Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
119° (2001), Vol. XXV, fasc. 1, pagg. 161-194

R. MOLLE - D. PASSASEO (*)

Multiplicity of Solutions for Variational Inequalities Involving Pointwise Constraints on the Derivatives (**)

Abstract. — We consider a class of nonlinear variational problems involving pointwise constraints on the second derivatives. We describe the set of data for which these problems have solutions and, for these data, we analyze the structure of the set of solutions (comparison properties of the solutions, existence of a minimal solution and its properties, number of solutions, etc.) under suitable assumptions on the asymptotic behaviour of the nonlinear term. Notice that the results presented in this paper show that the presence of constraints of this kind produces some phenomena which are typical of well known problems for semilinear elliptic equations with «jumping» nonlinearities. We also discuss the reasons which explain why this analogy with «jumping» problems occurs.

Molteplicità di soluzioni per disequazioni variazionali con vincoli puntuali sulle derivate

Sunto. — Si considera una classe di problemi variazionali non lineari con vincoli puntuali sulle derivate seconde. Si descrive l'insieme dei dati per cui tali problemi hanno soluzioni e, per tali dati, si analizza la struttura dell'insieme delle soluzioni (proprietà di confronto tra soluzioni, esistenza di una soluzione minima e sue proprietà, numero di soluzioni, ecc.) sotto opportune ipotesi sul comportamento asintotico del termine non lineare. Si osserva che i risultati presentati in questo lavoro mostrano che la presenza di vincoli di questo tipo produce alcuni fenomeni che sono tipici di ben noti problemi per equazioni ellittiche semilineari con nonlinearità di tipo «jumping». Si discutono inoltre le ragioni che spiegano perché si riscontra questa analogia con i problemi di tipo «jumping».

(*) Indirizzo degli Autori: Riccardo Molle, Dipartimento di Matematica, Università di Roma «Tor Vergata», Via della Ricerca Scientifica, 00133 - Roma (e-mail: molle@mat.uniroma2.it); Donato Passaseo, Dipartimento di Matematica «E. De Giorgi», Università di Lecce, Prov.le Lecce-Arnesano P.O. Box 193, 73100 - Lecce.
(**) Memoria presentata il 25 settembre 2001 da Giorgio Letta, uno dei XL.

1. - Introduction

Let Ω be a bounded connected domain of $\mathbb{R}^n$, $\psi \in H_0^1(\Omega)$, $h \in L^2(\Omega)$ and $g : \Omega \times \times \mathbb{R} \to \mathbb{R}$ be a given Carathéodory function.

Let us set

$$K_\psi = \{u \in H_0^1(\Omega) \,|\, \Delta u \leqslant \Delta\psi \ \text{(in weak sense)}\}$$

and consider the following problem: find $u \in K_\psi$ such that

$$\int_\Omega [Du\, D(v-u) - g(x, u)(v-u) + h(v-u)]\, dx \geqslant 0 \qquad \forall v \in K_\psi .$$

Our aim is to study the solvability of this problem for a generic pair (ψ, h) and estimate the number of solutions under suitable assumptions on the behaviour of the $\lim_{t \to +\infty} g(x, t)/t$ with respect to the eigenvalues $\lambda_1 < \lambda_2 \leqslant \lambda_3 \dots$ of the Laplace operator in $H_0^1(\Omega)$.

If $\lim_{t \to +\infty} g(x, t)/t < \lambda_1$, it is easy to prove that the problem has at least one solution for every $\psi \in H_0^1(\Omega)$ and $h \in L^2(\Omega)$ (exactly one solution under some additional assumptions on g). On the contrary, if $\lim_{t \to +\infty} g(x, t)/t > \lambda_1$, then there exist some pairs (ψ, h) for which the problem has no solution, while for other pairs there exist at least two solutions. For example, if we fix $\psi \in H_0^1(\Omega)$ and consider a term h of the form $h = \overline{h} + \tau e_1$, where $\overline{h}$ is a fixed function in $L^2(\Omega)$, τ a real parameter and e_1 a positive eigenfunction related to the first eigenvalue λ_1, under the assumption that $\lim_{t \to +\infty} g(x, t)/t > \lambda_1$, there exists $\overline{\tau} \in \mathbb{R}$ such that the problem has no solution for $\tau < \overline{\tau}$, has at least one solution for $\tau = \overline{\tau}$ and at least two solutions for $\tau > \overline{\tau}$.

In the particular case that $g(x, u) = \lambda u$, this problem has been studied in [14]. The general case, in which $g(x, \cdot)$ is not necessarily a linear function, is treated in [15] by using topological methods of Calculus of Variations applied in a non-smooth setting.

In the present paper we improve the results obtained in [15] by using more refined methods which gives new information on the structure of the set of pairs (ψ, h) for which our problem has solutions as well as on the properties of the set of solutions. In particular, under a natural monotonicity assumption on $g(x, \cdot)$, we prove some comparison properties of the solutions (mainly Theorem 4.1), which allow us to obtain improved multiplicity results and to evaluate the number of solutions. Also in this work an important tool is the notion of a new type of supersolution, different from the classical one, introduced in [14] because it seems appropriate to handle constraints on the derivatives.

The results we present in this paper show that the presence of the constraint K_ψ gives rise to some phenomena which are typical of nonlinear elliptic equations with «jumping» nonlinearity, that have been investigated in a large number of papers (see, for example, [1, 2, 3, 8, 11] and references therein). In particular, Theorems 5.1, 5.10, 5.12 and 5.14 point out an evident analogy with a well known result, stated by Ambrosetti and Prodi in [2], concerning semilinear elliptic equations of the form

$$\Delta u + g(u) = h \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega ,$$

where the nonlinear term satisfies a «jumping» condition involving the first eigenvalue, namely

$$\lim_{t \to -\infty} \frac{g(t)}{t} < \lambda_1 < \lim_{t \to +\infty} \frac{g(t)}{t} < \lambda_2 .$$

Indeed, although the result in [2] is obtained by using deeply different (non variational) methods, a comparison of the corresponding functionals shows that, in some sense, the presence of the constraint in our problem plays the same role as the condition

$$\lim_{t \to -\infty} \frac{g(t)}{t} < \lambda_1$$

in [2]. This observation explains the analogies with the problems with «jumping» nonlinearities.

The paper is organized as follows: in section 2 we state the problem, characterize its solutions as the lower critical points of a suitable functional and introduce some equivalent obstacle problems which, in particular, give information on the pointwise properties of the solutions; in section 3 we report the definition of supersolution (introduced in [14] and [15]) and some properties of the supersolutions which are used in section 4 to study the structure of the set of solutions and the set of data (ψ, h) for which there exist solutions; finally, section 5 is devoted to prove existence, nonexistence and multiplicity results.

The following notations will be used throughout this paper.

Notations 1.1

– We denote by e_1 the positive eigenfunction related to the first eigenvalue λ_1 of $-\Delta$ in $H_0^1(\Omega)$, satisfying $\int_\Omega e_1^2\, dx = 1$ (λ_1 is a simple eigenvalue since Ω is assumed to be a connected domain).

– For all $t \in \mathbb{R}$, we set

$$P_t = \left\{ u \in L^2(\Omega) \,\middle|\, \int_\Omega u e_1\, dx = t \right\} \text{ and } H_t = \left\{ u \in L^2(\Omega) \,\middle|\, \int_\Omega u e_1\, dx \leqslant t \right\}.$$

– The Hilbert space we shall consider in this paper will be $H_0^1(\Omega)$. We shall denote by $\|\cdot\|$ and $\|\cdot\|_p$ the usual norms in $H_0^1(\Omega)$ and in $L^p(\Omega)$ respectively.

– For any set E we denote by I_E the indicator function of E (i.e. $I_E(u)=0$ if $u \in E$ and $I_E(u)=+\infty$ if $u \notin E$).

2. - Preliminary results

Let Ω be a bounded domain of $\mathbb{R}^n$ and ψ a function in $H_0^1(\Omega)$; set

$$K_\psi = \left\{ u \in H_0^1(\Omega) \,\middle|\, \int_\Omega Du\,Dw\,dx \geq \int_\Omega D\psi\,Dw\,dx \ \ \forall w \in C_0^\infty(\Omega),\ w \geq 0 \right\} \tag{2.1}$$

(note that K_ψ is a convex cone with vertex in ψ).

The following lemma shows that the solutions we obtain when we introduce the constraint K_ψ in an elliptic problem, solve a suitable obstacle problem (i.e. a variational inequality involving only pointwise unilateral constraints on the solution and not on its derivatives).

Lemma 2.1: *Assume $\psi \in H_0^1(\Omega)$ and $k \in L^q(\Omega)$, with $q>1$ and $q \geq 2n/(n+2)$ if $n>2$.*

Let us set

$$\overline{K} = \{ u \in H_0^1(\Omega) \ | \ u \geq \Delta^{-1} k \ \text{a.e. in } \Omega \}.$$

Then a function $u \in H_0^1(\Omega)$ solves the problem

$$\begin{cases} u \in K_\psi \\ \int_\Omega Du\,D(v-u)\,dx + \int_\Omega k(v-u)\,dx \geq 0 \qquad \forall v \in K_\psi \end{cases} \tag{2.2}$$

if and only if it is solution of the variational inequality

$$\begin{cases} u \in \overline{K} \\ \int_\Omega Du\,D(w-u)\,dx - \int_\Omega D\psi\,D(w-u)\,dx \geq 0 \qquad \forall w \in \overline{K}. \end{cases} \tag{2.3}$$

Proof: Suppose that $u \in K_\psi$ solves problem (2.2).

If, for $\delta \in C_0^\infty(\Omega)$, $\delta \geq 0$ in Ω, we put $v = u - \Delta^{-1}\delta$, then $v \in K_\psi$ and so inequality (2.2) implies

$$\int_\Omega (u - \Delta^{-1}k)\,\delta dx \geq 0 \qquad \forall \delta \in C_0^\infty(\Omega), \qquad \delta \geq 0\,.$$

Therefore $u \in \overline{K}$.

Now, if w is in $\overline{K}$, then

$$(2.4)\qquad \int_\Omega Du\, D(w-u)\,dx - \int_\Omega D\psi\, D(w-u)\,dx =$$

$$= \int_\Omega (Du - D\psi)\, D(w-u)\,dx \geq \int_\Omega (Du - D\psi)\, D(\Delta^{-1}k - u)\,dx\,,$$

where the last inequality is true because $\Delta u \leq \Delta\psi$ (in weak sense) and $w \geq \Delta^{-1}k$. It holds

$$\int_\Omega (Du - D\psi)\, D(\Delta^{-1}k - u)\,dx = \int_\Omega Du\, D(\psi - u)\,dx + \int_\Omega k(\psi - u)\,dx\,,$$

which is nonnegative by assumption (notice that $\psi \in K_\psi$).

Conversely, let $u \in \overline{K}$ be a solution of problem (2.3).

If, for $\alpha \in C_0^\infty(\Omega)$, $\alpha \geq 0$ in Ω, we put $w = u + \alpha$, then $w \in \overline{K}$; therefore inequality (2.3) implies

$$\int_\Omega (Du - D\psi)\, D\alpha\,dx \geq 0 \qquad \forall \alpha \in C_0^\infty(\Omega), \qquad \alpha \geq 0\,,$$

that is $u \in K_\psi$.

Now, if v is in K_ψ, then

$$(2.5)\qquad \int_\Omega Du\, D(v-u)\,dx + \int_\Omega k(v-u)\,dx =$$

$$= \int_\Omega D(u - \Delta^{-1}k)\, D(v-u)\,dx \geq \int_\Omega D(u - \Delta^{-1}k)\, D(\psi - u)\,dx\,,$$

where the last inequality is true because $u \geq \Delta^{-1}k$ and $\Delta(\psi - u) \geq \Delta(v - u)$, in weak sense. It holds

$$\int_\Omega D(u - \Delta^{-1}k)\, D(\psi - u)\,dx = \int_\Omega Du\, D(\Delta^{-1}k - u)\,dx - \int_\Omega D\psi\, D(\Delta^{-1}k - u)\,dx\,,$$

which is nonnegative because u solves problem (2.3) and $\Delta^{-1}k \in \overline{K}$. *q.e.d.*

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for almost all $x \in \Omega$,

$$(2.6)\qquad |g(x, t)| \leq a(x) + b|t|^{p-1} \qquad \forall t \in \mathbb{R}\,,$$

for suitable $p, b \in \mathbb{R}$, $p > 1$, $a \in L^{p/(p-1)}(\Omega)$, with $p \leq 2^* = 2n/(n-2)$ if $n > 2$.

