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Quasi-Linear Relaxed Dirichlet Problems Involving a Dirichlet Form (**)

ABSTRACT. — We study the existence and the asymptotic behaviour of bounded solutions of a quasi-linear relaxed Dirichlet problem, involving a Dirichlet form. The classical problem, that is when the Dirichlet form is associated with the Laplace operator with Dirichlet boundary condition, has been studied by S. Finzi Vita, F.Murat and N.Tchou in [16]. The same classical problem, in the non-relaxed case, has been treated by L. Boccardo, F. Murat and J. P. Puel in [7]. We prove the existence result under a suitable assumption on non-linear term that, in [16], corresponds to the requirement of quadratic growth with respect to the gradient. As in [16], the proof is divided in five steps of which first three steps extend the techniques used in [7]. We also show a stability property of solutions with respect to the γ-convergence of measures when the limit measure is sufficiently regular. In this case, the assumption on the non-linear term corresponds, in [16], to the requirement of strictly subquadratic growth with respect to the gradient. The proof makes essential use of correctors result of M. Biroli, C. Picard and N. A. Tchou ([4]).

I problemi di dirichlet rilassati quasi-lineari per una forma di dirichlet

SUNTO. — Studiamo l'esistenza ed il comportamento asintotico delle soluzioni limitate del problema di Dirichlet rilassato, quasi-lineare, per una forma di Dirichlet. Il problema classico, ossia quando la forma di Dirichlet è associata all'operatore di Laplace con condizioni di Dirichlet al bordo, è stato studiato da S. Finzi Vita, F. Murat and N. Tchou in [16]. Lo stesso problema classico, nel caso non-rilassato, è stato trattato da L. Boccardo, F. Murat and J. P. Puel in [7]. Proviamo il risultato di esistenza sotto un'opportuna ipotesi sul termine non-lineare che, in [16], corrisponde alla richiesta di crescita quadratica rispetto al gradiente. Come in [16], la dimostrazione è divisa in cinque passi dei quali, i primi tre, estendono le tecniche usate in [7]. Mostriamo anche una proprietà di stabilità delle soluzioni rispetto alla γ-convergenza di misure, quando la misura limite è sufficientemente regolare. In questo caso, l'ipotesi sul termine non-lineare corrisponde, in [16], alla richiesta di crescita strettamente sottoquadratica ri-

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spetto al gradiente. Nella dimostrazione si fa un uso essenziale del risultato dei correttori di M. Biroli, C. Picard e N. A. Tchou ([4]).

1. - Introduction

In this paper we are interested in the study of the existence and the asymptotic behaviour of bounded solutions of quasi-linear relaxed Dirichlet problems involving a Dirichlet form. The problem, in the classical case of the Dirichlet form $a(u, v) = \int_\Omega \nabla u \nabla v \, dx$ with domain $H_0^1(\Omega)$, was studied by S. Finzi Vita, F. Murat and N. Tchou in [16]. In this case, the Dirichlet form is associated with the Laplace operator with Dirichlet boundary condition on $\partial\Omega$ and the problem can be *formally* written as

$$\begin{cases} -\Delta u + \lambda_0 u + \mu u = f(x, u, Du) & \text{in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, λ_0 is a positive constant, f is a given function that satisfies a quadratic growth hypothesis with respect to Du and μ is a measure in the class $\mathcal{M}_0(\Omega)$ of all non-negative Borel measures on Ω that vanish on subsets of Ω with zero capacity.

Let us recall that the relaxed Dirichlet problem in the linear case have been introduced by G. Dal Maso and U. Mosco in [11] and [12] to study limits of Dirichlet problems in highly perturbed domains. The generalisation when a Dirichlet form appears has been studied by G. Dal Maso, V. De Cicco, L. Notarantonio, N. Tchou in [10] and by M. Biroli and N.Tchou in [5] in the symmetric case, by S. Mataloni and N. Tchou in [24] without any assumption of symmetry. In all these works a particular class of Dirichlet forms has been considered: the strongly local regular Dirichlet forms satisfying, as it will be explain with details in Section 2, a Poincaré inequality and a suitable duplication condition. These forms are called regular Poincaré-Dirichlet forms and their properties have been investigated by M. Biroli and U.Mosco in [2]. They prove that this framework is «rich enough» in the sense that the theory that they developed includes some aspects of the classical variational theory of second order elliptic equations and also a wide class of degenerate elliptic operators with discontinuous coefficients, such as weighted and subelliptic operators.

Let us come back to our problem to explain it more precisely.

Let X be an arbitrary connected locally compact separable Hausdorff space and let m be a given positive Radon measure supported on the whole of X. Let $(a, D(a))$ a regular Poincaré-Dirichlet form on the Lebesgue space $L^2_m(X)$, with energy measure $\alpha(u, u)(x) \in L^1_m(\Omega)$, and let Ω be a relatively compact open subset of X. Let us denote by $D_0(a, \Omega)$ the closure in $D(a)$ of $D(a) \cap C_0(\Omega)$. Let us define $V^0_\mu(\Omega)$ as the space $D_0(a, \Omega) \cap L^2_\mu(\Omega)$ where the measure μ belongs to the space of the non-negative

Borel measures on Ω that vanish on subsets of Ω with zero a-capacity (Definition 3.1).

We are interested in the bounded solutions of the following problem

$$(1.2)\qquad \begin{cases} u \in V_\mu^0(\Omega) \cap L_m^\infty(\Omega) \\ a(u, v) + \lambda_0 \int_\Omega uv\, dm + \int_\Omega uv\, d\mu = \int_\Omega \phi(x, u(x), \alpha^{1/2}(u, u)(x))\, v\, dm \\ \forall v \in V_\mu^0(\Omega) \cap L_m^\infty(\Omega) \end{cases}$$

where $\lambda_0 > 0$ and ϕ is a Carathéodory function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^+$ such that

$$|\phi(x, s, p)| \leq c_0 + b(|s|)\, p^2$$

for some constant c_0 and an increasing function b.

In order to prove the existence of bounded solutions of (1.2) we use the techniques of [16] that extend which ones used by L. Boccardo, F. Murat, J. P. Puel in [7] in the classical (non relaxed) case. More precisely, as in [7], we prove the existence of a sequence $\{u_\varepsilon\}$ of solutions of approximate problems and we show that such a sequence is uniformly bounded in $V_\mu^0(\Omega) \cap L_m^\infty(\Omega)$. In the last two steps, as in [16], we prove the strongly convergence of the sequence $\{u_\varepsilon\}$ in $D_0(a, \Omega)$ and in $L_\mu^2(\Omega)$ to a function u that we will prove to be the solution of the problem (1.2). In particular let us show that the uniform bound in $V_\mu^0(\Omega) \cap L_m^\infty(\Omega)$ of the solution of (1.2) is independent of the measure μ. This property will be essential to prove in Section 4 the homogenization result of Theorem 4.10. In this theorem we show a stability property of solutions of (1.2) with respect to the γ-convergence of measures (Definition 4.2), when the limit measure is sufficiently regular, under a suitable assumption on ϕ that in the classical case (1.1) correspond to the requirement on the growth of f with respect to Du to be strictly subquadratic. Let us underline that, in the proof of Theorem 4.10, an important role is played by the use of the correctors result in the linear case of M. Biroli, C. Picard, N. Tchou (Proposition 4.7).

2. - Setting and notation

Let X be an arbitrary connected locally compact separable Hausdorff space and let m be a given positive Radon measure supported on the whole of X. We denote by $L_m^2(X)$ the usual Lebesgue space endowed with the inner product $(u, v) = \int_X uv\, dm$ and the norm $\|u\|_{L_m^2}^2 = (u, u)$.

