,1/2} Xu| \, |D_{h_s^*,1/2} u| g^3 \, |Xg| \, dx + \int_{\Omega'} W^{p-1} |D_{h_s^*,1/2} u| g^3 \, |Xg| \, dx \,. \end{aligned} \tag{18}$$

The case $p \geq 2$. Applying Young's inequality to (18) gives

$$\begin{aligned} |I_2| &\leq \delta \int_{\Omega'} g^4 W^{p-2}_{h_s^*} |D_{h_s^*,1/2} Xu|^2 \, dx + \\ &+ C R^{-2} \int_{\Omega'} g^2 W^{p-2}_{h_s^*} |D_{h_s^*,1/2} u|^2 \, dx + C \int_{\Omega'} W^{p-1} |D_{h_s^*,1/2} u| g^3 \, |Xg| \, dx \,. \end{aligned} \tag{19}$$

Let's observe that

$$\int_{\Omega'} g^2 W_{h_s^*}^{p-2} |D_{h_s^*,1/2} u|^2 \, dx \leq C \int_{\Omega'} (W_{h_s^*}^p + g^p |D_{h_s^*,1/2} u|^p) \, dx \tag{20}$$

and

$$\int_{\Omega'} W_{h_s^*}^{p-1} |D_{h_s^*,1/2} u| g^3 |Xg| \, dx \leq C \int_{\Omega'} (R^{-2} W_{h_s^*}^p + g^p |D_{h_s^*,1/2} u|^p) \, dx . \tag{21}$$

Lemma 3.4 gives

$$\int_{\Omega'} |D_{h_s^*,1/2}(gu)|^p \, dx \leq C \int_{\Omega'} (V^p + |u|^p) \, dx . \tag{22}$$

On account of (16), (18), …, (22) we obtain

$$|I_2| \leq \delta \int_{\Omega'} g^4 W_{h_s^*}^{p-2} |D_{h_s^*,1/2} Xu|^2 \, dx + CR^{-2} \int_{\Omega'} (V^p + |u|^p) \, dx . \tag{23}$$

Combining the estimates (17) of I_1 with (23) of I_2 for small δ we finally obtain

$$\int_{\Omega'} g^4 W_{h_s^*}^{p-2} |D_{h_s^*,1/2} Xu|^2 \, dx \leq CR^{-2} \int_{\Omega'} (V^p + |u|^p) \, dx . \tag{24}$$

As $|s^{1/2} D_{h_s^*,1/2} Xu|^2 = |Xu(x \circ h_s^*) - Xu(x)|^2 \leq 4[|Xu(x \circ h_s^*)|^2 + |Xu(x)|^2] \leq$ $\leq 4 W_{h_s^*}^2$, then (24) gives

$$\int_{\Omega'} s^{\frac{p-2}{2}} |D_{h_s^*,1/2} X(g^4 u)|^p \leq CR^{-2p} \int_{\Omega'} (V^p + |u|^p) \, dx . \tag{25}$$

From (5), (25), Lemma 3.3 and Lemma 3.4 we have

$$\int_{\Omega'} s^{\frac{p-2}{2}} |D_{-h_s^*,1/2} D_{h_s^*,1/2}(g^4 u)|^p \leq CR^{-2p} \int_{\Omega'} (V^p + |u|^p) \, dx$$

that is

$$\int_{\Omega'} |\Delta_s^2 (g^4 u)|^p s^{-1-p/2} \, dx \leq CR^{-2p} \int_{\Omega'} (V^p + |u|^p) \, dx \tag{26}$$

where $\Delta_s^2 w = w(x \circ h_s^*) - 2w(x) + w(x \circ (h_s^*)^{-1})$, and then, for any $\alpha \in (0, 1)$,

$$\int_0^1 \int_{\Omega'} |\Delta_s^2 (g^4 u)|^p s^{-1-p\beta} \, dx \, ds \leq CR^{-2p} \int_{\Omega'} (V^p + |u|^p) \, dx \tag{27}$$

where $\beta = \dfrac{\alpha}{p} + \dfrac{1}{2}$.