DEFINITION 2.2: *Assume that the function g satisfies condition* (2.6). *Then, for all* $h \in L^2(\Omega)$ *and* $\psi \in H_0^1(\Omega)$, *we say that u is solution of problem* $P_\psi(h)$ *if*

$$P_\psi(h) \begin{cases} u \in K_\psi \\ \int_\Omega [Du\, D(v-u) - g(x, u)(v-u) + h(v-u)]\, dx \geq 0 \qquad \forall v \in K_\psi . \end{cases}$$

REMARK 2.3: Notice that Lemma 2.1 gives also information on the pointwise properties of the solutions of variational inequalities involving pointwise constraints on the laplacian. In fact, a function u which solves problem (2.2) must solve also the variational inequality (2.3), whose pointwise meaning is the following: $u \geq \Delta^{-1} k$ a.e. in Ω and $\Delta u = \Delta \psi$ where $u > \Delta^{-1} k$.

In analogous way, if we set

$$\overline{K}_u = \{ w \in H_0^1(\Omega) \,|\, w \geq -\Delta^{-1}(g(x, u) - h) \text{ a.e. in } \Omega \},$$

then we can say that a function u solves problem $P_\psi(h)$ if and only if

$$\text{(2.7)} \qquad \begin{cases} u \in \overline{K}_u \\ \int_\Omega Du\, D(w-u)\, dx - \int_\Omega D\psi\, D(w-u)\, dx \geq 0 \qquad \forall w \in \overline{K}_u , \end{cases}$$

which means that $u \geq -\Delta^{-1}(g(x, u) - h)$ a.e. in Ω and $\Delta u = \Delta \psi$ where $u > -$ $- \Delta^{-1}(g(x, u) - h)$.

Now let us introduce some notions of non-smooth analysis (see, for example, [4, 7, 9]), we shall use to describe the variational nature of problem $P_\psi(h)$.

Let H be an Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$.

For all $f: H \to \mathbb{R} \cup \{+\infty\}$ let us define the domain of f to be the set

$$\mathcal{D}(f) = \{ u \in H \mid f(u) < +\infty \}.$$

As usual, for all $c \in \mathbb{R}$ we denote by f^c the sublevel of f:

$$f^c = \{ u \in H \mid f(u) \leq c \}.$$

For all $u \in \mathcal{D}$, we call subdifferential of f at u the set $\partial^- f(u)$ of all α in H such that

$$\liminf_{v \to u} \frac{f(v) - f(u) - (\alpha, v-u)}{\|v-u\|} \geq 0 .$$

If $\partial^- f(u) \neq \emptyset$, then we call subgradient of f at u, and denote it by $\text{grad}^- f(u)$, the element of $\partial^- f(u)$ having minimal norm (it is easy to check that $\partial^- f(u)$ is a closed and convex subset of H).

We say that u is a lower critical point for f if $0 \in \partial^- f(u)$, that is if $\mathrm{grad}^- f(u) = 0$.

DEFINITION 2.4: *Assume that the function g satisfies condition* (2.6). *Then, for all $\psi \in H_0^1(\Omega)$ and $h \in L^2(\Omega)$, we consider the functionals $f_h, f_{h,\psi}$: $H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f_h(u) = \frac{1}{2} \int_\Omega (|Du|^2 - G(x, u))\, dx + \int_\Omega hu\, dx\,,$$

where $G(x, t) = \int_0^t g(x, s)\, ds$, *and* $f_{h,\psi} = f_h + I_{K_\psi}$ (*see Notations* 1.1).

Notice that condition (2.6) implies that there exist $A \in L^1(\Omega)$ and $B \in \mathbb{R}$ such that, for almost all $x \in \Omega$,

$$|G(x, t)| \leq A(x) + B|t|^p. \tag{2.8}$$

Hence, under this condition, f_h is a C^1 functional and $f_{h,\psi}$ is lower semicontinuous (because K_ψ is a closed subset of $H_0^1(\Omega)$). Moreover it is easy to verify that a function $u \in H_0^1(\Omega)$ solves problem $P_\psi(h)$ if and only if $0 \in \partial^- f_{h,\psi}(u)$.

3. - PROPERTIES OF THE SUPERSOLUTIONS

In this section we report (for the convenience of the reader) the definition of supersolution introduced in [14, 15] and some properties will be used in next sections. Notice that these properties allow us to use these supersolutions of new type as the classical ones. In particular, Propositions 3.6 and 3.7 show that they can be used as «upper ficticious obstacle» in our problem and that the existence of a supersolution $\overline{\psi}$ implies the existence of a solution $u \leq \overline{\psi}$.

DEFINITION 3.1: *Let $h \in L^2(\Omega)$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying condition* (2.6). *We say that $\overline{\psi} \in H_0^1(\Omega)$ is a supersolution for the operator $I + \Delta^{-1}(g(x, \cdot) - h)$ if*

$$\int_\Omega (D\overline{\psi}\, Dw - g(x, \overline{\psi})\, w + hw)\, dx \geq 0 \qquad \forall w \in K_0.$$

REMARK 3.2: It is evident that every solution of problem $P_\psi(h)$ is a supersolution for the operator $I + \Delta^{-1}(g(x, \cdot) - h)$. One could not say that it is a supersolution in the classical sense, i.e. for the operator $\Delta + g(x, \cdot) - h$.

REMARK 3.3: If $\overline{\psi}$ is a supersolution for the operator $I + \Delta^{-1}(g(x, \cdot) - h)$ and $h' \geq h$, then it is easy to verify that $\overline{\psi}$ is a supersolution for the operator $I + \Delta^{-1}(g(x, \cdot) - h')$ too.

An easy computation shows the following statement.

PROPOSITION 3.4: *Let g satisfy condition* (2.6); *then the function* $\overline{\psi}$ *is a supersolution for the operator* $I+\Delta^{-1}(g(x,\cdot)-h)$ *if and only if* $\overline{\psi}+\Delta^{-1}(g(x,\overline{\psi})-h)\geqslant 0$ *a.e. in* Ω.

PROPOSITION 3.5: *Assume that the function g satisfies condition* (2.6) *and that*

(3.1) $$g(x,\cdot) \quad \text{is a non-decreasing function for a.a.} \quad x\in\Omega\,.$$

Let $\overline{\psi}_1, \overline{\psi}_2$ *be supersolutions for the operator* $I+\Delta^{-1}(g(x,\cdot)-h)$. *Then* $\overline{\psi}_1\wedge\overline{\psi}_2$ *is a supersolution too.*

PROOF: Proposition 3.4 and assumption (3.1) guarantee that

$$\overline{\psi}_1\geqslant -\Delta^{-1}(g(x,\overline{\psi}_1)-h)\geqslant -\Delta^{-1}(g(x,\overline{\psi}_1\wedge\overline{\psi}_2)-h) \qquad \text{a.e. in } \Omega$$

$$\overline{\psi}_2\geqslant -\Delta^{-1}(g(x,\overline{\psi}_2)-h)\geqslant -\Delta^{-1}(g(x,\overline{\psi}_1\wedge\overline{\psi}_2)-h) \qquad \text{a.e. in } \Omega\,.$$

Therefore

$$\overline{\psi}_1\wedge\overline{\psi}_2\geqslant -\Delta^{-1}(g(x,\overline{\psi}_1\wedge\overline{\psi}_2)-h) \qquad \text{a.e. in } \Omega$$

that is, again by Proposition 3.4, $\overline{\psi}_1\wedge\overline{\psi}_2$ is a supersolution. *q.e.d.*

PROPOSITION 3.6: *Let g satisfy conditions* (2.6) *and* (3.1) *and* $\overline{\psi}\in K_\psi$ *be a supersolution for the operator* $I+\Delta^{-1}(g(x,\cdot)-h)$. *Set* $K=\{u\in K_\psi \mid u\leqslant\overline{\psi}$ *a.e. in* $\Omega\}$ *and assume that w is a lower critical point for* f_h+I_K. *Then w is a solution of problem* $P_\psi(h)$.

PROOF: First of all, let us remark that $\overline{\psi}\geqslant -\Delta^{-1}(g(x,\overline{\psi})-h)$ a.e., by Proposition 3.4. Moreover $g(x,w)-h\leqslant g(x,\overline{\psi})-h$, because $w\in K$ and (3.1) holds. So we obtain:

(3.2) $$-\Delta^{-1}(g(x,w)-h)\leqslant -\Delta^{-1}(g(x,\overline{\psi})-h)\leqslant\overline{\psi}.$$

The function w verifies

$$\int_\Omega [DwD(v-w)-g(x,w)(v-w)+h(v-w)]\,dx\geqslant 0 \qquad \forall v\in K\,;$$

therefore, if we put

(3.3) $$\tilde{f}(u)=\frac{1}{2}\int_\Omega |Du|^2\,dx+\int_\Omega [h-g(x,w)]\,u\,dx\,,$$

w is a lower critical point for $\tilde{f} + I_K$. The functional $\tilde{f} + I_{K_\psi}$ is strictly convex, lower semicontinuous and coercive; so there exists only one minimum point for $\tilde{f}$ on K_ψ; let us call it $\tilde{w}$.

The function $\tilde{w}$ verifies

$$\int_\Omega D\tilde{w}\, D(v - \tilde{w})\, dx - \int_\Omega (g(x, w) - h)(v - \tilde{w})\, dx \geq 0 \qquad \forall v \in K_\psi . \tag{3.4}$$

The functional $\tilde{f} + I_K$ admits only one lower critical point (its unique minimum point), because it is strictly convex; so, if we show that $\tilde{w} \leq \overline{\psi}$, then we have $\tilde{w} = w$ and (3.4) gives us the desired conclusion.

Applying Lemma 2.1 with $k = -g(x, w) + h$, we have that $\tilde{w}$ is a lower critical point for the functional $F : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$F(u) = \frac{1}{2} \int_\Omega |Du|^2\, dx - \int_\Omega D\psi\, Du\, dx \tag{3.5}$$

and constrained on the set

$$\overline{K} = \{u \in H_0^1(\Omega)\, |\, u \geq -\Delta^{-1}(g(x, w) - h) \text{ a.e. in } \Omega\} .$$

The function $\overline{\psi}$ is in the set $\overline{K}$ by (3.2) and it verifies $\Delta \overline{\psi} \leq \Delta \psi$ (in weak sense) by assumption; then it is a supersolution for the operator F' (in the usual sense: see, for example, [16, 17]).

Therefore, as it is stated in [16], the functional $F + I_{\overline{K}}$ has a lower critical point, that we call w', satisfying $w' \leq \overline{\psi}$; but $F + I_{\overline{K}}$ has only one critical point, because it is strictly convex, so $\tilde{w} = w'$. This implies that $\tilde{w} \leq \overline{\psi}$ and so $\tilde{w} = w$, which completes the proof. *q.e.d.*

We can now prove the following assertion.

PROPOSITION 3.7: *Let g satisfy conditions* (2.6) *and* (3.1). *If $\overline{\psi} \in K_\psi$ is a supersolution for the operator $I + \Delta^{-1}(g(x, \cdot) - h)$, then problem $P_\psi(h)$ has at least one solution w such that $w \leq \overline{\psi}$ a.e. in Ω.*

PROOF: In this proof we use the notations introduced in Proposition 3.6. It is clear that to prove the assertion it is sufficient to find a minimum point w for the functional $f_h + I_K$ and then apply Proposition 3.6. Such a minimum point there exists because $K \subset \{u \in H_0^1(\Omega)\, |\, \psi \leq u \leq \overline{\psi}\}$ with ψ and $\overline{\psi}$ in $H_0^1(\Omega)$; taking also into account condition (2.6), it follows that the sublevels of $f_h + I_K$ are bounded in $H_0^1(\Omega)$; hence every minimizing sequence for $f_h + I_K$ converges, up to a subsequence, weakly in $H_0^1(\Omega)$ and strongly in $L^p(\Omega)$; since $f_h + I_K$ is weakly lower semicontinuous, the limit point gives the minimum of $f_h + I_K$, as desired. *q.e.d.*

4. - Properties of the set of solutions

The following theorem will play a crucial role to analyse the structure of the set of solutions and to evaluate their number.

Theorem 4.1: *Let g satisfy conditions* (2.6) *and* (3.1). *If u_1 and u_2 are solutions of $P_\psi(h)$ and $u_1 \leqslant u_2$ a.e. in Ω, then $\Delta u_1 \geqslant \Delta u_2$ in weak sense in Ω.*

To prove this theorem, we need the following lemma.

Lemma 4.2: *Under the assumptions of Theorem* 4.1 *we have*:

$$u_1 \geqslant u_2 - \Delta^{-1}(g(x, u_1) - g(x, u_2)) \quad \text{a.e. in } \Omega .$$

Proof: Let us define

$$\overline{K}_1 = \{u \in H_0^1(\Omega) \mid u \geqslant -\Delta^{-1}(g(x, u_1) - h) \text{ a.e. in } \Omega\}$$

and set

$$\tilde{u} = u_2 - \Delta^{-1}(g(x, u_1) - g(x, u_2)) .$$

Since u_2 is a solution of $P_\psi(h)$, then we have $u_2 \geqslant -\Delta^{-1}(g(x, u_2) - h)$ and so it follows immediately that $\tilde{u} \in \overline{K}_1$.