We assume that we are given a *strongly local, regular, symmetric Dirichlet* form $a(\cdot, \cdot)$ on $L_m^2(X)$ whose domain will be denoted by $D(a)$. Such a form admits the integral representation $a(u, v) = \int_X \alpha(u, v)(dx)$ for every $u, v \in D(a)$ where $\alpha(u, v)$ is a non-negative Radon measure on X, uniquely associated with the functions u and v and is called the energy measure of the form a. Let us recall that a form $a(\cdot,\cdot)$ is regular if there exists a core $C\subset C_0(X)\cap D(a)$, which is dense in $C_0(X)$ with respect to the uniform norm and in $D(a)$ with respect to the intrinsic norm:

$$\|u\|_{a_1}=(a(u,u)+\|u\|^2_{L^2_m})^{1/2}.$$

We assume that C is an *m-separating core*, see [2], that is for every $x, y\in X$, $x\neq y$, there exists $\phi\in C$ such that $\phi(x)\neq\phi(y)$ and $\alpha(\phi,\phi)\leqslant m$ where the last inequality is understood in the sense of measures on X.

By the strong locality of the form, for any open subset A of X the restriction of the energy measure $\alpha(u,v)$ to A depends only on the restrictions of u and v to A, then this property allows us to define $D(a,A)$ as the set of all functions $u|_A$ when $u\in D(a)$. We define $D_0(a,A)$ as the closure of $D(a)\cap C_0(A)$ in $D(a)$ endowed with the intrinsic norm. We refer to [2], [17], [22] for the properties of $\alpha(u,v)$ with respect to Leibnitz, chain and truncation rules.

We define a distance d associated with the form by

$$d(x,y)=\sup\{\varphi(x)-\varphi(y):\varphi\in D(a)\cap C_0(X)\ \text{ with }\ \alpha(\varphi,\varphi)\leqslant m\ \text{ on } X\}$$

and denote by $B(x,r)$ the ball $\{y\in X: d(x,y)<r\}$.

Moreover, for every compact set $K\subset X$, we make the following assumptions:

(D) The metric topology induced by the distance d on X is equivalent to the given topology of X. Further we assume that there exist three constants ν, $R_0>0$ and $C_0>1$ (with ν independent of K) such that a *duplication property* holds for the balls $B(x,r)$ with $x\in K$ and $r\leqslant 2r\leqslant R\leqslant R_0$, that is

$$m(B(x,r))\geqslant C_0\left(\frac{r}{R}\right)^{\nu} m(B(x,R)).$$

Then metric space (X,d), together with this doubling measure m is a locally space of homogeneous type or a *homogeneous space* in the sense of Coifman-Weiss (see [9]).

Let us remark that if X is the union of a sequence of balls of radius R_0, then the separability of X is a consequence of the homogenity.

(P) For every ball $B(x,r)$, $x\in K$, $(r\leqslant R_0)$ and every $f\in D_{loc}(B(x,kr))$ the *Poincaré inequality*

$$\int_{B(x,r)}|f-f_{x,r}|^2\,dm\leqslant C_1 r^2\int_{B(x,kr)}\alpha(f,f)(dx)$$

holds, where $C_1>0$ and $k\geqslant 1$ are constants independent of x, $r\leqslant R_0$ and $f_{x,r}$ is the average of f on $B(x,r)$ with respect to the measure m.

From property (P), assuming that $B(x,r)\subseteq B(x,2r)\neq X$ $\left(r\leqslant\frac{R_0}{2}\right)$, we obtain, by standard methods, the inequality

$$(P_0) \qquad \int_{B(x,\,r)} |f|^2 \, dm \leq C_2 r^2 \int_{B(x,\,r)} \alpha(f,\, f)(dx)$$

for every $f \in D_0(a,\, B(x,\, r))$; by a covering argument it is easy to prove that the inequality (P_0) holds also if $r \geq \frac{R_0}{2}$, with a constant C_2 that depends on R_0.

We recall that in this assumption the following embedding result holds:

THEOREM 2.1 Compact embedding property: *Let B_R a ball in X. Then the property*

$$D_0(a,\, B_R) \text{ is compactly embedded into } L^2_m(B_R)$$

is fulfilled.

PROOF: See Lemma 2.5 in [5]. ■

Finally, for any open subset A of X, we assume the existence of Radon Nikodym derivative

$$(\mathrm{A}) \qquad \alpha(u,\, u)(\cdot) = \frac{\alpha(u,\, u)(dx)}{dm} \in L^1_{\mathrm{loc}}(A)\,.$$

Let us conclude this section with some examples in which our results can be applied:

(a) forms connected with second order elliptic operators for $X = \mathbb{R}^n$, $n \geq 2$ and $dx := m$ the Lebesgue measure on X. Here the distance is equivalent to the usual Euclidean distance;

(b) forms connected with degenerate elliptic operators with a weight w in the Muckenhoupt's class A_2; let us recall that in the model case $X = \mathbb{R}^n$ and $n \geq 2$, $w(x) = |x|^\alpha$ the requirement $w \in A_2$ means that $-n < \alpha < n$. Here the distance is equivalent to the usual Euclidean distance (refer to [15] for the validity of properties (D) and (P));

(c) forms connected with subelliptic operators in the case of smooth or non-smooth coefficients. Here the distance is defined in relation with the operator (we refer to [19] for properties (D) and (P));

(d) forms connected with vector fields satisfying Hörmander condition in the case of smooth or non-smooth coefficients, given by a matrix, that is uniformly elliptic with respect to a weight in the intrinsic Muckenhoupt's class: here the distance is the same as in the non-weighted case, property (D) derives from the definition of the intrinsic Muckenhoupt's class and we refer to [23] for property (P) – see also [18], [25] for the non-weighted case.

3. - THE PROBLEM. EXISTENCE RESULT

From now on let Ω be a relatively compact open set in X such that $\Omega \subseteq B_R \subseteq B_{2R}$ with $B_{2R} \neq X$. For every Borel subset E, let

$$\mathrm{cap}^a(E, \Omega) = \inf\{a(v, v)/v \in D(a) \cap C_0(\Omega), \quad v \geqslant 1 \text{ on a neighbourhood of } E\}.$$

We refer, for all properties holding for the capacity related to a Dirichlet form, to the book of Fukushima [17], only observing that they hold again, in our case, due to validity of property (P_0).

In particular we say that a property $P(X)$ holds *quasi everywhere* (abridged as q.e.) in a set $E \subset \overline{E} \subset \Omega$, if it holds for all $x \in E$ except of a subset N of E with $\mathrm{cap}^a(N, \Omega) = 0$. A function $u : \Omega \to \mathbb{R}$ is said to be *quasi continuous* if for every $\varepsilon > 0$ there exists a set $A \subset \Omega$, with $\mathrm{cap}^a(A, \Omega) < \varepsilon$, such that the restriction of u to $\Omega \backslash A$ is continuous.

Every $u \in D(a, \Omega)$ has a quasi continuous representative, which is uniquely defined up to a set of capacity zero. In the sequel we shall always identify u with is quasi continuous representative, so the pointwise values of a function $u \in D(a, \Omega)$ are defined quasi everywhere.

DEFINITION 3.1: *For a relatively compact open set $\Omega \subset X$, let $\mathcal{M}_0^a(\Omega)$ be the set of all non-negative Borel measures μ on Ω which are absolutely continuous with respect to cap^a, i.e.,* $\mu(E) = 0$ for every Borel set $E \subset \Omega$ with $\mathrm{cap}^a(E, \Omega) = 0$.