Let's now briefly recall some notions about some known functional spaces on lR. We refer the readers to [12] and [13] for the details. Let us denote by $B^{\theta}_{p,p}$, $\theta \in (0, 1)$, $p > 1$, the Besov space defined as the completion of $C_0^{\infty}(lR)$ with respect to the norm $\|\varphi\|_{L^p} + \left(\int \|\Delta_s^2 \varphi\|_{L^p}^p |s|^{-1-p\theta} ds\right)^{1/p}$, where $\Delta_s^2 \varphi(t) = \varphi(t+s) - 2\varphi(t) + \varphi(t-s)$. It results ([13]: (9) p. 37; (9) p. 90) $B^{\theta}_{p,p} = W^{\theta, p}$, where $W^{\theta, p}$ denotes the fractional Sobolev space. For the same values of θ, p let us consider the Bessel potential spaces $H^{\theta, p}$ defined as the completion of $C_0^{\infty}(lR)$ with respect to the norm $\|F^{-1}((1+ + |\xi|^2)^{\theta/2} F\varphi)\|_{L^p}$, where F denotes the Fourier transform. It results $H^{\theta+\tau, p} \subset W^{\theta, p} \subset H^{\theta-\tau, p}$ for any small $\tau > 0$. Moreover the interpolation spaces $(L^p, W^{1,p})_{\theta, \infty} = \left\{\varphi \in L^p \colon \sup\limits_{0<|s|<\sigma} \dfrac{\|\varphi(\cdot+s) - \varphi(\cdot)\|_{L^p}}{|s|^{\theta}} < \infty\right\}$, for some positive constant $\sigma > 0$, satisfy $H^{\theta, p} \subset (L^p, W^{1,p})_{\theta, \infty} \subset H^{\tilde{\theta}, p}$, for any $0 < \tilde{\theta} < \theta$ ([12]: Theorem 1 p. 64; (1), (4) p. 25; (11) p. 185). Collecting the previous inclusions we obtain

$$(L^p, W^{1,p})_{\theta+2\tau, \infty} \subset H^{\theta+\tau, p} \subset W^{\theta, p} = B^{\theta}_{p,p} \subset H^{\theta-\tau, p} \subset (L^p, W^{1,p})_{\theta-\tau, \infty} \tag{28}$$

for any $\theta \in (0, 1)$, $p > 1$ and any small $\tau > 0$. It follows in particular

$$\|\varphi\|_{(L^p, W^{1,p})_{\beta-\tau, \infty}} \leq C \|\varphi\|_{B^{\beta}_{p,p}} \tag{29}$$

for any $\varphi \in C_0^{\infty}(lR)$, any small $\tau > 0$ and any $\alpha \in (0, 1)$, where $\beta = \dfrac{\alpha}{p} + \dfrac{1}{2}$.

From (27) and (29) we obtain

$$\sup_{s<\sigma} \int_{\Omega'} |D_{b^*_{s, \beta-\tau}}(g^4 u)|^p dx \leq C R^{-2p} \int_{\Omega'} (V^p + |u|^p)\, dx\,. \tag{30}$$

Let's multiply equation (1) by the function $D_{-b^*_{s,\beta-\tau}}(g^4 D_{b^*_{s,\beta-\tau}} u)$. Let's observe that it is a right test function for (1) in virtue of (5). Using (30) in place of Lemma 3.4 (possibly modifying the domain of the cut-off function g) we can repeat the argument from (9) until (25). More precisely, in place of (25) we have now

$$\int_{\Omega'} s^{(p-2)(\beta-\tau)} |D_{b^*_{s,\beta-\tau}} X(g^4 u)|^p dx \leq C R^{-4p} \int_{\Omega'} (V^p + |u|^p)\, dx \tag{31}$$

from which, arguing as for (26), we obtain

$$\int_{\Omega'} |\Delta_s^2 (g^4 u)|^p s^{-2(\beta-\tau) - p/2} dx \leq C R^{-4p} \int_{\Omega'} (V^p + |u|^p)\, dx \tag{32}$$

and then

$$\int_0^1 \int_{\Omega'} |\Delta_s^2 (g^4 u)|^p s^{-\sigma} dx\, ds \leq C R^{-4p} \int_{\Omega'} (V^p + |u|^p)\, dx \tag{33}$$

where $\sigma = 2(\beta - \tau) + \frac{p}{2} + \alpha$.