The function u_1 is the unique minimum point of the functional F (see (3.5)) constrained on the closed convex set $\overline{K}_1$, because F is strictly convex and (by Lemma 2.1) the solution u_1 is a critical point for F constrained on $\overline{K}_1$.

The functional F has a unique minimum point, we call u', on the convex closed set $\{u \in \overline{K}_1 \mid u \geqslant \tilde{u} \text{ a.e. in } \Omega\}$; if we show that $u' = u_1$, then $u_1 > \tilde{u}$ and the lemma will be proved.

For every $w \in H_0^1(\Omega)$, let us define $\tilde{P}(w) = w \vee \tilde{u}$. Now suppose $w \in \overline{K}_1$; since $\tilde{u} \in \overline{K}_1$, then $\tilde{P}(w) \in \{u \in \overline{K}_1 \mid u \geqslant \tilde{u}\}$.

From the convexity of F we infer that

$$F(w) \geqslant F(\tilde{P}(w)) + F'(\tilde{P}(w))[w - \tilde{P}(w)] \tag{4.1}$$

and that

$$F(\tilde{P}(w)) \geqslant F(u') + F'(u')[\tilde{P}(w) - u'] . \tag{4.2}$$

Since u' is the minimum point for F on $\{u \in \overline{K}_1 \mid u \geqslant \tilde{u}\}$ and $\tilde{P}(w) \in \{u \in \overline{K}_1 \mid u \geqslant \tilde{u}\}$, it follows that

$$F'(u')[\tilde{P}(w) - u'] \geqslant 0 . \tag{4.3}$$

Furthermore we claim that

$$F'(\tilde{P}(w))[w-\tilde{P}(w)] \geq 0 . \tag{4.4}$$

In fact, since $\tilde{P}(w)=\tilde{u}$ where $w-\tilde{P}(w)\neq 0$, we have

$$F'(\tilde{P}(w))[w-\tilde{P}(w)] = \int_\Omega D\tilde{P}(w)\, D(w-\tilde{P}(w))\, dx - \int_\Omega D\psi D(w-\tilde{P}(w))\, dx =$$

$$= \int_\Omega D\tilde{u} D(w-\tilde{P}(w))\, dx - \int_\Omega D\psi D(w-\tilde{P}(w))\, dx = \int_\Omega Du_2\, D(w-\tilde{P}(w))\, dx +$$

$$+ \int_\Omega [g(x,u_1)-g(x,u_2)](w-\tilde{P}(w))\, dx - \int_\Omega D\psi D(w-\tilde{P}(w))\, dx .$$

Now observe that $u_2+(w-\tilde{P}(w)) \geq -\Delta^{-1}(g(x,u_2)-h)$, as one can easily verify taking into account that u_2 is a solution of problem $P_\psi(h)$, and $w \geq -\Delta^{-1}(g(x,u_1)-$ $-h)$ since $w\in \overline{K}_1$.

Moreover we have also

$$u_2-(w-\tilde{P}(w)) \geq -\Delta^{-1}(g(x,u_2)-h)$$

because $w-\tilde{P}(w)\leq 0$ and u_2 solves $P_\psi(h)$. Therefore, from Lemma 2.1 we infer that

$$\int_\Omega Du_2\, D(w-\tilde{P}(w))\, dx - \int_\Omega D\psi\, D(w-\tilde{P}(w))\, dx = 0 .$$

On the other hand condition (3.1) and the assumption $u_1\leq u_2$ imply that

$$\int_\Omega [g(x,u_1)-g(x,u_2)](w-\tilde{P}(w))\, dx \geq 0$$

because $(w-\tilde{P}(w))\leq 0$. Thus (4.4) is proved.

By (4.1), ...,(4.4), we can conclude that

$$F(w)\geq F(u') \qquad \forall w\in \overline{K}_1 ;$$

so $u'=u_1$, since u_1 is the unique critical point for F on $\overline{K}_1$. *q.e.d.*

Proof of Theorem 4.1: In this proof we use the notations introduced in the proof of Lemma 4.2.

Define $Q: H_0^1(\Omega)\to\mathbb{R}$ by

$$Q(u)=\frac{1}{2}\int_\Omega |D(u-u_2)|^2\, dx ;$$

our assertion is that

$$Q'(u_1)[\alpha] = \int_\Omega D(u_1 - u_2)\, D\alpha\, dx \leqslant 0 \qquad \forall \alpha \in H_0^1(\Omega),\ \alpha \geqslant 0 \text{ a.e. in } \Omega. \tag{4.5}$$

Now consider a fixed $\alpha \in H_0^1(\Omega)$, $\alpha \geqslant 0$ a.e. in Ω; from the convexity of Q we infer

$$Q(u_1 - \alpha) \geqslant Q(\tilde{P}(u_1 - \alpha)) + Q'(\tilde{P}(u_1 - \alpha))[u_1 - \alpha - \tilde{P}(u_1 - \alpha)] \tag{4.6}$$

and

$$Q(\tilde{P}(u_1 - \alpha)) \geqslant Q(u_1) + Q'(u_1)[\tilde{P}(u_1 - \alpha) - u_1)]. \tag{4.7}$$

The assumption $u_1 \leqslant u_2$ and condition (3.1) imply

$$\begin{aligned} Q'(\tilde{P}(u_1 - \alpha))[u_1 - \alpha - \tilde{P}(u_1 - \alpha)] &= Q'(\tilde{u})[u_1 - \alpha - \tilde{P}(u_1 - \alpha)] = \\ &= \int_\Omega D(\tilde{u} - u_2)\, D(u_1 - \alpha - \tilde{P}(u_1 - \alpha))\, dx = \\ &= \int_\Omega [g(x, u_1) - g(x, u_2)][u_1 - \alpha - \tilde{P}(u_1 - \alpha)]\, dx \geqslant 0, \end{aligned} \tag{4.8}$$

because $u_1 - \alpha - \tilde{P}(u_1 - \alpha) \leqslant 0$.

Furthermore

$$\begin{aligned} Q'(u_1)[\tilde{P}(u_1 - \alpha) - u_1)] &= \int_\Omega D(u_1 - u_2)\, D(\tilde{P}(u_1 - \alpha) - u_1)\, dx - \\ &- \int_\Omega D\psi\, D(\tilde{P}(u_1 - \alpha) - u_1)\, dx + \int_\Omega D\psi\, D(\tilde{P}(u_1 - \alpha) - u_1)\, dx = \\ &= F'(u_1)[\tilde{P}(u_1 - \alpha) - u_1] - F'(u_2)[\tilde{P}(u_1 - \alpha) - u_1], \end{aligned} \tag{4.9}$$

with

$$-F'(u_2)[\tilde{P}(u_1 - \alpha) - u_1] \geqslant 0 \tag{4.10}$$

because $u_2 \in K_\psi$ and $-[\tilde{P}(u_1 - \alpha) - u_1)] \geqslant 0$ (since $\tilde{u} \leqslant u_1$ by Lemma 4.2).

Now we claim that

$$F'(u_1)[\tilde{P}(u_1 - \alpha) - u_1] = 0; \tag{4.11}$$

indeed, arguing as before, we infer that

$$-F'(u_1)[\tilde{P}(u_1 - \alpha) - u_1] \geqslant 0; \tag{4.12}$$

moreover we have

$$F'(u_1)[\tilde{P}(u_1-\alpha)-u_1] \geqslant 0 \tag{4.13}$$

because u_1 is the minimum point for F constrained on $\overline{K}_1$ and $\tilde{P}(u_1-\alpha)\in\overline{K}_1$ (since $\tilde{u}\in\overline{K}_1$).

From (4.6), ...,(4.11) it follows that

$$Q(u_1-\alpha)\geqslant Q(u_1) \qquad \forall\alpha\in H_0^1(\Omega),\ \alpha\geqslant 0 \text{ a.e. in } \Omega\,;$$

this implies (4.5) and completes the proof. *q.e.d.*

In next propositions we use also the supersolutions to describe some properties of the set of solutions and of the set of data for which the problem has solutions.

PROPOSITION 4.3: *Under the same assumptions as in Proposition* 3.7, *if problem* $P_\psi(h)$ *has solution, then there exists a solution for every problem* $P_{\psi'}(h')$ *such that* $h'\geqslant h$ *a.e. in* Ω *and* $\Delta\psi'\geqslant\Delta\psi$ *in weak sense.*

The proof follows easily from Proposition 3.7, taking into account Remarks 3.2 and 3.3.

Another consequence of Proposition 3.7 is the following result, which can be easily proved using Proposition 3.5.

PROPOSITION 4.4: *Under the same assumptions as in Proposition* 3.7, *if* u_1 *and* u_2 *are solutions of problem* $P_\psi(h)$, *then there exists a solution* u *such that* $u\leqslant u_1\wedge u_2$.

In the proof of next proposition, we need also the following result (see [7], for example).

LEMMA 4.5: *Let* H *be an Hilbert space and* $f: H\to\mathbb{R}\cup\{+\infty\}$ *a lower semicontinuous function. Suppose that there exists* $c\in\mathbb{R}$ *such that*

$$f(v)\geqslant f(u)+(\alpha,\, v-u)-c\|v-u\|^2 \qquad \forall u,\, v\in\mathcal{D}(f),\ \ \forall\alpha\in\partial^- f(u)\,.$$

Let $(u_m)_m$ *and* $(\alpha_m)_m$ *be two sequences such that* $u_m\in\mathcal{D}(f)$, $\alpha_m\in\partial^- f(u_m)$ *for every* $m\in\mathbb{N}$, $\lim_{m\to\infty}u_m=u$ *and* $\alpha_m\rightharpoonup\alpha$ *weakly in* H.

Then $u\in\mathcal{D}(f)$, $\lim_{m\to\infty}f(u_m)=f(u)$ *and* $\alpha\in\partial^- f(u)$.

PROPOSITION 4.6: *Assume that g satisfies conditions* (2.6) *and* (3.1) *and, in addition, there exists* $\lambda\in\mathbb{R}$ *such that, for almost all* $x\in\Omega$,

$$\frac{g(x,\,t_1)-g(x,\,t_2)}{t_1-t_2}\leqslant\lambda \quad \text{for } t_1\neq t_2. \tag{4.14}$$

If problem $P_\psi(h)$ *has solution, then there exists a solution* $\overline{u}$ *such that* *i*) $\Delta \bar{u} \geqslant \Delta u$ *in weak sense in* Ω, *for every u solution of* $P_\psi(h)$

(*hence* $\bar{u}$ *is the minimal solution, i.e.*

ii) $\bar{u} \leqslant u$ *a.e. in* Ω, *for every u solution of* $P_\psi(h)$).

Proof: It is clear that property (*i*) could be sufficient to obtain also (*ii*). However, in this proof we need to prove first (*ii*) and then (*i*) will be a direct consequence of (*ii*) and Theorem 4.1.

Notice that condition (4.14) implies

$$f_{h,\psi}(v) \geqslant f_{h,\psi}(u) + f'_{h,\psi}(u)[v-u] + \frac{1}{2}\|v-u\|^2 - \frac{\lambda}{2}\|v-u\|_2^2 \geqslant \tag{4.15}$$

$$\geqslant f_{h,\psi}(u) + (\alpha, v-u) + \frac{1}{2}\|v-u\|^2 - \frac{\lambda}{2}\|v-u\|_2^2$$

$$\forall u, v \in K_\psi \ \ \forall \alpha \in \partial^- f_{h,\psi}(u).$$

Therefore, since

$$\|v-u\|_2^2 \leqslant c(\Omega)\|v-u\|^2 \qquad \forall u, v \in H_0^1(\Omega)$$

for a suitable constant $c(\Omega)$, we can apply Lemma 4.5 with $H = H_0^1(\Omega)$ and $f = f_{h,\psi}$.

Let $(u_m)_m$ be a sequence of solutions of $P_\psi(h)$ such that

$$\lim_{m\to\infty} \int_\Omega u_m\, dx = \inf\left\{ \int_\Omega u dx \,\middle|\, u \text{ solution of } P_\psi(h)\right\}$$

(notice that this infimum is finite because there exists at least one solution and $K_\psi \subset \{u \in H_0^1(\Omega) \mid u \geqslant \psi\}$).

Let us fix $v \in K_\psi$. By (4.14) and (4.15) we obtain

$$f_h(v) \geqslant f_h(u_m) - \frac{\lambda}{2}\|v-u_m\|_2^2 \geqslant \tag{4.16}$$

$$\geqslant \frac{1}{2}\int_\Omega |Du_m|^2 dx - \int_\Omega |g(x,0)|\,|u_m|\, dx - \frac{\lambda}{2}\int_\Omega u_m^2\, dx + \int_\Omega h u_m\, dx - \frac{\lambda}{2}\|v-u_m\|_2^2,$$

with $g(\cdot, 0) \in L^{p/(p-1)}(\Omega)$ because of condition (2.6).