Let us consider a Carathéodory function ϕ on $\Omega \times \mathbb{R} \times \mathbb{R}^+$ that is ϕ is such that

$$\text{(E1)} \qquad \begin{cases} i) \ \forall (s, p) \in \mathbb{R} \times \mathbb{R}^+, \quad x \to \phi(x, s, p) \ \text{ is a measurable function} \\ ii) \ \text{for a.e. } \ x \in \Omega, \ (s, p) \to \phi(x, s, p) \ \text{ is a continuous function} \end{cases}$$

and let us assume that the following inequality holds:

$$\text{(E2)} \qquad \text{for a.e. } \ x \in \Omega, \forall s \in \mathbb{R}, \ \forall p \in \mathbb{R}^+, \qquad |\phi(x, s, p)| \leqslant c_0 + b(|s|)\, p^2$$

where $b(\cdot)$ is an increasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and $c_0 \in \mathbb{R}^+$. Let us remark that the assumption «b increasing» is not a restriction indeed we can replace b with $\overline{b}$ defined as $\overline{b}(s) = \sup\limits_{0 \leqslant r \leqslant s} b(r)$ for any $s \in \mathbb{R}^+$. Moreover let us consider

$$\text{(E3)} \qquad \lambda_0 > 0 \qquad \mu \in \mathcal{M}_0^a(\Omega).$$

We are interested in bounded solutions of the following problem

$$(3.1) \qquad \begin{cases} u \in V_\mu^0(\Omega) \cap L_m^\infty(\Omega) \\ a_{\lambda_0}(u, v) + \displaystyle\int_\Omega uv\, d\mu = \int_\Omega \phi(x, u, \alpha^{1/2}(u, u))\, v\, dm \\ \forall v \in V_\mu^0(\Omega) \cap L_m^\infty(\Omega) \end{cases}$$

where $V_\mu^0(\Omega) = D_0(a, \Omega) \cap L_\mu^2(\Omega)$, $a_{\lambda_0}(u, v) = a(u, v) + \lambda_0(u, v)$. $(a, D(a))$ is a Dirichlet form with the assumptions of the previous section and $\alpha(u, u)$ is its energy measure. Let us observe that, since $u \in D(a, \Omega) \cap L_m^\infty(\Omega)$, the function $\phi(x, u(x), \alpha^{1/2}(u, u)(x))$ belongs to $L_m^1(\Omega)$ by (E2), indeed:

$$\int_\Omega |\phi(x, u(x), \alpha^{1/2}(u, u)(x))|\, dm \leq c_0\, m(\Omega) + b(\|u\|_\infty) \int_\Omega \alpha(u, u)\, dm < \infty\ .$$

THEOREM 3.2: *Under assumptions* (E1), (E2), (E3), *there exists at least a bounded solution u of problem* (3.1).

REMARK 3.3: Theorem 3.2 extends the result of [16] (Theorem 3.2) when the relaxed Dirichlet problem involves a Dirichlet form.

PROOF OF THEOREM 3.2.

Step 1: Existence of approximate solutions.

For sake of simplicity we introduce the operator Φ from $D(a, \Omega) \cap L_m^\infty(\Omega)$ to $L_m^1(\Omega)$ defined as

$$\Phi(u) = \phi(x, u, \alpha^{1/2}(u, u))\,.$$

We construct a sequence of problems that approximate (3.1) by introducing for any $\varepsilon > 0$ the bounded operator

$$\Phi_\varepsilon(u) = \frac{\Phi(u)}{1 + |\Phi(u)|\varepsilon}\ .$$

Let us note that

$$|\Phi_\varepsilon(u)| \leq \frac{1}{\varepsilon} \qquad |\Phi_\varepsilon(u)| \leq |\Phi(u)| \ \text{ a.e.}\,. \tag{3.2}$$

We shall first prove, for $\varepsilon > 0$ fixed, the existence of a solution u_ε of the quasi-linear problem

$$\begin{cases} u_\varepsilon \in V_\mu^0(\Omega) \cap L_m^\infty(\Omega) \\ a_{\lambda_0}(u_\varepsilon, v) + \displaystyle\int_\Omega u_\varepsilon v\, d\mu = \int_\Omega \Phi_\varepsilon(u_\varepsilon)\, v\, dm \qquad \forall v \in V_\mu^0(\Omega)\,. \end{cases} \tag{3.1_ε}$$

Claim: the mapping $S: V_\mu^0(\Omega) \to V_\mu^0(\Omega)$ that associates with each function $w \in V_\mu^0(\Omega)$ the unique solution $\overline{w} = Sw$ of the linear problem (whose existence is assured by [10] (Prop. 4.2)

$$\begin{cases} \overline{w} \in V^0_\mu(\Omega) \\ a_{\lambda_0}(\overline{w}, v) + \int\limits_\Omega \overline{w} v \, d\mu = \int\limits_\Omega \Phi_\varepsilon(w) \, v \, dm \qquad \forall v \in V^0_\mu(\Omega) \end{cases}$$

satisfies the hypothesis of Schauder fixed-point theorem. Indeed let us denote by V the following set

$$V := \left\{ v \in V^0_\mu(\Omega) : \|v\|_{V^0_\mu(\Omega)} \leqslant \frac{m(\Omega)^{1/2}}{\tilde{\lambda}_0 \varepsilon} \right\}$$

where $\tilde{\lambda}_0 = \min(1, \lambda_0)$ and

$$\|v\|_{V^0_\mu(\Omega)} := \left(a_1(v, v) + \int\limits_\Omega |v|^2 \, d\mu \right)^{1/2}$$

and let us observe that, for every $\lambda \geqslant 0$, $a_\lambda(v, v)$ and $a_1(v, v)$ are equivalent norms on $D_0(a, \Omega)$. It results that, if $w \in V^0_\mu(\Omega)$ then $Sw \in V^0_\mu(\Omega)$ and

$$a_{\lambda_0}(Sw, Sw) + \int\limits_\Omega |Sw|^2 \, d\mu \leqslant \|\Phi_\varepsilon w\|_{L^2_m} \|Sw\|_{L^2_m} \leqslant \frac{m(\Omega)^{1/2}}{\varepsilon} \|Sw\|_{V^0_\mu}$$

then

$$\|Sw\|_{V^0_\mu} \leqslant \frac{m(\Omega)^{1/2}}{\tilde{\lambda}_0 \varepsilon}$$

that is $Sw \in V$.

S is a compact operator: let $\{w_n\}$ be a bounded sequence of $V^0_\mu(\Omega)$. For ε fixed, $\Phi_\varepsilon(u) \in L^\infty_m(\Omega)$ hence $\Phi_\varepsilon(w_n)$ is bounded in $L^2_m(\Omega)$ uniformly with respect to n and $\Phi_\varepsilon(w_n) \rightharpoonup \Phi^0_\varepsilon$ weakly in $L^2_m(\Omega)$. Let $\{\overline{w}_n\} := \{Sw_n\} \in V^0_\mu(\Omega)$. Since $\overline{w}_n$ is bounded in $V^0_\mu(\Omega)$, there exists a subsequence $\{\overline{u}_n\}$ such that $\overline{u}_n \rightharpoonup \overline{u}$ weakly in $D_0(a, \Omega)$, $\overline{u}_n \rightharpoonup \overline{u}$ weakly in $L^2_\mu(\Omega)$ and $\overline{u}_n \to \overline{u}$ strongly in $L^2_m(\Omega)$ -since the embedding of $D_0(a, \Omega)$ in $L^2_m(\Omega)$ is compact, see Theorem 2.1. Let us denote by $\{u_n\}$ the subsequence of $\{w_n\}$ such that $Su_n = \overline{u}_n$ for every $n \in \mathbb{N}$. It results that

$$a_{\lambda_0}(\overline{u}_n, \overline{u}_n - \overline{u}) + \int\limits_\Omega \overline{u}_n(\overline{u}_n - \overline{u}) \, d\mu = \int\limits_\Omega \Phi_\varepsilon(u_n)(\overline{u}_n - \overline{u}) \, dm$$

then

$$a_{\lambda_0}(\overline{u}_n - \overline{u}, \overline{u}_n - \overline{u}) + \int\limits_\Omega (\overline{u}_n - \overline{u})^2 \, d\mu + a_{\lambda_0}(\overline{u}, \overline{u}_n - \overline{u}) + \int\limits_\Omega \overline{u}(\overline{u}_n - \overline{u}) \, d\mu = \int\limits_\Omega \Phi_\varepsilon(u_n)(\overline{u}_n - \overline{u}) \, dm .$$