Let now $E' \subset lR^{2n}$, $F' \subset lR$ be intervals such that $\Omega' \subset E' \times F'$ and let $q, q' > 1$ be such that $\frac{1}{q} + \frac{1}{q'} = 1$. Let's choose q in such a way that $\frac{2p'}{q'} < 1$, that is $\frac{2p}{q} > p+1$. As $B^1_{p,1}(lR) \subset W^{1,p}(lR)$ ([13]: (5), (10), p. 90), then we have

$$(34) \qquad \int_{\Omega'} |T(g^4 u)|^p \, dx = 4 \int_{E'} \int_{F'} |\partial_t (g^4 u)|^p \, dt \, dx' \leqslant$$

$$\leqslant C \int_{E'} \left(\int_0^1 \| \Delta_s^2 (g^4 u) \|_{L^p(F')} s^{-2} \, ds \right)^p dx' \leqslant$$

$$\leqslant C \int_{E'} \left[\left(\int_0^1 \left(\int_{F'} |\Delta_s^2 (g^4 u)|^p \, dt \right) s^{-\frac{2p}{q}} \, ds \right) \left(\int_0^1 s^{\frac{-2p'}{q'}} \, ds \right)^{\frac{p}{p'}} \right] dx' \leqslant$$

$$\leqslant C \int_{E'} \int_0^1 \left(\int_{F'} |\Delta_s^2 (g^4 u)|^p \, dt \right) s^{-\frac{2p}{q}} \, ds \, dx' =$$

$$= C \int_0^1 \int_{\Omega'} |\Delta_s^2 (g^4 u)|^p s^{-\frac{2p}{q}} \, dx \, ds \, .$$

Let's now set $\frac{2p}{q} = \sigma$. This can be done provided $\sigma > p+1$, and this holds for any $2 \leqslant p < \alpha - 2\tau + \sqrt{(\alpha - 2\tau)^2 + 4\alpha}$. As $(\alpha - 2\tau) + \sqrt{(\alpha - 2\tau)^2 + 4\alpha} \to 1 + \sqrt{5}$ when $\alpha \to 1$, $\tau \to 0$, then, for any $2 \leqslant p < 1 + \sqrt{5}$, (3) follows from (33) and (34).

The case $p < 2$. As

$$|s^{1/2} D_{h^*_s, 1/2} Xu| \leqslant 2 W_{h^*_s}$$

then, for $\gamma = \frac{2-p}{p}$, we obtain

$$\left| \int_{\Omega'} D_{h^*_s, 1/2} A^k D_{h^*_s, 1/2} u g^3 X_k g \, dx \right|$$

$$\leqslant \int_{\Omega'} W_{h^*_s}^{p-2} |D_{h^*_s, 1/2} Xu| \, |D_{h^*_s, 1/2} u| g^3 \, |X_k g| \, dx$$

$$\leqslant C \int_{\Omega'} W_{h^*_s}^{p-2+\gamma} s^{-\gamma/2} |D_{h^*_s, 1/2} Xu|^{1-\gamma} \, |D_{h^*_s, 1/2} u| g^3 \, |X_k g| \, dx$$

(by Young's inequality with exponents $\frac{p}{p-1}$ and p)

(35) $$\leq \delta \int_{\Omega'} g^4 W_{h_s^*}^{p-2} |D_{h^*_{s,1/2}} Xu|^2 \, dx + CR^{-p} \delta^{-1} \int_{\Omega'} (|D_{h^*_{s,1/2}}(gu)|^p + |u|^p) \, s^{-\gamma p/2} \, dx \, .$$

Using Lemma 3.4 the second integral in the right hand-side of (35) can be estimated by

$$cR^{-p} \delta^{-1} \int_{\Omega'} (V^p + |u|^p) \, s^{-\gamma p/2} \, dx \, .$$

Hence, on account of (17) and (35) with small δ, we obtain

(36) $$\int_{\Omega'} g^4 W_{h_s^*}^{p-2} |D_{h^*_{s,1/2}} Xu|^2 \, dx \leq CR^{-p} \int_{\Omega'} (V^p + |u|^p) \, s^{-\gamma p/2} \, dx \, .$$