We claim that $\sup_{m\in\mathbb{N}} \|u_m\|_2 < +\infty$. Contrary to our claim, suppose that, up to a subsequence, $\lim_{m\to\infty} \|u_m\|_2 = +\infty$. Let us set $z_m = u_m / \|u_m\|_2$; from (4.16) we deduce that $\sup_{m\in\mathbb{N}} \|z_m\| < +\infty$ and, consequently, $(z_m)_m$ (or a subsequence) converges in $L^2(\Omega)$ and a.e. in Ω to a function $z \in H_0^1(\Omega)$ with the properties that $\|z\|_2 = 1$ and $z \geqslant 0$ because

$u_m \geqslant \psi$. Hence

$$\lim_{m\to\infty} \int_\Omega z_m\, dx = \int_\Omega z dx > 0 \, .$$

But this is impossible, since

$$\lim_{m\to\infty} \int_\Omega z_m\, dx = \lim_{m\to\infty} \frac{1}{\|u_m\|_2} \int_\Omega u_m\, dx \leqslant 0$$

because $\lim_{m\to\infty} \int_\Omega u_m\, dx < +\infty$ and $\lim_{m\to\infty} \|u_m\|_2 = +\infty$.

So $(u_m)_m$ must be bounded in $L^2(\Omega)$ and, using (4.16), we obtain that it is bounded in $H_0^1(\Omega)$ too. Then, up to a subsequence, $(u_m)_m$ converges in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$ to a function $\bar{u}$.

Now consider (4.16) with v replaced by $\bar{u}$. Then, taking into account (2.6), (4.14) and Fatou's Lemma, it is a simple matter to see that

$$\limsup_{m\to\infty} \int_\Omega |Du_m|^2\, dx \leqslant \int_\Omega |D\bar{u}|^2\, dx \, .$$

This implies that, really, $u_m \to \bar{u}$ in $H_0^1(\Omega)$.

Hence, applying Lemma 4.5, we have that $\bar{u}$ is a solution of $P_\psi(h)$.

It is easily seen that

$$\int_\Omega \bar{u}\, dx = \min\left\{ \int_\Omega u\, dx \,\middle|\, u \text{ solution of } P_\psi(h) \right\}; \tag{4.17}$$

so we deduce that $\bar{u}$ is a solution verifying (*ii*): suppose, arguing by contradiction, that there is a solution u such that $\bar{u} \wedge u \neq \bar{u}$; then there exists another solution $w \leqslant \bar{u} \wedge u$, by Proposition 4.4; therefore $\int_\Omega w\, dx < \int_\Omega \bar{u}\, dx$, in contradiction with (4.17).

Thus property (*ii*) is proved (and (*i*) follows immediately taking into account Theorem 4.1). *q.e.d.*

Also next assertion follows from Lemma 4.5 in a straightforward way.

PROPOSITION 4.7: *Assume that g satisfies conditions* (2.6) *and* (4.14), *for a* $\lambda \in \mathbb{R}$. *If* $h \in L^2(\Omega)$ *and* $\psi \in H_0^1(\Omega)$, *then the set of the solutions of* $P_\psi(h)$ *is closed in* $H_0^1(\Omega)$.

REMARK 4.8: Actually, the set of the solutions of problem $P_\psi(h)$ is closed in $L^2(\Omega)$ too. To see this, we have to consider the functional f_h defined on $L^2(\Omega)$ instead of $H_0^1(\Omega)$, setting $f_h = +\infty$ on $L^2(\Omega) \backslash H_0^1(\Omega)$.

Another consequence of the properties of the supersolutions is the following result, that holds when $g(x, \cdot)$ is convex.

PROPOSITION 4.9: *Under the same assumptions of Proposition* 3.7, *if we assume in addition that*

(4.18) $$g(x,\cdot) \text{ is a convex function for a.a. } x \in \Omega\,,$$

then the set of the pairs (ψ, h) *such that* $P_\psi(h)$ *has solution is convex.*

PROOF: It suffices to observe that, if u_i is a solution of problem $P_{\psi_i}(h_i)$, $i = 1, 2$, then $tu_1 + (1-t)u_2$ is in $K_{t\psi_1 + (1-t)\psi_2}$ and is a supersolution for the operator $I + \Delta^{-1}(g(x,\cdot) - th_1 - (1-t)h_2)$ for all $t \in [0, 1]$, because of the convexity of $g(x,\cdot)$. Hence we can complete the proof using Proposition 3.7. *q.e.d.*

Now, using supersolutions, let us describe some closure properties of the set of data ψ and h for which $P_\psi(h)$ has solution.

LEMMA 4.10: *Let g satisfy conditions* (2.6) *and* (4.14) *and assume, in addition, that there exist* $\bar\lambda \in \mathbb{R}$ *and* $c \in L^2(\Omega)$ *such that, for almost all* $x \in \Omega$,

(4.19) $$g(x, t) \geq \bar\lambda t - c(x) \qquad \forall t \geq 0\,.$$

Then for every $u \in K_\psi$ *and* $\alpha \in \partial^- f_{h,\psi}(u)$ *we have:*

$$\int_\Omega [(\lambda_1 - \bar\lambda)\, u^+ - (\lambda_1 - \lambda)\, u^- + c + h]\, e_1\, dx \geq (\alpha, e_1)\,.$$

PROOF: Set $v = u + e_1$ (note that $v \in K_\psi$). Then, for every $\alpha \in \partial^- f_{h,\psi}(u)$, it holds

(4.20) $$(\alpha, e_1) = (\alpha, v-u) \leq f_h'(u)[v-u] =$$
$$= \int_\Omega [Du\, De_1 - g(x,u)\, e_1 + he_1]\, dx = \int_\Omega [\lambda_1 u - g(x,u) + h]\, e_1\, dx\,.$$

Furthermore, if $\Omega^+ = \{x \in \Omega \,|\, u(x) \geq 0\}$, then

$$\int_\Omega g(x,u)\, e_1\, dx = \int_{\Omega^+} g(x,u)\, e_1\, dx + \int_{\Omega\setminus\Omega^+} g(x,u)\, e_1\, dx \geq$$
$$\geq \int_{\Omega^+} (\bar\lambda u - c)\, e_1\, dx + \int_{\Omega\setminus\Omega^+} (\lambda u - c)\, e_1\, dx = \int_\Omega (\bar\lambda u^+ - \lambda u^- - c)\, e_1\, dx\,,$$

which, together with (4.20), completes the proof. *q.e.d.*

PROPOSITION 4.11: *Suppose that g satisfy conditions* (2.6), (4.14) *and* (4.19) *with* $\bar\lambda > \lambda_1$. *Let* $(\psi_m)_m$ *and* $(h_m)_m$ *be two sequences such that, for all* $m \in \mathbb{N}$, $\psi_m \in H_0^1(\Omega)$, $h_m \in L^2(\Omega)$ *and* $P_{\psi_m}(h_m)$ *has at least one solution* u_m. *Furthermore assume that* $\psi_m \to$

$\to \psi$ in $H^1_0(\Omega)$ and $h_m \to h$ in $L^2(\Omega)$, as $m \to \infty$. Then:

a) the sequence $(u_m)_m$ is bounded in $H^1_0(\Omega)$;

b) if $(u_m)_m$ (or a subsequence) converges to u in $L^2(\Omega)$ and weakly in $H^1_0(\Omega)$, then u solves problem $P_\psi(h)$;

c) problem $P_\psi(h)$ has at least one solution.

PROOF: Taking into account (4.15), we have

$$(4.21)\qquad f_{h_m}(\psi_m) \geq f_{h_m}(u_m) - \frac{\lambda}{2}\|\psi_m - u_m\|_2^2 \geq$$

$$\geq \frac{1}{2}\int_\Omega |Du_m|^2\,dx - \int_\Omega |g(x,0)|\,|u_m|\,dx - \frac{\lambda}{2}\int_\Omega u_m^2\,dx + \int_\Omega h_m u_m\,dx - \frac{\lambda}{2}\|\psi_m - u_m\|_2^2$$

with $\int_\Omega |g(x,0)|\,|u_m|\,dx \leq c_1\|u_m\|$, for a suitable $c_1 \in \mathbb{R}$, because of condition (2.6). Let us prove that the sequence $(u_m)_m$ is bounded in $L^2(\Omega)$. In fact assume by contradiction that, up to a subsequence, $\lim_{m\to\infty}\|u_m\|_2 = +\infty$. If we put $z_m = u_m/\|u_m\|_2$, from (4.21) we deduce that $(z_m)_m$ is bounded in $H^1_0(\Omega)$; so, up to a subsequence, it converges in $L^2(\Omega)$ and a.e. in Ω, to a function $z \in H^1_0(\Omega)$. The function z verifies:

$$(4.22)\qquad \|z\|_2 = 1 \quad \text{and} \quad z \geq 0 \ \text{in } \Omega$$

(since $u_m \geq \psi_m$ a.e. in Ω and $\psi_m \to \psi$ in $H^1_0(\Omega)$).

By Lemma 4.10 we have

$$\frac{1}{\|u_m\|_2}\int_\Omega [(\lambda_1 - \bar\lambda)\,u_m^+ - (\lambda_1 - \lambda)\,u_m^- + c + h_m]\,e_1\,dx \geq 0\,,$$

from which, as $m \to \infty$, we obtain $(\lambda_1 - \bar\lambda)\int_\Omega z e_1\,dx \geq 0$, that is impossible because $\bar\lambda > \lambda_1$ and (4.22) holds. So the sequence $(u_m)_m$ must be bounded in $L^2(\Omega)$ and then, from (4.21), it follows that it is bounded in $H^1_0(\Omega)$ too. Thus (a) is proved.

Let us prove (b). For all $v \in K_\psi$, set $v_m = v + \psi_m - \psi$; then $v_m \in K_{\psi_m}$ $\forall m \in \mathbb{N}$ and so, by (4.15),

$$(4.23)\qquad f_{h_m}(v_m) \geq f_{h_m}(u_m) - \frac{\lambda}{2}\|v_m - u_m\|_2^2 \qquad \forall m \in \mathbb{N}\,.$$

Now let $m \to \infty$ in (4.23); since $v_m \to v$ in $H^1_0(\Omega)$, we get

$$f_h(v) \geq f_h(u) - \frac{\lambda}{2}\|v - u\|_2^2 \qquad \forall v \in K_\psi$$

and this gives (b).

Assertion (c) is a direct consequence of (a) and (b). *q.e.d.*

Let us remark that in Proposition 4.11 the condition $\bar{\lambda} > \lambda_1$ cannot be removed, as shown by the following example.

EXAMPLE 4.12: Let $g(x, t) = \lambda_1 t$; choose $\psi \in H_0^1(\Omega)$ such that $\sup_{\Omega} (\psi/e_1) = +\infty$ and set $\psi_m = \psi$ and $h_m = e_1/m$ for all $m \in \mathbb{N}$. Then Theorem 6.1 of [14] guarantees that $P_{\psi_m}(h_m)$ has a unique solution u_m for all $m \in \mathbb{N}$, while the limit problem $P_\psi(0)$ has no solution. In fact, by Theorem 6.1 of [14], every solution of $P_\psi(0)$ should be an eigenfunction related to the first eigenvalue λ_1 (which cannot belong to K_ψ under our assumptions). In this case the sequence of solutions $(u_m)_m$ is not bounded in $H_0^1(\Omega)$.

5. - EXISTENCE AND MULTIPLICITY RESULTS

In this section we use mini-max arguments and topological methods of Calculus of Variations for non-smooth functionals in order to evaluate the number of solutions of problem $P_\psi(h)$.

THEOREM 5.1: *Let g satisfy conditions* (2.6), (3.1) *and* (4.14); *let $\psi \in H_0^1(\Omega)$ and $\bar{h} \in L^2(\Omega)$. Then there exists $\tau_1 \in [-\infty, +\infty[$ such that problem $P_\psi(\bar{h} + \tau e_1)$ has at least one solution for every $\tau > \tau_1$, while it has no solution if $\tau < \tau_1$.*

Furthermore, if we assume in addition that condition (4.19) *holds with $\bar{\lambda} > \lambda_1$, then $\tau_1 > -\infty$, $P_\psi(\bar{h} + \tau_1 e_1)$ has solution and there exists $\tau_2 \geq \tau_1$ such that problem $P_\psi(\bar{h} + \tau e_1)$ has at least two solutions for every $\tau > \tau_2$.*

REMARK 5.2: Without the additional assumption that condition (4.19) holds with $\bar{\lambda} > \lambda_1$, we cannot say that problem $P_\psi(\bar{h} + \tau_1 e_1)$ has solution. In fact, arguing as in Example 4.12, it is simple to give an example in which $P_\psi(\bar{h} + \tau_1 e_1)$ has no solution: it suffices to choose ψ as in Example 4.12; then, for $g(x, t) = \lambda_1 t$ and $\bar{h} = 0$, we have $\tau_1 = 0$ (as one can easily verify) and problem $P_\psi(0)$ has no solution.