Letting n to infinity we have

$$\lim_{n\to\infty} a_{\lambda_0}(\bar{u}_n - \bar{u}, \bar{u}_n - \bar{u}) = 0$$

and

$$\lim_{n\to\infty} \int_\Omega (\bar{u}_n - \bar{u})^2 \, d\mu = 0$$

that is $\bar{u}_n \to \bar{u}$ in $V^0_\mu(\Omega)$. Thus the mapping S has a fixed point that is the problem (3.1_ε) has at least a solution that belongs to $V^0_\mu(\Omega)$. Moreover, this solution is in $L^\infty_m(\Omega)$, with

$$-\frac{1}{\lambda_0 \varepsilon} \leqslant u_\varepsilon \leqslant \frac{1}{\lambda_0 \varepsilon}\,. \tag{3.3}$$

To prove this, we use the function $\left(u_\varepsilon - \dfrac{1}{\lambda_0 \varepsilon}\right)^+ = z_\varepsilon$ as the test function in problem (3.1_ε). This is possible since $z_\varepsilon \in D_0(a, \Omega)$ and since $0 \leqslant z_\varepsilon \leqslant u_\varepsilon^+$ then z_ε belongs to $L^2_\mu(\Omega)$ and thus to $V^0_\mu(\Omega)$. Then, since $\mu \geqslant 0$ and by (3.2) we have

$$0 \leqslant a(z_\varepsilon, z_\varepsilon) = \int_\Omega \alpha(z_\varepsilon, z_\varepsilon)\, dm = \int_\Omega \chi_{\{u_\varepsilon > (1/\lambda_0 \varepsilon)\}}\, \alpha(u_\varepsilon, z_\varepsilon)\, dm \leqslant$$

$$\int_\Omega \alpha(u_\varepsilon, z_\varepsilon)\, dm + \int_\Omega z_\varepsilon u_\varepsilon^+ \, d\mu = a(u_\varepsilon, z_\varepsilon) + \int_\Omega z_\varepsilon u_\varepsilon \, d\mu =$$

$$\int_\Omega (-\lambda_0 u_\varepsilon + \Phi_\varepsilon(u_\varepsilon))\, z_\varepsilon \, dm \leqslant \int_\Omega \left(-\lambda_0 u_\varepsilon + \frac{1}{\varepsilon}\right) z_\varepsilon \, dm$$

$$= \int_\Omega \left(-\lambda_0 u_\varepsilon + \frac{1}{\varepsilon}\right)\left(u_\varepsilon - \frac{1}{\lambda_0 \varepsilon}\right)^+ dm \leqslant 0$$

so that $z_\varepsilon = 0$. This proves one of the inequality of (3.3). The other inequality is proved in the same way by considering the test function $-\left(u_\varepsilon + \dfrac{1}{\lambda_0 \varepsilon}\right)^-$. Thus $u_\varepsilon \in L^\infty_m(\Omega)$. We have also proved that $u_\varepsilon \in L^\infty_\mu(\Omega)$ since $V^0_\mu(\Omega) \cap L^\infty_m(\Omega) \subset V^0_\mu(\Omega) \cap L^\infty_\mu(\Omega)$. Indeed if $u \in V^0_\mu(\Omega) \cap L^\infty_m(\Omega)$, its quasi-continuous representative $\tilde{u}$ is such that $|\tilde{u}| \leqslant k$ q.e..Since $u \in L^2_\mu(\Omega)$ then u is μ-measurable and since $\mu \in \mathcal{M}^a_0(\Omega)$ hence $|\tilde{u}| \leqslant k$ μ-a.e. that is $u \in L^\infty_\mu(\Omega)$.

Step 2: *The solutions* u_ε *of problem* (3.1_ε) *are uniformly bounded in* $L^\infty_m(\Omega)$.

More precisely we can prove that

$$\|u_\varepsilon\|_\infty \leqslant \frac{c_0}{\lambda_0} \tag{3.4}$$

where c_0 is the constant that appears in (E2). Let us denote by

$$t_\varepsilon = \frac{b^2(\|u_\varepsilon\|_\infty)}{2} \quad \text{and} \quad T_\varepsilon(v) = v \exp(t_\varepsilon v^2)\,.$$

Let us consider $z_\varepsilon = u_\varepsilon - \frac{c_0}{\lambda_0}$ and let us note that $T_\varepsilon(z_\varepsilon^+) \in V^0_\mu(\Omega)$ by the chain rule and since $z_\varepsilon^+ \in V^0_\mu(\Omega) \cap L^\infty_m(\Omega) \cap L^\infty_\mu(\Omega)$. To prove the uniform bound in $L^\infty_m(\Omega)$ (3.4) we use $T_\varepsilon(z_\varepsilon^+)$ as test function in (3.1_ε):

$$a_{\lambda_0}(z_\varepsilon, T_\varepsilon(z_\varepsilon^+)) + \int_\Omega z_\varepsilon T_\varepsilon(z_\varepsilon^+)\, d\mu =$$

$$a(z_\varepsilon, T_\varepsilon(z_\varepsilon^+)) + \lambda_0(z_\varepsilon, T_\varepsilon(z_\varepsilon^+)) + \int_\Omega z_\varepsilon T_\varepsilon(z_\varepsilon^+)\, d\mu =$$

$$a(u_\varepsilon, T_\varepsilon(z_\varepsilon^+)) + \lambda_0\left(u_\varepsilon - \frac{c_0}{\lambda_0}\,,\, T_\varepsilon(z_\varepsilon^+)\right) + \int_\Omega z_\varepsilon T_\varepsilon(z_\varepsilon^+)\, d\mu =$$

$$a_{\lambda_0}(u_\varepsilon, T_\varepsilon(z_\varepsilon^+)) + \int_\Omega u_\varepsilon T_\varepsilon(z_\varepsilon^+)\, d\mu - c_0 \int_\Omega T_\varepsilon(z_\varepsilon^+)\, dm - \frac{c_0}{\lambda_0}\int_\Omega T_\varepsilon(z_\varepsilon^+)\, d\mu =$$

$$\int_\Omega \Phi_\varepsilon(u_\varepsilon)\, T_\varepsilon(z_\varepsilon^+)\, dm - c_0 \int_\Omega T_\varepsilon(z_\varepsilon^+)\, dm - \frac{c_0}{\lambda_0}\int_\Omega T_\varepsilon(z_\varepsilon^+)\, d\mu$$

since a is a strongly local form. Let us observe now that $T'_\varepsilon(v) = \exp(t_\varepsilon v^2) + 2\,t_\varepsilon v^2 \exp(t_\varepsilon v^2)$ hence, denoted by $e_\varepsilon = \exp[t_\varepsilon (z_\varepsilon^+)^2]$, by chain and truncation rules it results that

$$\int_\Omega \alpha(z_\varepsilon, T_\varepsilon(z_\varepsilon^+))\, dm = \int_\Omega T'_\varepsilon(z_\varepsilon^+)\, \alpha(z_\varepsilon, z_\varepsilon^+)\, dm$$

$$= \int_\Omega e_\varepsilon\, \alpha(z_\varepsilon, z_\varepsilon^+)\, dm + 2\,t_\varepsilon \int_\Omega (z_\varepsilon^+)^2 e_\varepsilon\, \alpha(z_\varepsilon, z_\varepsilon^+)\, dm\,.$$

