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Errata Corrige to the Paper: Some Results on Minimal Barriers in the Sense of De Giorgi Applied to Driven Motion by Mean Curvature

ABSTRACT. — We provide the complete argument for the proof of step 2 in the demonstration of Lemma 4.2 in the paper [1].

Errata Corrige al lavoro: Some results on minimal barriers in the sense of De Giorgi applied to driven motion by mean curvature

SUNTO. — Diamo la dimostrazione completa del passo 2 all'interno della dimostrazione del Lemma 4.2 nel lavoro [1].

1. - SHORT EXPLANATION

The proof of step 2 in the demonstration of Lemma 4.2 at pag. 52 of the paper [1] is not completely correct, since formula (4.3) (and consequently formula (4.4)) does not hold in general in the case that $\frac{dp_\tau}{d\tau}\Big|_{\tau=0}$ is tangent to $\partial f(t)$. To convince oneself of this assertion, let us consider the following example: let $f:[0,1] \to \mathcal{P}(\mathbb{R}^2)$ be the smooth flow consisting of the initial circle $f(0) = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ which translates in the positive x_1-direction. Let p_τ be the intersection of $\partial f(\tau)$ with the half-line $\{(x_1, x_2) : x_1 = 1,\ x_2 > 0\}$, and $p := (1, 0) = p_0$. We have $(p_\tau - p)\cdot \nu = 0$, where ν denotes the outward unit normal to $\partial f(0)$ at p; in particular $\lim_{\tau \to 0^+}\left(\frac{p_\tau - p}{\tau}\right)\cdot \nu = 0$. On

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the other hand, the normal velocity of $\partial f(0)$ at p is clearly nonzero, and this is a contradiction with formula (4.3) in [1]. Note that $\lim_{\tau\to 0^+}\left|\frac{p_\tau - p}{\tau}\right| = +\infty$. The above constructed flow is not a curvature flow, but it is clear that, by adding a suitable forcing term, similar examples can be easily constructed. Since we miss the proof that $\frac{dp_\tau}{d\tau}\Big|_{\tau=0}$ is not tangent to $\partial f(t)$ in the situation considered in step 2 of Lemma 4.2, we provide here a different argument to prove the same statement.

2. - Proof of step 2 inside the proof of Lemma 4.2 in [1]

We begin by recalling that the forcing term g belongs to $\mathcal{C}^\infty(\mathbb{R}^n \times I) \cap L^\infty(\mathbb{R}^n \times I)$, and satisfies the following property: there exists a constant $G > 0$ such that

$$|g(x,t) - g(y,t)| \le G|x-y|, \qquad x, y \in \mathbb{R}^n,\ t \in I. \tag{2.1}$$

We are now in a position to prove step 2 in Lemma 4.2 in [1], i.e., if $\phi : I \to \mathcal{P}(\mathbb{R}^n)$, $\phi \in \mathrm{Barr}(\mathcal{F}_g)$, $f : [a,b] \subseteq I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}_g$ and $f(a) \subseteq \phi(a)$, then

$$\limsup_{\tau\to 0^+} \frac{\delta(t) - \delta(t+\tau)}{\tau} \le G\delta(t), \qquad t \in [a, b[,$$

where

$$\delta(t) := \mathrm{dist}(f(t), \mathbb{R}^n \setminus \phi(t)), \qquad t \in [a,b].$$

Suppose by contradiction that there exist $t_0 \in [a, b[$ with $\delta(t_0) > 0$ and $\Lambda < -G\delta(t_0)$ such that

$$\liminf_{\tau\to 0^+} \frac{\delta(t_0+\tau) - \delta(t_0)}{\tau} < \Lambda .$$

Observe that we can always assume

$$\delta(t_0) = \liminf_{\tau\to 0^+} \delta(t_0+\tau)$$

because, if $\delta(t_0) < \liminf_{\tau\to 0^+} \delta(t_0+\tau)$ then $\limsup_{\tau\to 0^+} \frac{\delta(t_0) - \delta(t_0+\tau)}{\tau} = -\infty$, and the assertion of step 2 is trivially satisfied. On the other hand, using the definition of barrier it is not difficult to prove that $\delta(t) \le \liminf_{\tau\to 0^+} \delta(t+\tau)$ for any $t \in [a, b[$.