In order to prove Theorem 5.1, we need some preliminary results. In particular, Lemma 5.4 gives us a compactness property for the (non-smooth) functional $f_{h,\psi}$, which is analogous to the Palais-Smale condition.

LEMMA 5.3: *Let $\psi \in H_0^1(\Omega)$, $\bar{h} \in L^2(\Omega)$ and g satisfy conditions* (2.6) *and* (4.14). *Then, for all $t \in \mathbb{R}$ such that $K_\psi \cap H_t \neq \emptyset$* (*see Notations* 1.1), *the sublevels of the func-*

tional $f_{\overline{h},\psi} + I_{H_t}$ *are bounded in* $H_0^1(\Omega)$ (*hence the minimum of* $f_{\overline{h},\psi}$ *on any closed subset of* H_t *is achieved*).

PROOF: Condition (4.14) implies

$$f_{\overline{h}}(u) \geq \frac{1}{2}\int_\Omega |Du|^2\,dx - \frac{\lambda}{2}\int_\Omega u^2\,dx - \int_\Omega |g(x,0)\,u|\,dx + \int_\Omega \overline{h}u\,dx \quad \forall u \in H_0^1(\Omega) \tag{5.1}$$

with $\int_\Omega |g(x,0)\,u|\,dx \leq c_1\|u\|$ because of (2.6).

By (5.1), in order to prove the lemma it is sufficient to show that the sublevels of $f_{\overline{h},\psi} + I_{H_t}$ are bounded in $L^2(\Omega)$. Suppose, contrary to our claim, that there exists a sequence $(u_m)_m$ in a sublevel of $f_{\overline{h},\psi} + I_{H_t}$ such that $\lim_{m\to\infty}\|u_m\|_2 = +\infty$ and set $z_m = u_m/\|u_m\|_2$. Inequality (5.1) implies

$$\limsup_{m\to\infty}\int_\Omega |Dz_m|^2\,dx < +\infty\,.$$

Hence there exists a function $z \in H_0^1(\Omega)$ such that (up to a subsequence) $z_m \to z$ in $L^2(\Omega)$ and a.e. in Ω. Furthermore

$$\|z\|_2 = 1 \quad \text{and} \quad z \geq 0 \quad (\text{because } u_m \geq \psi)\,. \tag{5.2}$$

On the other hand we have

$$\lim_{m\to\infty}\frac{1}{\|u_m\|_2}\int_\Omega u_m e_1\,dx = \lim_{m\to\infty}\int_\Omega z_m e_1\,dx = \int_\Omega z e_1\,dx\,,$$

which is not possible, because $\int_\Omega u_m e_1\,dx \leq t$ implies

$$\lim_{m\to\infty}\frac{1}{\|u_m\|_2}\int_\Omega u_m e_1\,dx \leq 0\,,$$

while $\int_\Omega z e_1\,dx > 0$ by (5.2). *q.e.d.*

LEMMA 5.4: *Let g satisfy conditions* (2.6) *and* (4.14) *and assume that condition* (4.19) *holds with* $\overline{\lambda} > \lambda_1$. *If* $(u_m)_m$ *is a sequence in* K_ψ *such that* $\partial^- f_{h,\psi}(u_m) \neq \emptyset\ \forall m \in \mathbb{N}$ *and* $\sup_{m\in\mathbb{N}}\|\mathrm{grad}^- f_{h,\psi}(u_m)\| < +\infty$, *then* $(u_m)_m$ *is bounded in* $H_0^1(\Omega)$.

If we assume in addition that $\lim_{m\to\infty}\|\mathrm{grad}^- f_{h,\psi}(u_m)\| = 0$, *then the sequence* $(u_m)_m$ *is relatively compact in* $H_0^1(\Omega)$.

PROOF: Set $\alpha_m = \mathrm{grad}^- f_{h,\psi}(u_m)$ and fix $\overline{u} \in K_\psi$. Then, arguing as in (4.15), we have

$$(5.3)\qquad f_{h,\psi}(\overline{u}) \geq f_{h,\psi}(u_m) + (\alpha_m, \overline{u} - u_m) - \frac{\lambda}{2}\|\overline{u} - u_m\|_2^2 \geq \frac{1}{2}\int_\Omega |Du_m|^2\,dx -$$

$$-\frac{\lambda}{2}\int_\Omega u_m^2\,dx - \int_\Omega |g(x,0)\,u_m|\,dx + \int_\Omega hu_m\,dx + (\alpha_m, \overline{u} - u_m) - \frac{\lambda}{2}\|\overline{u} - u_m\|_2^2.$$

Let us prove that $(u_m)_m$ is bounded in $L^2(\Omega)$. Arguing by contradiction, assume that (up to a subsequence) $\lim\limits_{m\to\infty}\|u_m\|_2 = +\infty$.

Set $z_m = u_m/\|u_m\|_2$. From (5.3) it follows that $(z_m)_m$ is bounded in $H_0^1(\Omega)$; so there is a subsequence (still denoted by $(z_m)_m$) converging in $L^2(\Omega)$ and a.e. in Ω to a function $z \in H_0^1(\Omega)$. Furthermore

$$(5.4)\qquad \|z\|_2 = 1 \quad \text{and } z \geq 0 \text{ in } \Omega \quad (\text{because } u_m \geq \psi).$$

On the other hand, from Lemma 4.10 it follows that

$$(\lambda_1 - \overline{\lambda})\int_\Omega ze_1\,dx \geq 0$$

which is not possible because $\overline{\lambda} > \lambda_1$ and (5.4) holds.

Hence $(u_m)_m$ is bounded in $L^2(\Omega)$ and, by (5.3), in $H_0^1(\Omega)$ too.

Now, to prove the second claim, suppose also that $\alpha_m \to 0$ in $H_0^1(\Omega)$. Since $(u_m)_m$ is bounded in $H_0^1(\Omega)$, up to a subsequence we have that $u_m \to u$ in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$ and $L^p(\Omega)$, for a suitable $u \in H_0^1(\Omega)$. By (4.15) it is

$$f_{h,\psi}(u) \geq f_{h,\psi}(u_m) + (\alpha_m, u - u_m) - \frac{\lambda}{2}\|u - u_m\|_2^2,$$

with $(\alpha_m, u - u_m) \to 0$ (as $\|\alpha_m\| \to 0$ and $(\|u_m\|)_m$ is bounded).

Therefore

$$\limsup_{m\to\infty} f_{h,\psi}(u_m) \leq f_{h,\psi}(u).$$

Taking into account (2.6) and (4.14), by Fatou's Lemma, we infer

$$\limsup_{m\to\infty}\int_\Omega |Du_m|^2\,dx \leq \int_\Omega |Du|^2\,dx,$$

which implies $u_m \to u$ in $H_0^1(\Omega)$. *q.e.d.*

PROOF OF THEOREM 5.1: Taking into account Proposition 4.3, we have only to prove that there exists $\tau \in \mathbb{R}$ such that problem $P_\psi(\overline{h} + \tau e_1)$ has solution.

Choose $t \in \mathbb{R}$ such that $t > \int_\Omega \psi e_1\,dx$. Since the functional $f_{h,\psi}$ is weakly lower semicontinuous, Lemma 5.3 implies that, for all $h \in L^2(\Omega)$, the minimum of $f_{h,\psi}$ constrained on the subsets H_t and P_t is achieved. Moreover, for τ sufficiently large, we have

$$\min\{f_{\overline{b}+\tau e_1,\psi}(u)\,|\,u\in H_t\}\leq f_{\overline{b}+\tau e_1,\psi}(\psi)=f_{\overline{b}}(\psi)+\tau\int_\Omega \psi e_1\,dx< \tag{5.5}$$

$$<\min\{f_{\overline{b},\psi}(u)\,|\,u\in P_t\}+\tau t=\min\{f_{\overline{b}+\tau e_1,\psi}(u)\,|\,u\in P_t\}\,.$$

Hence a solution of problem $P_\psi(\overline{b}+\tau e_1)$ can be obtained as minimum point of the functional $f_{\overline{b}+\tau e_1,\psi}$ on the open subset $H_t\backslash P_t$.

Now let us prove the second part of the theorem.

Assume, contrary to our claim, that $\tau_1=-\infty$. Then problem $P_\psi(\overline{b}-me_1)$ has a solution u_m for every $m\in\mathbb{N}$.

Let us fix $\overline{u}\in K_\psi$. By (4.14) we have

$$f_{\overline{b}}(\overline{u})-m\int_\Omega \overline{u}\,e_1\,dx=f_{\overline{b}-me_1}(\overline{u})\geq \tag{5.6}$$

$$\geq f_{\overline{b}-me_1}(u_m)-\frac{\lambda}{2}\|\overline{u}-u_m\|_2^2\geq\frac{1}{2}\int_\Omega|Du_m|^2\,dx-\int_\Omega|g(x,0)\,u_m|\,dx-$$

$$-\frac{\lambda}{2}\int_\Omega u_m^2\,dx+\int_\Omega(\overline{b}-me_1)\,u_m\,dx-\frac{\lambda}{2}\|\overline{u}-u_m\|_2^2.$$

Call $z_m=u_m/m$; by (5.6), taking also into account (2.6), it is easy to verify that there exists a constant $c_2>0$ such that

$$\int_\Omega|Dz_m|^2\,dx\leq c_2\left(1+\int_\Omega z_m^2\,dx\right)\qquad\forall m\in\mathbb{N}\,. \tag{5.7}$$

Now, if $(z_m)_m$ is bounded in $L^2(\Omega)$, it follows from (5.7) that it is bounded in $H_0^1(\Omega)$ too. So, up to a subsequence, it converges in $L^2(\Omega)$ and a.e. in Ω to a function $z\in H_0^1(\Omega)$ such that $z\geq 0$ (because $u_m\geq\psi$). Then Lemma 4.10 yields

$$(\lambda_1-\overline{\lambda})\int_\Omega ze_1\,dx-\int_\Omega e_1^2\,dx\geq 0\,,$$

which is not possible because $\overline{\lambda}>\lambda_1$, $z\geq 0$ and $\int_\Omega e_1^2\,dx=1$.

Hence let us consider the other case, i.e. $(z_m)_m$ not bounded in $L^2(\Omega)$. Up to a subsequence we can assume that $\lim_{m\to\infty}\|z_m\|_2=+\infty$.

Set $z_m'=z_m/\|z_m\|_2$; from (5.7) it follows that $(z_m')_m$ is bounded in $H_0^1(\Omega)$; so, up to a subsequence, it converges in $L^2(\Omega)$ and a.e. in Ω to a function $z'\in H_0^1(\Omega)$ such that

$\|z'\|_2 = 1$ and $z' \geq 0$. Then Lemma 4.10 yields

$$(\lambda_1 - \overline{\lambda}) \int_\Omega z' e_1 \, dx \geq 0 \,,$$

which is not possible because $\overline{\lambda} > \lambda_1$ and $\int_\Omega z' e_1 \, dx > 0$ (since $z' \geq 0$ and $\|z'\|_2 = 1$).

So it must be $\tau_1 > -\infty$.

The solvability of problem $P_\psi(\overline{h} + \overline{\tau}_1 e_1)$ is a straightforward consequence of Proposition 4.11.

Our next claim is the existence of two distinct solutions, for τ sufficiently large.

Choose τ_2 large enough in such a way that (5.5) holds for all $\tau > \tau_2$ and set $h = \overline{h} + \tau e_1$ for a fixed $\tau > \tau_2$. Let $\overline{u}$ be a minimum point for the functional $f_{h,\psi}$ on the set $K_\psi \cap H_t$. Since $\Delta e_1 < 0$ on Ω, $\overline{u} + se_1 \in K_\psi$ for all $s \geq 0$. Let us prove that

$$\lim_{s \to \infty} f_{h,\psi}(\overline{u} + se_1) = -\infty \,. \tag{5.8}$$

In fact condition (4.19) implies

$$\begin{aligned} f_{h,\psi}(\overline{u} + se_1) \leq \frac{1}{2} \int_\Omega |D(\overline{u} + se_1)|^2 \, dx - \frac{\overline{\lambda}}{2} \int_\Omega (\overline{u} + se_1)^2 \, dx \, + \\ + \int_\Omega c(\overline{u} + se_1)^+ \, dx + \int_\Omega \overline{u} h \, dx + s \int_\Omega h e_1 \, dx + k_1 \leq \\ \leq \frac{\lambda_1}{2} s^2 - \frac{\overline{\lambda}}{2} s^2 + k_2 s + k_3 \qquad \forall s \geq 0 \,, \end{aligned}$$

where k_1, k_2 and k_3 are suitable positive numbers, which do not depend on s (recall that $-\psi^- \leq -v^- \leq 0$ for every $v \in K_\psi$). Hence (5.8) follows since $\overline{\lambda} > \lambda_1$.