Moreover let us note that $\alpha(z_\varepsilon, z_\varepsilon^+) = \chi_{\{z_\varepsilon > 0\}}\, \alpha(z_\varepsilon, z_\varepsilon) = \alpha(z_\varepsilon^+, z_\varepsilon^+)$, then by previous computation, by (3.2) and by (E2), we have

$$\int_\Omega e_\varepsilon \alpha(z_\varepsilon^+, z_\varepsilon^+)\, dm + 2t_\varepsilon \int_\Omega (z_\varepsilon^+)^2 e_\varepsilon \alpha(z_\varepsilon^+, z_\varepsilon^+)\, dm + \lambda_0 \int_\Omega z_\varepsilon z_\varepsilon^+ e_\varepsilon\, dm$$

$$+ \int_\Omega z_\varepsilon z_\varepsilon^+ e_\varepsilon\, d\mu = \int_\Omega \Phi_\varepsilon(u_\varepsilon)\, e_\varepsilon z_\varepsilon^+\, dm - c_0 \int_\Omega e_\varepsilon z_\varepsilon^+\, dm - \frac{c_0}{\lambda_0} \int_\Omega e_\varepsilon z_\varepsilon^+\, d\mu$$

$$\leqslant b(\|u_\varepsilon\|_\infty) \int_\Omega \alpha(u_\varepsilon, u_\varepsilon)\, e_\varepsilon z_\varepsilon^+\, dm - \frac{c_0}{\lambda_0} \int_\Omega e_\varepsilon z_\varepsilon^+\, d\mu$$

$$\leqslant b(\|u_\varepsilon\|_\infty) \int_\Omega e_\varepsilon \alpha(u_\varepsilon, u_\varepsilon)^{1/2} z_\varepsilon^+ \alpha(u_\varepsilon, u_\varepsilon)^{1/2}\, dm$$

$$= b(\|u_\varepsilon\|_\infty) \int_\Omega e_\varepsilon \alpha(z_\varepsilon^+, z_\varepsilon^+)^{1/2} z_\varepsilon^+ \alpha(z_\varepsilon^+, z_\varepsilon^+)^{1/2}\, dm$$

$$\leqslant \frac{1}{2} \int_\Omega e_\varepsilon \alpha(z_\varepsilon^+, z_\varepsilon^+)\, dm + \frac{b^2(\|u_\varepsilon\|_\infty)}{2} \int_\Omega e_\varepsilon (z_\varepsilon^+)^2 \alpha(z_\varepsilon^+, z_\varepsilon^+)\, dm$$

where we have used the fact that

$$z_\varepsilon^+ \alpha(z_\varepsilon^+, z_\varepsilon^+) = z_\varepsilon^+ \alpha(z_\varepsilon, z_\varepsilon) = z_\varepsilon^+ \alpha(u_\varepsilon, u_\varepsilon).$$

Finally, by the definition of t_ε the last two terms are involved in the corresponding terms that reduce the first term in the left hand-side, so

$$\frac{1}{2} \int_\Omega e_\varepsilon \alpha(z_\varepsilon^+, z_\varepsilon^+)\, dm + \frac{b^2(\|u_\varepsilon\|_\infty)}{2} \int_\Omega e_\varepsilon (z_\varepsilon^+)^2 \alpha(z_\varepsilon^+, z_\varepsilon^+)\, dm \leqslant 0$$

then, since $e_\varepsilon \geqslant 1$ we have $z_\varepsilon^+ = 0$ and $u_\varepsilon \leqslant \frac{c_0}{\lambda_0}$. The inequality $u_\varepsilon \geqslant -\frac{c_0}{\lambda_0}$ can be proved by the same methods.

Step 3: Uniform estimate in $D_0(a, \Omega)$.

Let $c_1 = b\left(\frac{c_0}{\lambda_0}\right)$. To show the uniform estimate in $D_0(a, \Omega)$, we use as test function in (3.1_ε) the function $T(u_\varepsilon) \in V_\mu^0(\Omega)$ where $T(v) = v \exp(tv^2)$ and $t = \frac{c_1^2}{2}$. In the following we denote by $E_\varepsilon = \exp(tu_\varepsilon^2)$. We have

$$\int_\Omega \alpha(u_\varepsilon, T(u_\varepsilon)) + \lambda_0 \int_\Omega u_\varepsilon T(u_\varepsilon)\, dm + \int_\Omega u_\varepsilon T(u_\varepsilon)\, d\mu = \int_\Omega \Phi_\varepsilon(u_\varepsilon)\, T(u_\varepsilon)\, dm$$

then, by chain rule,

$$\int_\Omega (E_\varepsilon + 2tu_\varepsilon^2 E_\varepsilon)\, \alpha(u_\varepsilon, u_\varepsilon)\, dm + \lambda_0 \int_\Omega u_\varepsilon^2 E_\varepsilon\, dm + \int_\Omega u_\varepsilon^2 E_\varepsilon\, d\mu$$

$$= \int_\Omega \Phi_\varepsilon(u_\varepsilon)\, E_\varepsilon u_\varepsilon\, dm$$

$$\leqslant \int_\Omega (c_0 + b(|u_\varepsilon|)\, \alpha(u_\varepsilon, u_\varepsilon))\, E_\varepsilon\, |u_\varepsilon|\, dm$$

$$\leqslant \int_\Omega c_0\, |u_\varepsilon| E_\varepsilon\, dm + \int_\Omega c_1\, |u_\varepsilon| E_\varepsilon\, \alpha(u_\varepsilon, u_\varepsilon)\, dm$$

$$\leqslant c_0 \exp\left(t \frac{c_0^2}{\lambda_0^2}\right) \frac{c_0}{\lambda_0} m(\Omega) + \int_\Omega \frac{E_\varepsilon}{2} \alpha(u_\varepsilon, u_\varepsilon)\, dm + \int_\Omega \frac{E_\varepsilon c_1^2 u_\varepsilon^2}{2} \alpha(u_\varepsilon, u_\varepsilon)\, dm$$

then

$$\frac{1}{2}\int_\Omega E_\varepsilon\, \alpha(u_\varepsilon, u_\varepsilon)\, dm + \frac{1}{2}\int_\Omega E_\varepsilon c_1^2 u_\varepsilon^2\, \alpha(u_\varepsilon, u_\varepsilon)\, dm + \lambda_0 \int_\Omega u_\varepsilon^2 E_\varepsilon\, dm + \int_\Omega u_\varepsilon^2 E_\varepsilon\, d\mu$$

$$\leqslant c_0 \exp\left(t \frac{c_0^2}{\lambda_0^2}\right) \frac{c_0}{\lambda_0} m(\Omega)\,.$$