Pick a decreasing sequence $\{\tau_m\}$ of positive times converging to 0 as $m \to +\infty$ such that $t_0 + \tau_m \in [a, b[$ and

$$\liminf_{\tau\to 0^+} \frac{\delta(t_0+\tau) - \delta(t_0)}{\tau} = \lim_{m\to+\infty} \frac{\delta(t_0+\tau_m) - \delta(t_0)}{\tau_m},$$

and

$$\frac{\delta(t_0+\tau_m)-\delta(t_0)}{\tau_m} < \Lambda\,, \qquad m \in \mathbb{N}\,. \tag{2.2}$$

Recalling that $\partial f(\cdot)$ is compact, there exist two sequences $\{x^m\}$, $\{y^m\}$ of points with $x^m \in \partial f(t_0+\tau_m)$ and $y^m \in \partial\phi(t_0+\tau_m)$ such that

$$|x^m - y^m| = \delta(t_0+\tau_m) < \Lambda\tau_m + \delta(t_0), \qquad m \in \mathbb{N}\,. \tag{2.3}$$

Possibly passing to suitable subsequences, we can suppose that

$$x^m \to x^\infty \in \partial f(t_0), \qquad y^m \to y^\infty \in \overline{\mathbb{R}^n \setminus \phi(t_0)} \qquad \text{as } m \to \infty\,.$$

Notice that the conclusion $y^\infty \in \overline{\mathbb{R}^n \setminus \phi(t_0)}$ is ensured by the properties of barriers. Moreover, using also (2.3) we have

$$\delta(t_0) = \liminf_{\tau\to 0^+} \delta(t_0+\tau) \le \liminf_{m\to+\infty} \delta(t_0+\tau_m) = \liminf_{m\to+\infty} |x^m - y^m| \le \delta(t_0),$$

so that $|x^\infty - y^\infty| = \liminf_{m\to+\infty} \delta(t_0+\tau_m) = \delta(t_0)$.

We now localize our problem as follows. We take an open ball $B(x^\infty)$ centered at x^∞ and small enough, and we define $B(y^\infty) := B(x^\infty) + (y^\infty - x^\infty)$. Clearly $B(y^\infty)$ is a ball centered at y^∞. We can assume that $B(x^\infty) \cap B(y^\infty) = \emptyset$ (recall that $\delta(t_0) > 0$) and that $x^m \in B(x^\infty)$ and $y^m \in B(y^\infty)$ for any $m \in \mathbb{N}$.

We now need to introduce a notation. Given a nonzero vector $\xi \in \mathbb{R}^n$, we denote by f_ξ the mean curvature flow with forcing term g of the translated set $f(t_0) + \xi$. Note that $f_\xi \in \mathcal{F}_g$ and that f_ξ is defined on a time interval of the form $[t_0, c]$, with $c > t_0$ possibly smaller than b. Note also that, as g may depend on the space variable x, $f_\xi(t)$ does not coincide, in general, with $f(t) + \xi$, $t \in [t_0, c]$.

We can suppose that, if ξ is any translation vector of the form $w - z$, for $z \in B(x^\infty)$ and $w \in B(y^\infty)$, with $|\xi| = \delta(t_0)$, the two evolutions $f(\cdot) \cap B(x^\infty)$ and $f_\xi(\cdot) \cap B(y^\infty)$ are the subgraphs of two real-valued smooth functions F, F_ξ defined on $A \times [t_0, t_0 + \bar\tau]$, with A an open subset of $\mathbb{R}^{n-1}$ and where $\bar\tau > 0$ can be chosen independently of ξ. We can further assume that the evolutions $F(\cdot,\cdot)$ and $F_\xi(\cdot,\cdot)$ are also subgraphs with respect to any direction indicated by the vectors ξ described above.

Fix now any integer $m > 0$. Without loss of generality we fix a coordinate system in $\mathbb{R}^n$ depending on m as follows: we suppose that x^m is at the origin of $\mathbb{R}^n$; moreover, as $|x^m - y^m| = \delta(t_0+\tau_m)$, the tangent hyperplane $T_{x^m}(\partial f(t_0+\tau_m))$ to $\partial f(t_0+\tau_m)$ at the origin x^m is horizontal and

$$y^m = (0, \delta(t_0+\tau_m)) \in T_{x^m}(\partial f(t_0+\tau_m)) \times \mathcal{V}_{x^m} = \mathbb{R}^{n-1} \times \mathbb{R}\,,$$

where $\mathcal{V}_{x^m}$ denotes the vertical axis $\{x^m + (y^m - x^m)\, r : r \in \mathbb{R}\}$.