If $2\sigma = p(2-p)$, then, by Young's inequality with exponents $\frac{2}{2-p}$ and $\frac{2}{p}$, (36) gives

$$\int_{\Omega'} g^4 |D_{h^*_{s,1/2}} Xu|^p \, dx = \int_{\Omega'} g^4 W_{h_s^*}^{\sigma} W_{h_s^*}^{-\sigma} |D_{h^*_{s,1/2}} Xu|^p \, dx$$

$$\leq C \int_{\Omega'} g^2 (W_{h_s^*}^p + W_{h_s^*}^{p-2} |D_{h^*_{s,1/2}} Xu|^2) \, dx \leq CR^{-p} \int_{\Omega'} (V^p + |u|^p) \, s^{-\gamma p/2} \, dx$$

and then

(37) $$\int_{\Omega'} |D_{h^*_{s,1/2}} X(g^4 u)|^p \, dx \leq CR^{-2p} \int_{\Omega'} (V^p + |u|^p) \, s^{-\gamma p/2} \, dx \, .$$

From (37) and Lemma 3.4 we obtain

(38) $$\int_0^1 \int_{\Omega'} |\Delta_s^2 (g^4 u)|^p \, s^{-p-\alpha} \, dx \, ds \leq CR^{-2p} \int_{\Omega'} (V^p + |u|^p) \, dx$$

for any $0 < \alpha < 1/2$, where $\Delta_s^2 w := w(x \circ h_s^*) - 2w(x) + w(x \circ (h_s^*)^{-1})$, and then

(39) $$\int_0^1 \int_{\Omega'} |\Delta_s^2 (g^4 u)|^p \, s^{-1-p\beta} \, dx \, ds \leq CR^{-2p} \int_{\Omega'} (V^p + |u|^p) \, dx$$

where $\beta := \alpha + \left(1 - \frac{1}{p}\right)$.

From (29) and (39) we obtain

(40) $$\sup_{s < \sigma} \int_{B_{R/2}} |D_{h^*_{s,\beta-\tau}} (g^4 u)|^p \, dx \leq CR^{-2p} \int_{\Omega'} (V^p + |u|^p) \, dx \, .$$

Let's multiply equation (1) by the function $D_{-h^*_{s,\beta-\tau}}(g^4 D_{h^*_{s,\beta-\tau}} u)$. Let us observe that it is a right test function in virtue of (5). If we repeat the argument from (9) until (35), then we obtain

$$\left| \int_{\Omega'} D_{h^*_{s,\beta-\tau}} a^k D_{h^*_{s,\beta-\tau}} u g^3 X_k g \, dx \right| \leq \delta \int_{\Omega'} g^4 W^{p-2}_{h^*_s} |D_{h^*_{s,\beta-\tau}} Xu|^2 \, dx + \tag{41}$$

$$+ C\delta^{-1} R^{-p} \int_{\Omega'} (|D_{h^*_{s,\beta-\tau}}(gu)|^p + |u|^p) \, s^{-\gamma p(\beta-\tau)} \, dx \, .$$

The second integral in the right hand-side of (41) can now be estimated using (40) in place of Lemma 3.4 (possibly modifying the domain of the cut-off function) by

$$CR^{-3p} \delta^{-1} \int_{\Omega'} (V^p + |u|^p) \, s^{-\gamma p(\beta-\tau)} \, dx \, .$$

Hence, on account of (41) with small δ and the above estimate we obtain now in place of (36)

$$\int_{\Omega'} g^4 W^{p-2}_{h^*_s} |D_{h^*_{s,\beta-\tau}} Xu|^2 \, dx \leq cR^{-3p} \int_{\Omega'} (V^p + |u|^p) \, s^{-\gamma p(\beta-\tau)} \, dx \, . \tag{42}$$