Since $E_\varepsilon > 1$ we have

$$\int_\Omega u_\varepsilon^2\, d\mu + \int_\Omega \alpha(u_\varepsilon, u_\varepsilon)\, dm \leqslant K := 2\, \frac{c_0^2}{\lambda_0} \exp\left(\frac{c_0^2 c_1^2}{2\lambda_0^2}\right) m(\Omega)\,, \tag{3.5}$$

which means that u_ε is uniformly bounded in $V_\mu^0(\Omega)$. Extracting a subsequence (still denoted by u_ε), we have proved the existence of a function $u \in V_\mu^0(\Omega) \cap L^\infty(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ weakly in $D_0(a, \Omega)$, $u_\varepsilon \rightharpoonup u$ weakly in $L_\mu^2(\Omega)$, weakly * in $L_m^\infty(\Omega)$ and $u_\varepsilon \to$ $\to u$ a.e. in Ω, then $u_\varepsilon \to u$ strongly in $L_m^p(\Omega)$ for any $1 \leqslant p < \infty$. Hence we conclude in particular that

$$\|u\|_\infty \leqslant \frac{c_0}{\lambda_0} \quad \text{and} \quad \|u\|_{a_1} \leqslant K\,.$$

Note that the $L_m^\infty(\Omega)$ bound as well as the $D_0(a, \Omega)$ bound do not depend on the measure μ but only on c_0, λ_0, c_1 and Ω.

Step 4: The sequence u_ε converges strongly in $D_0(a, \Omega)$ and in $L^2_\mu(\Omega)$ to the function u.

Let ε and η be two positive parameters and u_ε and u_η be the corresponding solutions of (3.1_ε) and (3.1_η). Let $T(v) = v \exp(tv^2)$, $t = 16 c_1^2$ and $c_1 = b(c_0/\lambda_0)$. Subtracting (3.1_η) from (3.1_ε) and using the test function $T(u_\varepsilon - u_\eta)$ which belongs to $V^0_\mu(\Omega)$, we obtain

$$
\begin{aligned}
&\int_\Omega \alpha(u_\varepsilon - u_\eta, T(u_\varepsilon - u_\eta))\, dm + \lambda_0 \int_\Omega (u_\varepsilon - u_\eta)\, T(u_\varepsilon - u_\eta)\, dm \\
&+ \int_\Omega (u_\varepsilon - u_\eta)\, T(u_\varepsilon - u_\eta)\, d\mu \\
&= \int_\Omega [\Phi_\varepsilon(u_\varepsilon) - \Phi_\eta(u_\eta)]\, T(u_\varepsilon - u_\eta) \\
&\leqslant \int_\Omega [2c_0 + c_1 \alpha(u_\varepsilon, u_\varepsilon) + c_1 \alpha(u_\eta, u_\eta)]\, |T(u_\varepsilon - u_\eta)|\, dm
\end{aligned} \tag{3.6}
$$

where we have use the hypothesis (E2) on $\Phi_\eta(u_\eta)$ and on $\Phi_\varepsilon(u_\varepsilon)$.

Let us observe now that for any $u, v \in D(a)$, since the density $\alpha(u, v)(\cdot)$ is a symmetric bilinear form such that $\alpha(u, u)(\cdot) \geqslant 0$, then

$$
|\alpha(u, v)(\cdot)| \leqslant \frac{1}{2} \alpha(u, u)(\cdot) + \frac{1}{2} \alpha(v, v)(\cdot) \tag{3.7}
$$

that implies

$$
\begin{aligned}
&\alpha(u, u)(\cdot) = \alpha(u - v + v, u - v + v)(\cdot) \\
&= \alpha(u - v, u - v)(\cdot) + \alpha(v, v)(\cdot) + 2\alpha(u - v, v)(\cdot) \\
&\leqslant 2\alpha(u - v, u - v)(\cdot) + 2\alpha(v, v)(\cdot)
\end{aligned}
$$

hence

$$
\begin{aligned}
&\int_\Omega \alpha(u_\varepsilon - u_\eta, T(u_\varepsilon - u_\eta))\, dm + \lambda_0 \int_\Omega (u_\varepsilon - u_\eta)\, T(u_\varepsilon - u_\eta)\, dm \\
&+ \int_\Omega (u_\varepsilon - u_\eta)\, T(u_\varepsilon - u_\eta)\, d\mu \leqslant \\
&\int_\Omega 2c_0\, |T(u_\varepsilon - u_\eta)|\, dm + 2 \int_\Omega c_1\, |T(u_\varepsilon - u_\eta)|\, [\alpha(u_\varepsilon - u_\eta, u_\varepsilon - u_\eta) + \alpha(u_\varepsilon, u_\varepsilon)]\, dm \\
&+ \int_\Omega c_1\, |T(u_\varepsilon - u_\eta)|\, \alpha(u_\varepsilon, u_\varepsilon)\, dm \leqslant
\end{aligned}
$$

$$\int_\Omega 2c_0\,|T(u_\varepsilon - u_\eta)|\,dm + \int_\Omega 3c_1\,|T(u_\varepsilon - u_\eta)|\,\alpha(u_\varepsilon, u_\varepsilon)\,dm$$

$$+2\int_\Omega c_1\,|T(u_\varepsilon - u_\eta)|\,\alpha(u_\varepsilon - u_\eta, u_\varepsilon - u_\eta)\,dm\,.$$

Since the second and third integrals of the left hand side are non-negative, we get by the chain rule

$$\tag{3.8} \begin{aligned} &\int_\Omega [T'(u_\varepsilon - u_\eta) - 2c_1\,|T(u_\varepsilon - u_\eta)|\,]\,\alpha(u_\varepsilon - u_\eta, u_\varepsilon - u_\eta)\,dm \\ &\leqslant \int_\Omega 2c_0\,|T(u_\varepsilon - u_\eta)|\,dm + \int_\Omega 3c_1\,|T(u_\varepsilon - u_\eta)|\,\alpha(u_\varepsilon, u_\varepsilon)\,dm\,. \end{aligned}$$

Let us observe that the left-hand side is weak lower semicontinuous. Indeed,

$$\int_\Omega \alpha(u_\varepsilon - u_\eta, u_\varepsilon - u_\eta)\,F(u_\varepsilon - u_\eta)\,dm =$$

$$\int_\Omega \alpha(G(u_\varepsilon - u_\eta), G(u_\varepsilon - u_\eta))\,dm$$

where $F(u_\varepsilon - u_\eta) := T'(u_\varepsilon - u_\eta) - 2c_1\,|T(u_\varepsilon - u_\eta)| > 0$ (by the choice of t) and $G'(u_\varepsilon - u_\eta) := F^{1/2}(u_\varepsilon - u_\eta)$, hence by the weak lower semicontinuity of the form a, it results that

$$a(G(u_\varepsilon - u), G(u_\varepsilon - u)) \leqslant \liminf_{\eta\to 0} a(G(u_\varepsilon - u_\eta), G(u_\varepsilon - u_\eta))$$

$$= \liminf_{\eta\to 0} \int_\Omega \alpha(u_\varepsilon - u_\eta, u_\varepsilon - u_\eta)\,F(u_\varepsilon - u_\eta)\,dm\,.$$