Define

$$\xi_m := \delta(t_0)\,\frac{y^m - x^m}{|y^m - x^m|}\,,$$

and set

$$\bar{x}(t_0+\tau) = (0, \ldots, 0, \bar{x}_n(t_0+\tau)) := \partial f(t_0+\tau) \cap \mathcal{V}_{x^m},$$

$$\bar{y}(t_0+\tau) = (0, \ldots, 0, \bar{y}_n(t_0+\tau)) := \partial f_{\xi_m}(t_0+\tau) \cap \mathcal{V}_{x^m},$$

for $\tau \geq 0$ small enough. Note that

$$\bar{x}(t_0+\tau_m) = x^m = 0\,, \tag{2.4}$$

and moreover $\bar{y}(t_0) = \bar{x}(t_0) + \xi_m$; in particular

$$|\bar{x}(t_0) - \bar{y}(t_0)| = \bar{y}_n(t_0) - \bar{x}_n(t_0) = \delta(t_0)\,. \tag{2.5}$$

Observe that by construction we have $f(t_0) + \xi_m = f_{\xi_m}(t_0) \subseteq \overline{\phi(t_0)}$. Assume first that $f_{\xi_m}(t_0) \subseteq \phi(t_0)$. Since $\phi \in \mathrm{Barr}(\mathcal{F}_g)$ it follows that $f_{\xi_m}(t_0+\tau) \subseteq \phi(t_0+\tau)$ for $\tau > 0$ small enough, so that the vertical component of y^m is larger than or equal to the value of $\partial f_{\xi_m}(t_0+\tau_m)$ viewed as a function from $A \subset \mathbb{R}^{n-1}$ to $\mathbb{R}$ computed at the origin of $\mathbb{R}^{n-1}$. Therefore

$$\bar{y}_n(t_0+\tau_m) \leq \delta(t_0+\tau_m)\,. \tag{2.6}$$

The two functions $F(\cdot, t_0)$ and $F_{\xi_m}(\cdot, t_0)$ do not have, in general, zero gradient at $(\bar{x}_1(t_0), \ldots, \bar{x}_{n-1}(t_0)) = (\bar{y}_1(t_0), \ldots, \bar{y}_{n-1}(t_0)) = 0 \in \mathbb{R}^{n-1}$, but still we can show that this gradient is quite small. Since $T_{x^m}(\partial f(t_0+\tau_m))$ is horizontal, from the regularity of the evolution of $\partial f(\cdot)$ we get that the angle θ_m formed by the normal to $\partial f(t_0)$ at $\bar{x}^m$ and the vertical axis is bounded by $|\theta_m| < \mathcal{O}(\tau_m)$, so that

$$\cos\theta_m = 1 + \mathcal{O}(\tau_m^2)\,. \tag{2.7}$$

The vertical velocity $\frac{d\bar{x}_n}{dt}(t_0) := \bar{x}_n'(t_0)$ of $\bar{x}(t_0)$ at $\tau = 0$ is given by

$$\bar{x}_n'(t_0) = \frac{\overline{V}_f(\bar{x}(t_0), t_0)}{\cos\theta_m} = \overline{V}_f(\bar{x}(t_0), t_0)(1 + \mathcal{O}(\tau_m^2))\,, \tag{2.8}$$

where $\overline{V}_f(\bar{x}(t_0), t_0)$ is the outer normal velocity of $\partial f(t_0)$ computed at $\bar{x}(t_0)$, and we made use of (2.7). Similarly, the vertical velocity $\frac{d\bar{y}_n}{dt}(t_0) := \bar{y}_n'(t_0)$ of $\bar{y}(t_0)$ at $\tau = 0$ is given by

$$\bar{y}_n'(t_0) = \frac{\overline{V}_{f_{\xi_m}}(\bar{y}(t_0), t_0)}{\cos\theta_m} = \overline{V}_{f_{\xi_m}}(\bar{y}(t_0), t_0)(1 + \mathcal{O}(\tau_m^2))\,, \tag{2.9}$$

where $\overline{V}_{f_{\xi_m}}(\bar{y}(t_0), t_0)$ is the outer normal velocity of $\partial f_{\xi_m}(t_0)$ computed at $\bar{y}(t_0)$.