We proceed from (42) as in (36), ..., (38) to obtain

$$\int_0^1 \int_{\Omega'} |\Delta^2_s (g^4 u)|^p \, s^{-\frac{p}{2} - p(\beta-\tau) - \mu p} \, dx \, ds \leq CR^{-4p} \int_{\Omega'} (V^p + |u|^p) \, dx \tag{43}$$

where $\mu \in (0, 1)$ satisfies $\mu p + \gamma p(\beta - \tau) < 1$. For any $1 + \frac{1}{\sqrt{5}} < p < 2$ we can choose μ and τ in such a way that $\frac{p}{2} + p(\beta - \tau) + \mu p = 1 + p$. In this case (43) gives $(g^4 u)(x', \cdot) \in B^1_{p,p}(lR)$. As for any $1 < p < 2$ we have $B^1_{p,p}(lR) \subset W^{1,p}(lR)$ ([12]: (7) p. 169; (8) p. 179) then (3) follows and the proof of Theorem 1.1 is finally concluded.

6. - Proof of Theorem 1.2

Let's set $\varphi = D_{-h^*_{s,1}}(g^4 D_{h^*_{s,1}} u)$ in (2). Taking into account Theorem 1.1 in place of Lemma 3.4 we can proceed as in Section 4 obtaining now

$$\int_{\Omega'} g^4 W^{p-2}_{h^*_s} |D_{h^*_{s,1}} Xu|^2 \, dx \leq CR^{-2} \int_{\Omega'} (V^p + |u|^p) \, dx \, . \tag{44}$$

We deduce from (44) that $D_{h^*_{s,1}} Xu$ is bounded in $L^2(B(R))$. By Lemma 3.2, possibly up to a subsequence, $D_{h^*_{s,1}} Xu$ converges in $L^2_{\mathrm{loc}}(B(R))$ to $\partial_t Xu$ as $s \to 0$ and then $Tu \in S^{1,2}_{\mathrm{loc}}(B(R))$. Moreover we can extract from it a subsequence converging for a.e. $x \in B(R)$. By Lemma 3.2 $W_{h^*_s} \to (1 + 2|Xu|^2)^{1/2}$ for a.e $x \in B(R)$ as $s \to 0$. Passing to the limit $s \to 0$ on (44) we conclude the proof.

REFERENCES

[1] L. Capogna, *Regularity of quasilinear equations in the Heisenberg group*, Comm. Pure Appl. Math., L (1997), 867-889.

[2] A. Cutrí - M. G. Garroni, *Existence, uniqueness and regularity results for integro-differential Heisenberg equations*, Adv. in Differential Equations, 1 (1996), 920-939.

[3] E. Di Benedetto, $C^{1+\alpha}$ *local regularity of weak solutions of certain degenerate elliptic equations*, Nonlin. Anal., 7 (1983), 827-373.

[4] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat., 13 (1975), 161-207.

[5] G. B. Folland - E. M. Stein, *Estimates for the complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math., 27 (1974), 459-522.

[6] E. Giusti, *Metodi diretti nel Calcolo delle Variazioni*, UMI, Bologna (1994).

[7] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math., 119 (1967), 147-171.

[8] S. Marchi, $C^{1,\alpha}$ *local regularity for the solutions of the p-Laplacian on the Heisenberg group for* $2 \leq p < 1 + \sqrt{5}$, Z. Anal. Anwendungen, Vol, 20, No. 3 (2001), 617-636.

[9] S. Marchi, *Regularity for the solutions of double obstacle problems involving nonlinear elliptic operators on the Heisenberg group*, Le Matematiche, in print.

[10] J. Mu - P. Ziemer, *Smooth regularity of solutions of double obstacle problems involving degenerate elliptic equations*, Comm. Partial Differential Equations, 16 (1991), 821-843.

[11] A. Nagel - E. M. Stein - S. Wainger, *Balls and metrics defined by vector fields I: basic properties*, Acta Math., 155 (1985), 103-147.

[12] H. Triebel, *Interpolation theory, function spaces, differential operators*, North Holland (1978).

[13] H. Triebel, *Theory of function spaces*, Birkhäuser Verlag (1983).

[14] N. N. Ural'tzeva, *Degenerate quasilinear elliptic systems*, Zap. Nauchno Sem. Leningrad Otdel. Mat. Inst. Steklov, 7 (1968), 184-222.

Direttore responsabile: Prof. A. BALLIO - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5 - Bologna