Now let η go to zero in (3.8). By the results of Step 3 on the sequence u_η, the continuity of functions T and T' and the weak lower semicontinuity of the left-hand side, we easily pass to the limit in (3.8), so that

$$\begin{aligned} &\int_\Omega [T'(u_\varepsilon - u) - 2c_1\,|T(u_\varepsilon - u)|\,]\,\alpha(u_\varepsilon - u, u_\varepsilon - u)\,dm \\ &\leqslant \int_\Omega 2c_0\,|T(u_\varepsilon - u)|\,dm + \int_\Omega 3c_1\,|T(u_\varepsilon - u)|\,\alpha(u_\varepsilon, u_\varepsilon)\,dm \\ &\leqslant \int_\Omega 2c_0\,|T(u_\varepsilon - u)|\,dm + \int_\Omega 6c_1\,|T(u_\varepsilon - u)|\,[\alpha(u_\varepsilon - u, u_\varepsilon - u) + \alpha(u, u)]\,dm \end{aligned}$$

then, by the choice of t that implies $T'(v) - 8c_1 |T(v)| \geq \frac{1}{2}$, we have

$$\frac{1}{2}\int_\Omega \alpha(u_\varepsilon - u, u_\varepsilon - u)\, dm \leq \int_\Omega [T'(u_\varepsilon - u) - 8c_1 |T(u_\varepsilon - u)|]\, \alpha(u_\varepsilon - u, u_\varepsilon - u)\, dm$$

$$\leq \int_\Omega 2c_0 |T(u_\varepsilon - u)|\, dm + \int_\Omega 6c_1 |T(u_\varepsilon - u)|\alpha(u, u)\, dm .$$

Since the last integral tends to zero as ε tends to zero, we have proved that u_ε tends to u strongly in $D_0(a, \Omega)$. The convergence is strong in $L^2_\mu(\Omega)$ too, since, coming back to the first inequality in (3.6), we get

$$\int_\Omega |u_\varepsilon - u_\eta|^2\, d\mu \leq \int_\Omega |u_\varepsilon - u_\eta|\, |T(u_\varepsilon - u_\eta)|\, d\mu$$

$$\leq \int_\Omega [2c_0 + c_1\alpha(u_\varepsilon, u_\varepsilon) + c_1\alpha(u_\eta, u_\eta)]\, |T(u_\varepsilon - u_\eta)|\, dm$$

that tends to zero as $\varepsilon \to 0$ and $\eta \to 0$ since $u_\varepsilon \to u$ and $u_\eta \to u$ strongly in $D_0(a, \Omega)$.

Step 5: Passing to the limit in (3.1_ε) *and proving that u is a solution of problem* (3.1).

By Step 4 we know that u_ε tends to u strongly in $D_0(a, \Omega)$ (up to the extraction of a subsequence) that implies $\alpha(u_\varepsilon, u_\varepsilon) \to \alpha(u, u)$ a.e. in Ω. Then, by the continuity of ϕ with respect to (s, p),

$$\Phi_\varepsilon(u_\varepsilon) = \frac{\phi(x, u_\varepsilon, \alpha^{1/2}(u_\varepsilon, u_\varepsilon))}{1 + \varepsilon\phi(x, u_\varepsilon, \alpha^{1/2}(u_\varepsilon, u_\varepsilon))} \to \phi(x, u, \alpha^{1/2}(u, u)) = \Phi(u) \quad \text{a.e. in } \Omega .$$

Moreover, since $\alpha(u_\varepsilon, u_\varepsilon) \to \alpha(u, u)$ in $L^1_m(\Omega)$ hence, by Vitali's Theorem, for any subset $E \subset \Omega$,

$$\lim_{m(E)\to 0} \int_E \alpha(u_\varepsilon, u_\varepsilon)\, dm = 0$$

uniformly with respect to ε, and by (E2) with $c_1 = b\left(\frac{c_0}{\lambda_0}\right)$

$$\int_E |\Phi_\varepsilon(u_\varepsilon)|\, dm \leq \int_E [c_0 + c_1\alpha(u_\varepsilon, u_\varepsilon)]\, dm$$

then

$$\lim_{m(E)\to 0} \int_E |\Phi_\varepsilon(u_\varepsilon)|\, dm \leq \lim_{m(E)\to 0} c_0\, m(E) + \lim_{m(E)\to 0} c_1 \int_E \alpha(u_\varepsilon, u_\varepsilon)\, dm = 0$$

uniformly with respect to ε. Applying Vitali's theorem again,

$$\Phi_\varepsilon(u_\varepsilon) = \frac{\phi(x, u_\varepsilon, \alpha^{1/2}(u_\varepsilon, u_\varepsilon))}{1+\varepsilon\phi(x, u_\varepsilon, \alpha^{1/2}(u_\varepsilon, u_\varepsilon))} \to \phi(x, u, \alpha^{1/2}(u, u)) = \Phi(u) \text{ strongly in } L^1_m(\Omega).$$

This shows that u solves problem (3.1), as we wanted to prove. ■

REMARK 3.4: As already observed in [7] Remark 3.3 and in [16] Remark 2.2, the strict positivity of λ_0 is essential in the proof of Theorem 3.2 because it allows us to obtain the $L^\infty_m(\Omega)$ bound on u_ε. Since the term containing the measure could degenerate somewhere in Ω (either with $\mu \equiv 0$ or with $\mu \equiv +\infty$), the existence of a solution is no more guaranteed in the absence of the zero-order term in the operator. As a counterexample, one could consider which one considered by J. L. Kazdan in [20] (see also [7] Counter-ex. 3.1) with an extra term μu with either $\mu \equiv 0$ or or $\mu \equiv \infty_E$, E being a closed subset of Ω and ∞_E being the measure defined by

$$\infty_E(B) = \begin{cases} 0 & \text{if } \operatorname{cap}^a(B \cap E, \Omega) = 0 \\ +\infty & \text{otherwise}. \end{cases}$$

Of course, hypothesis (E3) can be replaced by

$$\lambda_0 \geq 0, \qquad \mu \in \mathcal{M}^a_0(\Omega), \qquad \mu + \lambda_0\, dm \geq a_0\, dm$$

for a strictly positive constant a_0.

4. - HOMOGENIZATION

In this section we study the convergence of the solutions of (3.1) when the measure μ varies. For sake of simplicity we consider only non-linear terms that do not depend explicitly on u_ε. More precisely, let us consider the sequence of problems:

$$(4.1_\varepsilon) \qquad \begin{cases} u_\varepsilon \in V^0_{\mu_\varepsilon}(\Omega) \cap L^\infty_m(\Omega) \\ a_{\lambda_0}(u_\varepsilon, v) + \int_\Omega u_\varepsilon v\, d\mu_\varepsilon = \int_\Omega \phi(x, \alpha^{1/2}(u_\varepsilon, u_\varepsilon))\, v\, dm \\ \forall v \in V^0_{\mu_\varepsilon}(\Omega) \cap L^\infty_m(\Omega) \end{cases}$$

where $\lambda_0 > 0$, $\{\mu_\varepsilon\} \in \mathcal{M}^a_0(\Omega)$ and Ω is a relatively compact subset of X. Let suppose that Ω is uniformly regular too, that is

DEFINITION 4.1: *Ω is an uniformly regular domain with respect to the form a if for any* $x \in \partial\Omega$

$$\lim_{r\to 0} \inf_{x\in\partial\Omega} \int_r^R \frac{\mathrm{cap}^a(\Omega \cap B(x, s), B(x, 2s))}{\mathrm{cap}^a(B(x, s), B(x, 2s))} \frac{ds}{s} = +\infty .$$

Let us underline that this assumption is necessary in order to prove the corrector result enunciated in Proposition 4.7 that play an important role in the proof of the homogenization theorem (Theorem 4.10). Let us recall the definition of γ-convergence of a sequence of measures in space $\mathcal{M}_0^a(\Omega)$. For any measure $\mu \in \mathcal{M}_0^a(\Omega)$, let us consider the following functional F^μ defined on $L^2_m(\Omega) \cap L^2_\mu(\Omega)$:

$$F^\mu(v) := \begin{cases} \int_\Omega \alpha(v, v)\, dm + \int_\Omega v^2\, d\mu \ \text{ if } \ v \in V^0_\mu(\Omega) \\ +\infty \ \text{ otherwise .} \end{cases} \tag{4.1}$$