Using (2.3), (2.4), (2.6), a Taylor expansion for $\overline{x}_n$ and $\overline{y}_n$, (2.5), (2.9), (2.8), (2.1), and finally (2.5) again we then get

$$\begin{aligned}
\delta(t_0+\tau_m) &= |x^m - y^m| \geqslant \overline{y}_n(t_0+\tau_m) - \overline{x}_n(t_0+\tau_m) \\
&= \delta(t_0) + (\overline{y}_n'(t_0) - \overline{x}_n'(t_0))\,\tau_m + o(\tau_m) \\
&= \delta(t_0) + (\overline{V}_{f_{\xi_m}}(\overline{y}(t_0), t_0) - \overline{V}_f(\overline{x}(t_0), t_0))\,\tau_m(1+\mathcal{O}(\tau_m^2)) + o(\tau_m) \\
&= \delta(t_0) + (g(\overline{y}(t_0), t_0) - g(\overline{x}(t_0), t_0))\,\tau_m(1+\mathcal{O}(\tau_m^2)) + o(\tau_m) \\
&\geqslant \delta(t_0) - G|\overline{x}(t_0) - \overline{y}(t_0)|\,\tau_m(1+\mathcal{O}(\tau_m^2)) + o(\tau_m) \\
&= \delta(t_0) - G\delta(t_0)\,\tau_m(1+\mathcal{O}(\tau_m^2)) + o(\tau_m) \\
&= \delta(t_0) - G\delta(t_0)\,\tau_m + o(\tau_m),
\end{aligned}$$

which is in contradiction with (2.2).

It remains to consider the general case when $f_{\xi_m}(t_0) \subseteq \overline{\phi(t_0)}$ (and $f_{\xi_m}(t_0)$ is not contained in $\phi(t_0)$). Given a set $C \subseteq \mathbb{R}^n$ and $\varrho > 0$, define $C_\varrho^- := \{x \in C : \mathrm{dist}(x, \mathbb{R}^n \setminus C) > \varrho\}$. Since $f_{\xi_m}(t_0)$ is a smooth compact set, if $\overline{\varrho} > 0$ is sufficiently small, we have that, for $\varrho \in [0, \overline{\varrho}]$, the set $(f_{\xi_m}(t_0))_\varrho^-$ is smooth, and the smooth mean curvature evolutions with forcing term g of $(f_{\xi_m}(t_0))_\varrho^-$ has an existence time which is independent of ϱ. Moreover, $\mathrm{int}(f_{\xi_m}(t_0)) = \bigcup_{\varrho \in [0,\overline{\varrho}]} (f_{\xi_m}(t_0))_\varrho^-$. Thanks to the fact that $f \in \mathcal{F}_g$, possibly reducing $\overline{\tau} > 0$ we also have

$$\mathrm{int}(f_{\xi_m}(t_0+\tau)) = \bigcup_{\varrho \in [0,\overline{\varrho}]} (f_{\xi_m}(t_0+\tau))_\varrho^-, \qquad \tau \in [0, \overline{\tau}].$$

Recalling our construction, the definition of $\delta(\cdot)$ and the assumption $\phi \in \mathrm{Barr}(\mathcal{F}_g)$, we then get $(f_{\xi_m}(t_0+\tau))_\varrho^- \subseteq \phi(t_0+\tau)$ for $\varrho \in [0, \overline{\varrho}]$ and $\tau \in [0, \overline{\tau}]$. It follows that $\mathrm{int}((f_{\xi_m}(t_0+\tau)) \subseteq \phi(t_0+\tau)$ for $\tau \in [0, \overline{\tau}]$. Repeating the previous arguments, we then conclude the proof.

REFERENCES

[1] G. Bellettini - M. Paolini, *Some results on minimal barriers in the sense of De Giorgi applied to driven motion by mean curvature*, Rend. Accad. Naz. Sci. XL Mem. Mat. (5), 19 (1995), 43-67.

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