Let us set

$$d_1 = \min \{ f_{h,\psi}(u) \,|\, u \in P_t \} \,; \tag{5.9}$$

(5.5) implies $f_{h,\psi}(\overline{u}) < d_1$; (5.8) allows us to choose $\overline{s}$ such that, if we put $v = \overline{u} + \overline{s} e_1$, then

$$f_{h,\psi}(v) < f_{h,\psi}(\overline{u}) \,. \tag{5.10}$$

Now set

$$d_2 = \sup_{s \in [0, \overline{s}]} f_{h,\psi}(\overline{u} + se_1) \,. \tag{5.11}$$

Notice that $d_2 \geq d_1$. In fact $\int_\Omega \overline{u} e_1 \, dx < t$ by the definition of $\overline{u}$ and $\int_\Omega v e_1 \, dx > t$ by (5.10); hence there exists $\tilde{s} \in]0, \overline{s}[$ such that $\int_\Omega (\overline{u} + \tilde{s} e_1) \, e_1 \, dx = t$.

Let us prove that there exists a lower critical value $c \in [d_1, d_2]$ for the functional $f_{h,\psi}$.

Arguing by contradiction, assume that $[d_1, d_2]$ does not contain any lower critical value. Taking into account the Palais-Smale type condition given by Lemma 5.4, it follows that, for all $\varepsilon > 0$ small enough, the sublevel $f_{h,\psi}^{d_1-\varepsilon}$ is a deformation retract of $f_{h,\psi}^{d_2}$ (see, for example, [7, 9, 10, 13, 17]). Then, if we choose $\varepsilon > 0$ small enough in such a way that $f_{h,\psi}(\bar{u}) < d_1 - \varepsilon$, we get a contradiction. In fact $f_{h,\psi}^{d_1-\varepsilon}$ contains $\bar{u}$ and v, but does not contain any continuous path connecting this two points, because $f_{h,\psi}^{d_1-\varepsilon} \cap P_t = \emptyset$. On the contrary $f_{h,\psi}^{d_2}$ contains the segment $\{\bar{u} + se_1 \mid s \in [0, \bar{s}]\}$. Therefore $f_{h,\psi}^{d_1-\varepsilon}$ cannot be a deformation retract of $f_{h,\psi}^{d_2}$, which is a contradiction.

Summarizing, the local minimum point $\bar{u}$ and the lower critical level c that we have found satisfy $f_{h,\psi}(\bar{u}) < d_1 \leqslant c$. This implies the existence of two distinct lower critical points for $f_{h,\psi}$, hence of two distinct solutions of $P_\psi(\bar{h} + \tau e_1)$, for all $\tau > \tau_2$. *q.e.d.*

Let us remark that, if in Theorem 5.1 we remove the assumption that condition (4.19) holds with $\bar{\lambda} > \lambda_1$, the solvability of Problem $P_\psi(h)$ does not present an analogous «folding type» behaviour (as in «jumping» problems). For example, if $g(x, t) = \lambda t$ with $\lambda < \lambda_1$, then it is easy to see that problem $P_\psi(h)$ has exactly one solution for every h in $L^2(\Omega)$ and ψ in $H_0^1(\Omega)$ (see [14]). When g has such an asymptotic growth, the following existence result holds.

Proposition 5.5: *Let g satisfy condition* (2.6) *and assume in addition that there exist $c_2 \in L^1(\Omega)$ and $\lambda' < \lambda_1$ such that, for almost all $x \in \Omega$,*

$$G(x, t) \leqslant c_2(x) + \frac{\lambda'}{2} t^2 \qquad \textit{for } t \geqslant 0 .$$

Then problem $P_\psi(h)$ has at least one solution for all $h \in L^2(\Omega)$ and $\psi \in H_0^1(\Omega)$.

Proof: It is sufficient to show that the sublevels of the functional $f_{h,\psi}$ are bounded in $H_0^1(\Omega)$; if it is so, in fact, one solution of $P_\psi(h)$ can be found by minimizing $f_{h,\psi}$ (which is weakly lower semicontinuous).

For all $u \in K_\psi$, we have:

$$f_{h,\psi}(u) = f_{h,\psi}(u^+) + f_{h,\psi}(-u^-) \geqslant$$

$$\geqslant \frac{1}{2}\int_\Omega |Du^+|^2\,dx - \frac{\lambda'}{2}\int_\Omega (u^+)^2\,dx - \int_\Omega c_2(x)\,dx + \int_\Omega hu^+\,dx +$$

$$+ \frac{1}{2}\int_\Omega |Du^-|^2\,dx - \int_\Omega G(x, -u^-)\,dx - \int_\Omega hu^-\,dx \geqslant$$

$$\geqslant \frac{1}{2}\left(1 - \frac{\lambda'}{\lambda_1}\right)\int_\Omega |Du^+|^2\,dx + \int_\Omega hu^+\,dx + \frac{1}{2}\int_\Omega |Du^-|^2\,dx - \bar{c}$$

for a suitable constant $\bar{c}$ independent of u (because $u^- \leq \psi^-$ and (2.8) holds). It follows that the sublevels of $f_{h,\psi}$ are bounded in $H_0^1(\Omega)$, which is the desired conclusion. *q.e.d.*

REMARK 5.6: If we assume that condition (4.14) holds with $\lambda < \lambda_1$, then the solution given by Proposition 5.5 is the unique solution, since in this case the functional f_h is strictly convex, so the unique minimum point of $f_{h,\psi}$ is the unique lower critical point. It is easy to verify that this case occurs if, under the assumptions of Proposition 5.5, we assume in addition that convexity condition (4.18) holds.

Now, under suitable assumptions, we study problem $P_\psi(h)$ by using mini-max methods. In particular, these methods allow us to prove that in Theorem 5.1 we have $\tau_1 = \tau_2$.

DEFINITION 5.7: *Let* $\psi \in H_0^1(\Omega)$, $h \in L^2(\Omega)$ *and* g *satisfy conditions* (2.6) *and* (4.14). *Taking into account Lemma* 5.3, *we can consider the function* $S_h\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *defined by*

$$S_h(t) = \min_{u \in P_t} f_{h,\psi}(u) \tag{5.12}$$

(*here we set* $\min \emptyset = +\infty$).

REMARK 5.8: Let $\bar{t} \in \mathbb{R}$ be such that $S_h(\bar{t}) < +\infty$; if condition (4.14) holds with $\lambda < \lambda_2$, then the minimum $S_h(\bar{t})$ in (5.12) is achieved by a unique function $\bar{u} \in P_{\bar{t}}$. In fact, if $\lambda < \lambda_2$, the functional $f_{h,\psi}$ is strictly convex on $P_{\bar{t}}$.

LEMMA 5.9: *Let* $\psi \in H_0^1(\Omega)$, $h \in L^2(\Omega)$ *and* g *satisfy conditions* (2.6) *and* (4.14). *Let* S_h *be the function introduced in Definition* 5.7. *Then:*

a) S_h *is lower semicontinuous and* $\mathcal{D}(S_h) = [\int_\Omega \psi e_1\, dx, +\infty[$;

b) *if* $u \in K_\psi$ *is a minimum point for* $f_{h,\psi} + I_{P_t}$ *and if* $k \in \partial^- S_h(t)$, *then* $ke_1 \in \partial^- f_{h,\psi}(u)$;

c) *if condition* (4.14) *holds with* $\lambda \leq \lambda_2$ *and* $ke_1 \in \partial^- f_{h,\psi}(u)$, *then:*

i) u *is a minimum point for* $f_{h,\psi} + I_{P_{\bar{t}}}$ *with* $\bar{t} = \int_\Omega ue_1\, dx$,

ii) $k \in \partial^- S_h(\bar{t})$.

PROOF: *a*) The lower semicontinuity of S_h follows easily from Lemma 5.3 and the weak lower semicontinuity of $f_{h,\psi}$.

To find $\mathcal{D}(S_h)$ it suffices to remark that $u \geq \psi$, for every u in K_ψ, and that $\psi + te_1$ is in K_ψ, for every $t \geq 0$.

b) It is a straightforward consequence of the definition of S_h.

c) To prove (*i*) it suffices to observe that $f_{h,\psi} + I_{P_{\bar{t}}}$ is convex if $\lambda \leq \lambda_2$ and that $0 \in \partial^-(f_{h,\psi} + I_{P_{\bar{t}}})(u)$ if $ke_1 \in \partial^- f_{h,\psi}(u)$.

To prove (*ii*), let us remark that

$$f_h(v) \geq f_h(u) + f_h'(u)[v-u] + \frac{1}{2}\|v-u\|^2 - \frac{\lambda}{2}\|v-u\|_2^2 \qquad \forall v \in H_0^1(\Omega)\,.$$

Then, if $\lambda \leq \lambda_2$,

$$(5.13) \quad f_{h,\psi}(v) \geq f_{h,\psi}(u) + k\int_\Omega (v-u)\, e_1\, dx - \frac{\lambda-\lambda_1}{2}\left(\int_\Omega (v-u)\, e_1\, dx\right)^2 \quad \forall v \in H_0^1(\Omega)\,.$$

Taking into account (*i*), by (5.13) we obtain

$$S_h(t) \geq S_h(\bar{t}) + k(t-\bar{t}) - \frac{\lambda-\lambda_1}{2}(t-\bar{t})^2 \qquad \forall t \in \mathbb{R}\,,$$

which obviously implies (*ii*). *q.e.d.*

Notice that Lemma 5.9 shows in particular that, if condition (4.14) holds with $\lambda \leq \lambda_2$, then problem $P_\psi(h)$ is equivalent to find lower critical points for S_h.
In this way we shall prove the following result.

THEOREM 5.10: *Let* $\psi \in H_0^1(\Omega)$, $\bar{h} \in L^2(\Omega)$ *and* g *satisfy condition* (2.6). *Moreover assume that* (4.14) *holds with* $\lambda \leq \lambda_2$ *and* (4.19) *with* $\bar{\lambda} > \lambda_1$.
Then there exists $\bar{\tau} = \bar{\tau}(\psi, \bar{h}) \in \mathbb{R}$ *such that problem* $P_\psi(\bar{h} + \tau e_1)$ *has*

i) *no solution for* $\tau < \bar{\tau}$,

ii) *at least one solution for* $\tau = \bar{\tau}$,

iii) *at least two solutions for all* $\tau > \bar{\tau}$.

To prove Theorem 5.10 we need some properties of the function S_h.

LEMMA 5.11: *Suppose that the hypotheses of Theorem* 5.10 *are fulfilled and set* $h = \bar{h} + \tau e_1$, *for* $\tau \in \mathbb{R}$. *If* S_h *is the function introduced in Definition* 5.7, *then:*

a) $S_h(t) + \frac{\lambda-\lambda_1}{2} t^2$ *is convex;*

b) S_h *is continuous on its domain and* $\partial^- S_h(t) \neq \emptyset$ *for every* $t > \int_\Omega \psi e_1\, dx$;

c) $\lim_{t \to +\infty} S_h(t) = -\infty\,.$

PROOF: *a*) This assertion follows easily taking into account that the functional $f_{h,\psi}(u) + \frac{\lambda-\lambda_1}{2}\left(\int_\Omega u e_1\, dx\right)^2$ is convex if condition (4.14) holds with $\lambda \leq \lambda_2$.

b) It is a straightforward consequence of (*a*) and (*a*) of Lemma 5.9.

c) It suffices to remark that, arguing as for (5.8), we obtain

$$\lim_{t \to +\infty} f_{h,\psi}(\psi + te_1) = -\infty . \qquad \textit{q.e.d.}$$

Proof of Theorem: 5.10: Define

$$\bar\tau = \inf\{\tau \mid P_\psi(\bar h + \tau e_1) \text{ has a solution}\} = \tag{5.14}$$

$$= \inf\{\tau \mid S_{\bar h + \tau e_1} \text{ has a lower critical point}\}.$$

Arguing as in the proof of Theorem 5.1, we can see that $\bar\tau > -\infty$ and $P_\psi(\bar h + \bar\tau e_1)$ has at least one solution.

Then $S_{\bar h + \bar\tau e_1}$ has at least one lower critical point $\bar t$; this means that

$$\liminf_{t \to \bar t} \frac{S_{\bar h + \bar\tau e_1}(t) - S_{\bar h + \bar\tau e_1}(\bar t)}{t - \bar t} \geqslant 0$$

and hence, for every $l > 0$, there exists $t_l > \bar t$ such that

$$S_{\bar h + \bar\tau e_1}(t_l) - S_{\bar h + \bar\tau e_1}(\bar t) > -l(t_l - \bar t). \tag{5.15}$$

From (5.15) it follows that

$$S_{\bar h + \bar\tau e_1}(\bar t) + l\bar t < S_{\bar h + \bar\tau e_1}(t_l) + lt_l,$$

which is equivalent to

$$S_{\bar h + (\bar\tau + l) e_1}(\bar t) < S_{\bar h + (\bar\tau + l) e_1}(t_l). \tag{5.16}$$

Taking into account Lemma 5.11 and (*a*) of Lemma 5.9, from (5.16) it is a simple matter to show that $S_{\bar h + (\bar\tau + l) e_1}$ has at least two critical points: a local minimum point $t_1 < t_l$ and a local maximum point $t_2 \geqslant t_l$ (where, indeed, $S_{\bar h}'(t_2) = 0$). *q.e.d.*

Under the additional assumption that $g(x,\cdot)$ is convex, we can specify the result given by Theorem 5.10.

Theorem 5.12: *Let* $\psi \in H_0^1(\Omega)$, $\bar h \in L^2(\Omega)$ *and* g *satisfy conditions* (2.6), (3.1), (4.14) *with* $\lambda < \lambda_2$ *and* (4.19) *with* $\bar\lambda > \lambda_1$. *If assumption* (4.18) *holds, then there exists* $\bar\tau = \bar\tau(\psi, \bar h) \in \mathbb{R}$ (*see Theorem* 5.10) *such that, for* $h = \bar h + \tau e_1$, *we have:*

i) *if* $\tau < \bar\tau$, *then problem* $P_\psi(h)$ *has no solution;*

ii) *if* $\tau = \bar\tau$, *then problem* $P_\psi(h)$ *either has a unique solution, or there exist two solutions* u_1 *and* u_2 *such that* $u_1 \leqslant u_2$ *and the set of the solutions is* $\Sigma = \{u_1 + \theta(u_2 - u_1) \mid 0 \leqslant \theta \leqslant 1\}$;

iii) *if* $\tau > \bar\tau$, *then problem* $P_\psi(h)$ *has exactly two solutions.*

For the proof we need the following lemma.

LEMMA 5.13: *Assume* $\psi \in H_0^1(\Omega)$, $h \in L^2(\Omega)$ *and let* g *satisfy conditions* (2.6), (3.1) *and* (4.18). *If* u_1 *and* u_2 *are solutions of problem* $P_\psi(h)$ *and* $u_1 \leqslant u_2$, *then the function* $\theta \mapsto f_h(u_1 + \theta(u_2 - u_1))$ *is non-decreasing in* $[0, 1]$.

PROOF: Let us remark that

$$\frac{d}{d\theta} f_h(u_1 + \theta(u_2 - u_1)) = f_h'(u_1 + \theta(u_2 - u_1))[u_2 - u_1] .$$

Now, arguing as in Proposition 4.9, we can say that $u_1 + \theta(u_2 - u_1)$ is a supersolution for the operator $I + \Delta^{-1}(g(x, \cdot) - h)$ for every $\theta \in [0, 1]$. So the conclusion follows taking into account Theorem 4.1. *q.e.d.*

PROOF OF THEOREM 5.12: In this proof we use the notations introduced for Theorem 5.10.

At first, let $\tau > \bar{\tau}$ (see (5.14)); by Theorem 5.10 we have two solutions, we call u_1 and u_2, such that $f_h(u_1) < f_h(u_2)$. Taking into account Proposition 4.6 and Lemma 5.13, we can assume that u_1 is the minimal solution, so $u_1 \leqslant u_2$ and $\Delta u_1 \geqslant \Delta u_2$ in weak sense in Ω.

Set $t_i = \int_\Omega u_i e_1 \, dx$, for $i = 1, 2$; we have that $t_1 < t_2$ and t_1, t_2 are lower critical points for $S_h(t)$ (see Lemma 5.9).

Now our goal is to prove that there is not any other solution of $P_\psi(h)$. Suppose, contrary to our claim, that there exists a solution u_3 different from u_1 and u_2. Set $t_3 = \int_\Omega u_3 e_1 \, dx$; by Remark 5.8 we have that t_3 is different from t_1 and t_2 and, using Lemma 5.13 and Proposition 4.6, we have that $t_1 < t_3$, since u_1 is the minimal solution.

We will show that under these assumptions there exist constants a, b, $c \in \mathbb{R}$ such that the function

$$S(t) = S_h(t) + a\frac{t^2}{2} + bt + c$$

satisfies the following property: there exist γ_2, γ_3, $\gamma_4 \in \mathbb{R}$ and a solution $\tilde{u}_3$ of $P_\psi(h)$ such that

$$t_1 < \gamma_2 < \gamma_3 < \gamma_4, \qquad \int_\Omega \tilde{u}_3 e_1 \, dx = \gamma_3,$$

$$S(t_1) < 0, \;\; S(\gamma_2) > 0, \;\; S(\gamma_3) < 0, \;\; S(\gamma_4) > 0 .$$

Suppose for the moment that this is true and set

$$\bar{f}(u) = f_{h,\psi}(u) + \frac{a}{2}\left(\int_\Omega u e_1 \, dx\right)^2 + b\left(\int_\Omega u e_1 \, dx\right) + c .$$

We have $S(t) = \min_{P_t} \overline{f}(u)$ and so $\overline{f}(u_1) = S(t_1) < 0$, $\overline{f}(\tilde{u}_3) = S(\gamma_3) < 0$, while

$$\overline{f}(u) \geqslant S(\gamma_2) > 0 \qquad \forall u \in P_{\gamma_2}, \tag{5.17}$$

$$\overline{f}(u) \geqslant S(\gamma_4) > 0 \qquad \forall u \in P_{\gamma_4}. \tag{5.18}$$

Notice that, because of (5.17) and (5.18), there exists a minimum point $\overline{u}_3$ for $\overline{f}$ constrained on

$$\left\{ v \in K_{u_1} \,\middle|\, \gamma_2 < \int_\Omega v e_1 \, dx < \gamma_4 \right\}$$

(see Lemma 5.3). Remark that $\tilde{u}_3$ belongs to this set because u_1 is the minimal solution and $\tilde{u}_3$ is a solution of $P_\psi(h)$. Therefore we have

$$\overline{f}(\overline{u}_3) < 0, \qquad \overline{f}'(\overline{u}_3)[v - \overline{u}_3] \geqslant 0 \qquad \forall v \in K_{u_1}. \tag{5.19}$$

Analogously, there exists a minimum point $\overline{u}_1$ for $\overline{f}$ constrained on

$$\left\{ v \in K_{u_1} \,\middle|\, \Delta v \geqslant \Delta \overline{u}_3, \int_\Omega v e_1 \, dx < \gamma_2 \right\}.$$

Remark that u_1 belongs to this set; so

$$\overline{f}(\overline{u}_1) < 0, \qquad \overline{f}'(\overline{u}_1)[v - \overline{u}_1] \geqslant 0 \qquad \forall v \in \{ v \in K_{u_1} \,|\, \Delta v \geqslant \Delta \overline{u}_3 \}; \tag{5.20}$$

in particular this inequality holds for $v = \overline{u}_3$.

For every $\theta \in [0, 1]$ set $u_\theta = \overline{u}_1 + \theta(\overline{u}_3 - \overline{u}_1)$; by convexity, taking into account (5.19) and (5.20), we have

$$\frac{d}{d\theta} \overline{f}(u_\theta) = \overline{f}'(u_\theta)[\overline{u}_3 - \overline{u}_1] \geqslant (1-\theta)\, \overline{f}'(\overline{u}_1)[\overline{u}_3 - \overline{u}_1] + \theta \overline{f}'(\overline{u}_3)[\overline{u}_3 - \overline{u}_1] \geqslant 0$$

that is the function $\theta \mapsto \overline{f}(u_\theta)$ is non-decreasing on $[0, 1]$. This is not possible because $\overline{f}(\overline{u}_3) < 0$ and $\overline{f}(u_{\theta_2}) \geqslant S(\gamma_2) > 0$ for

$$\theta_2 = \frac{\gamma_2 - \int_\Omega \overline{u}_1 e_1 \, dx}{\int_\Omega (\overline{u}_3 - \overline{u}_1)\, e_1 \, dx} \in (0, 1).$$

Now we have to prove the existence of the constants a, b, c satisfying the desired property.

There are two possible cases: (1) $t_3 > t_2$; (2) $t_1 < t_3 < t_2$.

Case (1) - We assume

$$S_h(t_2) = \max \{ S_h(t) \,|\, t \geqslant t_1 \} \tag{5.21}$$

(see the proof of Theorem 5.10 and Lemma 5.13); so we must have $S_h(t) \leqslant S_h(t_2)$ for every $t \in [t_2, t_3]$. Furthermore the function $\theta \mapsto f_h(u_1 + \theta(u_3 - u_1))$ is non-decreasing in $[0, 1]$ by Lemma 5.13; hence, if we set

$$\theta_2' = (t_2 - t_1)/(t_3 - t_1) \in (0, 1),$$

we have

$$S_h(t_3) = f_h(u_3) \geqslant f_h(u_1 + \theta_2'(u_3 - u_1)) \geqslant S_h(t_2)$$

and, as a consequence, we can say that $S_h(t_2) = S_h(t_3)$; an analogous argument shows that S_h is constant on the interval $[t_2, t_3]$. Thus it is now clear that there exist suitable constants a, b, c, with $a > 0$, and a solution $\tilde{u}_3$ satisfying the desired properties.

Case (2) - Since the functions $\theta \mapsto f_h(u_1 + \theta(u_i - u_1))$ $(i = 2, 3)$ are non-decreasing on $[0, 1]$ by Lemma 5.13, we must have $S_h(t_1) \leqslant S_h(t_3) \leqslant S_h(t_2)$ (recall also (5.21)).

If $S_h(t_3) = S_h(t_2)$ we conclude arguing as in the case (1).

If $S_h(t_3) = S_h(t_1)$, then S_h is constant on the interval $[t_1, t_3]$. In fact, since $\theta \mapsto f_h(u_1 + \theta(u_3 - u_1))$ is non-decreasing on $[0, 1]$, we obtain $S_h(t) \leqslant S_h(t_3)$ for every $t \in [t_1, t_3]$. Moreover, if we suppose that there exists $\bar{t} \in [t_1, t_3]$ such that $S_h(\bar{t}) < S_h(t_3)$, then there is a solution $\tilde{u}$ of $P_\psi(h)$, which is a minimum point for $f_{h,\psi}$ on H_{t_3} (see Notations 1.1) and verifies $f_h(\tilde{u}) < S_h(t_3) = f_h(u_1)$; but this is not possible because u_1 is the minimal solution and so, by Lemma 5.13, we must have $f_h(\tilde{u}) \geqslant f_h(u_1)$. Thus also in this case we can get the desired conclusion as before (now with $a < 0$).

Finally suppose $S_h(t_1) < S_h(t_3) < S_h(t_2)$. Since t_3 is a lower critical point for S_h, there exists $\varepsilon > 0$ small enough to have

$$\frac{S_h(t_3) - S_h(t_3 - \varepsilon)}{\varepsilon} < \min\left\{\frac{S_h(t_2) - S_h(t_3)}{t_2 - t_3}, \frac{S_h(t_3) - S_h(t_1)}{t_3 - t_1}\right\}.$$

Hence the existence of the desired constants a, b and c clearly follows (now we can choose $a = 0$).

Now let us prove the second claim of the theorem.

In the proof of (*ii*) of Theorem 5.10 the function S_h is non-increasing, because otherwise problem $P_\psi(\bar{h} + (\bar{\tau} - \varepsilon)e_1)$ would have two solutions for an $\varepsilon > 0$ sufficiently small (indeed, for such an ε the function $S_h(t) - \varepsilon t$ would have again a local minimum and a local maximum in the interior of the domain, where the derivative of S_h is zero).

Let us call u_1 the minimal solution of $P_\psi(h)$.

If there is another solution u_2, by Lemma 5.13 we have $f_h(u_1) \leqslant f_h(u_2)$; so, if we set

$t_i = \int_\Omega u_i e_1 \, dx$ for $i = 1, 2$, we must have

$$S_h(t_1) = f_h(u_1) = S_h(t_2) = f_h(u_2);$$

therefore S_h is constant on the interval $[t_1, t_2]$. Moreover the function $\theta \mapsto f_h(u_1 + \theta(u_2 - u_1))$ is non-decreasing on $[0, 1]$, so

$$f_h(u_1 + \theta(u_2 - u_1)) = S_h(t_1 + \theta(t_2 - t_1)) = S_h(t_1);$$

therefore $u_1 + \theta(u_2 - u_1)$ is a solution of $P_\psi(h)$ for every $\theta \in [0, 1]$.

Finally, taking into account (*c*) of Lemma 5.11 and Proposition 4.17, choose the solution u_2 in such a way that

$$\int_\Omega u_2 e_1 \, dx = \bar{t}_2 \quad \text{with} \quad \bar{t}_2 = \max \{ t \in \mathbb{R} \mid S_h(t) = S_h(t_1) \}.$$

Thus the conclusion of the proof of (*ii*) follows easily from Remark 5.8. *q.e.d.*

THEOREM 5.14: *Under the same assumptions of Theorem* 5.12, *but with* $\lambda = \lambda_2$ *in* (4.14), *there exists* $\bar{\tau} = \bar{\tau}(\psi, \bar{h}) \in \mathbb{R}$ (*see Theorem* 5.10) *such that, for* $h = \bar{h} + \tau e_1$, *we have:*

I) *if* $\tau < \bar{\tau}$, *then problem* $P_\psi(h)$ *has no solution;*

II) *if* $\tau = \bar{\tau}$, *then problem* $P_\psi(h)$ *either has a unique solution or there exists a convex set* Σ_2 *such that the set* Σ *of the solutions of problem* $P_\psi(h)$ *is*

$$\Sigma = \{ u_1 + \theta(u_2 - u_1) \mid \theta \in [0, 1], \ u_2 \in \Sigma_2 \}$$

where u_1 *is the minimal solution* (*see Proposition* 4.6); *furthermore* f_h *is constant on* Σ;

III) *if* $\tau > \bar{\tau}$, *then the set of solutions of problem* $P_\psi(h)$ *is* $\{u_1\} \cup \Sigma_2$, *where* u_1 *is the minimal solution and* Σ_2 *is a convex set; furthermore* f_h *is constant on* Σ_2 *and* $f_h(u_1) < f_h(u_2)$ *for every* $u_2 \in \Sigma_2$.

PROOF: Arguing as in the proof of Theorem 5.12, for $\tau > \bar{\tau}$ we obtain that the function $S_h(t)$ has exactly two lower critical points, we call t_1 and t_2, such that $t_1 < t_2$ and $S_h(t_1) < S_h(t_2)$. By Lemma 5.9 the set of the solutions of $P_\psi(h)$ is equal to $\Sigma_1 \cup \Sigma_2$, where Σ_i, for $i = 1, 2$, is the set of the minimum points of $(f_{h,\psi} + I_{P_{t_i}})$. It is easy to see that Σ_i are closed and convex sets.

Of course the minimal solution u_1 belongs to Σ_1. Let us prove that $\Sigma_1 = \{u_1\}$; in fact, if $u \in P_{t_1}$ is a solution of $P_\psi(h)$, then $u - u_1 \geqslant 0$ because u_1 is the minimal solution, and $\int_\Omega (u - u_1) e_1 \, dx = 0$ because $u_1, u \in P_{t_1}$; therefore we have $u = u_1$.

Now consider the case $\tau = \bar{\tau}$. Let u_1 be the minimal solution, set $t_1 = \int_\Omega u_1 e_1 \, dx$,

$$t_2 = \max \{ t \in \mathbb{R} \,|\, S_h(t) = S_h(t_1) \}$$

and define Σ_2 to be the set of the minimum points of $f_{h,\psi} + I_{P_{t_2}}$. Then, as in the proof of Theorem 5.12, we obtain that $u_1 + \theta(u_2 - u_1)$ is a solution of $P_\psi(h)$ for every $\theta \in [0, 1]$ and $u_2 \in \Sigma_2$; moreover $f_h(u_1 + \theta(u_2 - u_1)) = f_h(u_1)$ $\forall \theta \in [0, 1]$.

So we have to show that every solution u of $P_\psi(h)$ is in this form, that is if $u \neq u_1$ then $t_1 < \int_\Omega u e_1 \, dx \leq t_2$ and

$$u_1 + \bar{t}(u - u_1) \in \Sigma_2 \qquad \text{for } \bar{t} = \frac{t_2 - t_1}{\int_\Omega u e_1 \, dx - t_1} . \tag{5.22}$$

First notice that $u_1 + t(u - u_1) \in K_\psi$ $\forall t \geq 0$ because of Theorem 4.1. Let us consider the function $t \mapsto f_h(u_1 + t(u - u_1))$, for $t \geq 0$; by Lemma 5.13 and since $S_h(t)$ is non-increasing, we have

$$f_h(u_1 + t(u - u_1)) = f_h(u_1) \qquad \text{for } t \in [0, 1] . \tag{5.23}$$

Therefore $S_h(t_1) = f_h(u_1) = f_h(u) = S_h(\int_\Omega u e_1 \, dx)$, from which we obtain $\int_\Omega u e_1 \, dx \leq t_2$. Moreover, if $u \neq u_1$, we have $\int_\Omega u e_1 \, dx > t_1$ arguing as before (u_1 is the unique minimum point for $f_{h,\psi}$ on P_{t_1}).

In order to get (5.22) it is sufficient to prove that the function $t \mapsto f_h(u_1 + t(u - u_1))$ is non-increasing for $t \geq 0$. Indeed, if this is true, the function $u_{\bar{t}} = u_1 + \bar{t}(u - u_1)$, which belongs to $K_\psi \cap P_{t_2}$, verifies $f_h(u_{\bar{t}}) \leq f_h(u_1)$. Actually (see the definition of Σ_2) we must have $f_h(u) \geq f_h(u_1)$ for every $u \in K_\psi \cap P_{t_2}$, so $f_h(u_{\bar{t}}) = f_h(u_1) = S_h(t_2)$, that is $u_{\bar{t}} \in \Sigma_2$.

To prove our claim let us remark that

$$\frac{d}{dt} f_h(u_1 + t(u - u_1)) = f_h'(u_1 + t(u - u_1))[u - u_1] ; \tag{5.24}$$

furthermore, by (5.23), $f_h'(u_1 + t(u - u_1))[u - u_1] = 0$ for $t \in (0, 1]$. Thus, if we fix $t' \in (0, 1)$, we have

$$\begin{aligned} &f_h'(u_1 + t(u - u_1))[u - u_1] = \\ &= f_h'(u_1 + t(u - u_1))[u - u_1] - f_h'(u_1 + t'(u - u_1))[u - u_1] = (t - t') \int_\Omega |D(u - u_1)|^2 \, dx - \\ &\quad - \int_\Omega [g(x, u_1 + t(u - u_1)) - g(x, u_1 + t'(u - u_1))](u - u_1) \, dx . \end{aligned} \tag{5.25}$$

For $t = 1$, (5.23) implies

$$(1-t')\int_\Omega |D(u-u_1)|^2\,dx - \int_\Omega [g(x,u)-g(x,u_1+t'(u-u_1))](u-u_1)\,dx = 0\,; \tag{5.26}$$

from the convexity of $g(x,\cdot)$ we infer that, for every $t \geq 1$,

$$\frac{1}{t-t'}[g(x,u_1+t(u-u_1))-g(x,u_1+t'(u-u_1))] \geq \tag{5.27}$$

$$\geq \frac{1}{1-t'}[g(x,u)-g(x,u_1+t'(u-u_1))]\,.$$

So, using (5.26) and (5.27) in (5.25), we obtain $f_h'(u_1+t(u-u_1))[u-u_1] \leq 0$ for $t \geq 0$, that is the desired conclusion. *q.e.d.*

PROPOSITION 5.15: *The function* $\bar\tau : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ *introduced in Theorems* 5.10, 5.12 *and* 5.14 *satisfies the following properties:*

(*i*) *if* $\Delta\psi' \geq \Delta\psi$ *in weak sense and* $h' \geq h$ *a.e. in* Ω, *then* $\bar\tau(\psi', h') \leq \bar\tau(\psi, h)$;

(*ii*) *if assumption* (4.18) *holds, then* $\bar\tau$ *is a convex function.*

PROOF: Property (*i*) is a straightforward consequence of Proposition 4.3, while (*ii*) follows easily from Proposition 4.9. *q.e.d.*

REMARK 5.16: The proof of Theorems 5.1, 5.10, 5.12 and 5.14 make evident that the deep reason, which explains why the «jumping» type phenomena described by these theorems occur in problem $P_\psi(h)$, is that the sublevels of the functional $f_{h,\psi}$ have the same topological properties as the ones of the functional f_h when we assume that

$$\lim_{t\to-\infty} \frac{g(x,t)}{t} < \lambda_1\,. \tag{5.28}$$

In this sense we can say that the presence of the constraint K_ψ in our problem plays the same role as condition (5.28) in the «jumping» problems.

Notice that similar phenomena have been also pointed out in some problems with unilateral pointwise constraints on the function (see [12, 16, 17, 18, 19, 20]); on the contrary, no phenomenon of this kind arises when we consider unilateral constraints on the first derivatives (as in [5] and [10]).

Finally let us mention that constraints on the second derivatives have been considered, for example by Brezis and Stampacchia in [6], for problems involving the biharmonic operator while only obstacles on the function or on its first derivatives have been usually considered in the literature for problems involving the Laplace operator.

Acknowledgments. This paper was mainly developed during the Concentration Period on Nonlinear Elliptic Problems, organized in Rome on May 2001, sponsored by the Istituto Nazionale di Alta Matematica «Francesco Severi» and by the CEE-TMR Programme on Nonlinear Partial Differential Equations - Methods and Applications. The results reported in this paper have been presented in some lectures held at the University of Rome «La Sapienza» (in the general framework of the CEE-TMR Programme) and at the University of Rome «Tor Vergata».

The research of the authors is also supported by M.U.R.S.T. project «Metodi variazionali e topologici nello studio di fenomeni non lineari».

REFERENCES

[1] A. AMBROSETTI, *Elliptic equations with jumping nonlinearities*, J. Math. Phys. Sci., 18 (1984), 1-12.
[2] A. AMBROSETTI - G. PRODI, *On the inversion of some differentiable mappings with singularities between Banach spaces*, Ann. Mat. Pura Appl., 93 (1972), 231-246.
[3] M. S. BERGER - E. PODOLAK, *On the solutions of a nonlinear Dirichlet problem*, Indiana Univ. Math. J., 24 (1975), 837-846.
[4] H. BREZIS, *Opérateurs maximaux monotones et semigroupes des contractions dans les espaces de Hilbert*, North-Holland Mathematics Studies 5, Notas de Matematica 50, Amsterdam - London (1973).
[5] H. BREZIS - G. STAMPACCHIA, *Sur la régularité de la solution d'inéquations elliptiques*, Bull. Soc. Math. France, 96 (1968), 153-180.
[6] H. BREZIS - G. STAMPACCHIA, *Remarks on some fourth order variational inequalities*, Ann. Scuola Norm. Sup. Pisa Cl. Sci (4), 4 n. 2 (1977), 363-371.
[7] G. ČOBANOV - A. MARINO - D. SCOLOZZI, *Multiplicity of eigenvalues for the Laplace operator with respect to an obstacle, and non-tangency conditions*, Nonlinear Analysis T. M. A., vol. 15, 4 (1990), 199-215.
[8] E. N. DANCER, *Multiple solutions of asymptotically homogeneous problems*, Ann. Mat. Pura Appl., IV-152 (1988), 63-78.
[9] E. DE GIORGI - A. MARINO - M. TOSQUES, *Problemi di evoluzione in spazi metrici e curve di massima pendenza*, Atti Accad. Naz. Lincei, Rend. Cl. Sc. Fis. Mat. Natur., 68 (1980), 180-187.
[10] M. DEGIOVANNI - A. MARINO - C. SACCON, *A pointwise gradient constraint for the Laplace operator. Eigenvalues and bifurcation*, Nonlinear Analysis - A tribute in honour of G. Prodi, Quad. Scuola Norm. Sup. Pisa (1991), 189-203.
[11] A. C. LAZER - P. J. MCKENNA, *On the number of solutions of a nonlinear Dirichlet problem*, J. Math. Anal. Appl., 84 (1981), 282-294.
[12] A. MARINO - D. PASSASEO, *A jumping behaviour induced by an obstacle*, Progress in Variational Methods in Hamiltonian Systems and Elliptic Equations (M. Girardi, M. Matzeu, F. Pacella Ed.) Pitman (1992), 127-143.
[13] A. MARINO - D. SCOLOZZI, *Punti inferiormente stazionari ed equazioni di evoluzione con vincoli unilaterali non convessi*, Rend. Sem. Mat. e Fis. di Milano, 52 (1982), 393-414.

[14] R. Molle - D. Passaseo, *An elliptic problem with pointwise constraint on the laplacian*, Topological Methods in Nonlinear Analysis, 8-1 (1996), 1-23.

[15] R. Molle - D. Passaseo, *Variational problems with pointwise constraints on the derivatives*, J. of Convex Analysis, 6-2 (1999), 215-233.

[16] D. Passaseo, *Molteplicità di soluzioni per certe disequazioni variazionali di tipo ellittico*, Boll. U.M.I., 7 3-B (1989), 639-667.

[17] D. Passaseo, *Molteplicità di soluzioni per disequazioni variazionali non lineari di tipo ellittico*, Rend. Acc. Naz. Sci. detta dei XL, Memorie di mat., 109° Vol. XV, fasc. 2 (1991), 19-56.

[18] A. Szulkin, *On a class of variational inequalities involving gradient operators*, J. Math. Anal. Appl., 100 (1984), 486-499.

[19] A. Szulkin, *On the solvability of a class of semilinear variational inequalities*, Rend. Math., 4 (1984), 121-137.

[20] A. Szulkin, *Positive solutions of variationals inequalities: a degree-theoretic approach*, J. Differential Equations, 57 (1985), 90-111.

Direttore responsabile: Prof. A. Ballio - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5 - Bologna