DEFINITION 4.2: *Let ε be a sequence of positive numbers converging to zero, $\{\mu^\varepsilon\}$ a sequence of measures in the space $\mathcal{M}_0^a(\Omega)$ and $\mu \in \mathcal{M}_0^a(\Omega)$. Let F^{μ_ε} and F^μ the functionals associated with $\{\mu_\varepsilon\}$ and μ, as in* (4.1). *Then*

$$\mu_\varepsilon\, \gamma\text{-converges to } \mu$$

if the sequence of functionals F^{μ_ε} Γ-converges in the sense of De-Giorgi and Franzoni [14] *to the functional F^μ.*

As in the classical case (i.e. $a(u, v) = \int_\Omega \nabla u \nabla v\, dx$), it is possible to prove that the Γ-convergence of F^{μ_ε} to F^μ is equivalent to the $L^2_m(\Omega)$-convergence of the solutions $\bar{u}_\varepsilon$ of the homogeneous relaxed Dirichlet problem with respect to the form a, the function $f \in D^{-1}(a, \Omega)$ -dual space of $D_0(a, \Omega)$- and the sequence $\{\mu_\varepsilon\}$:

$$\begin{cases} \bar{u}_\varepsilon \in V^0_{\mu_\varepsilon}(\Omega) \\ a(\bar{u}_\varepsilon, v) + \int_\Omega \bar{u}_\varepsilon v\, d\mu_\varepsilon = \langle f, v\rangle \qquad \forall v \in V^0_{\mu_\varepsilon}(\Omega) \end{cases} \tag{4.2$_\varepsilon$}$$

to the solution $\bar{u}$ of the homogeneous relaxed Dirichlet problem with respect to the form a, the function $f \in D^{-1}(a, \Omega)$ and the sequence $\{\mu\}$:

$$\begin{cases} \bar{u} \in V^0_\mu(\Omega) \\ a(\bar{u}, v) + \int_\Omega \bar{u} v\, d\mu = \langle f, v\rangle \qquad \forall v \in V^0_\mu(\Omega) \end{cases} \tag{4.2}$$

for every $f \in D^{-1}(a, \Omega)$. It is easy to prove that $\bar{u}_\varepsilon \rightharpoonup \bar{u}$ weakly in $D_0(a, \Omega)$.

We want to prove a stability property for bounded solutions with respect to the γ-convergence of measures when the limit measure is sufficiently regular, making essential use of the correctors result of Biroli, Picard, Tchou. To this aim we recall here some definitions and properties on Kato measures and correctors.

Firstly, let us recall (see [3]) the notion of Kato measure associated with a regular Dirichlet form.

DEFINITION 4.3: *Let Ω be a relatively compact open subset of X with* $2\,\mathrm{diam}(\Omega) = R < R_0$ *where R_0 is the constant which appears in doubling condition* (D) *of Section* 2. *Assume that there exists $x_0 \in \Omega$ with $B(x_0, 4R) \subset\subset X$ and $B(x_0, 4R) \neq X$. We say that μ is a Kato measure on Ω if μ is a Radon measure on Ω such that*

$$\lim_{r\to 0} \sup_{x\in\Omega} \int_{\Omega\cap B(x,r)} \left(\int_{d(x,y)}^{R} \frac{s}{m(B(x,s))}\, ds \right) |\mu|(dy) = 0$$

where $|\mu|$ denotes the total variation of the measure μ. In [1], M. Biroli has studied some properties of weak Kato measure associated with a regular Dirichlet form.

DEFINITION 4.4: *Let Ω and X as in the previous definition. We say that μ is a weak Kato measure on Ω if μ is a Radon measure on Ω such that*

$$\lim_{r\to 0} \sup_{x\in\Omega} \int_{\Omega\cap B(x,r)} \left(\int_{d(x,y)}^{40r} \frac{s}{m(B(x,s))}\, ds \right) |\mu|(dy) = 0\,.$$

Let now z_ε be the solutions of the problems

$$\text{(4.3)}\qquad \begin{cases} z_\varepsilon \in V_{\mu_\varepsilon}(\Omega) = D(a,\Omega) \cap L^2_{\mu_\varepsilon}(\Omega) \quad z_\varepsilon - 1 \in D_0(a,\Omega) \\ a(z_\varepsilon, v) + \displaystyle\int_\Omega z_\varepsilon v\, d\mu_\varepsilon = 0 \quad \forall v \in V^0_{\mu_\varepsilon}(\Omega) \end{cases}$$

and let z be the solution of the problem

$$\text{(4.4)}\qquad \begin{cases} z \in V_\mu(\Omega) = D(a,\Omega) \cap L^2_\mu(\Omega), \quad z - 1 \in D_0(a,\Omega) \\ a(z, v) + \displaystyle\int_\Omega zv\, d\mu = 0 \quad \forall v \in V^0_\mu(\Omega)\,. \end{cases}$$

REMARKS 4.5: As in the classical case we always suppose that

$$D(a,\Omega) \cap L^2_{\mu_\varepsilon}(\Omega) \cap \{v \in D(a,\Omega) : v - 1 \in D_0(a,\Omega)\} \neq \emptyset$$

and

$$D(a,\, \Omega) \cap L^2_\mu(\Omega) \cap \{v \in D(a,\, \Omega) : v - 1 \in D_0(a,\, \Omega)\} \neq \emptyset\,.$$

By comparison principle it is simple to show that

$$0 \leqslant z_\varepsilon,\, z \leqslant 1\,.$$

Moreover, as in [11] (Prop. 5.13) it is easy to prove that if

$$\text{supp}\, \mu_\varepsilon \subset\subset \Omega \text{ and } \mu_\varepsilon\, \gamma\text{-converges to } \mu$$

then the sequence z_ε converges to z strongly in $L^2_m(\Omega)$ and weakly in $D(a,\, \Omega)$.

We are now in position to state the definition of correctors.

Definition 4.6: *If the sequence μ_ε γ-converges to μ, a sequence w_ε in V_{μ_ε} is said to be a sequence of correctors for the problem (4.1_ε) if for any ϕ, defining u_ε and u as solutions of (4.1_ε) for μ_ε and μ respectively, one has, as ε tends to* 0

$$u_\varepsilon - w_\varepsilon u \to 0 \ \text{ strongly in } D_0(a,\, \Omega)\,.$$

In the following correctors result we would like to define the correctors as the quotient $w_\varepsilon := z_\varepsilon / z$ where z_ε and z are respectively the solutions of (4.3) and (4.4). In order to do it we have to assume that there exists a positive constant $\delta > 0$ such that

$$z \geqslant \delta > 0 \ \text{ m-a.e. in } \Omega\,.$$

Actually, using continuity arguments and Harnack inequality proved in Theorem 4.3 in [1], this assumption is satisfied if μ is a weak Kato measure and Ω is uniformly regular.

Proposition 4.7: *Let Ω be an uniformly regular domain* (*see Definition* 4.1), *μ a weak positive Kato measure* (*see Definition* 4.4) *and f a weak Kato measure. Then the functions*

$$w_\varepsilon := \frac{z_\varepsilon}{z} \tag{4.5}$$

where z_ε and z are defined in (4.3) *and* (4.4), *belongs to $V_{\mu_\varepsilon}(\Omega) \cap L^\infty_m(\Omega)$. This sequence is a corrector sequence since there exist a function $\theta(t) : [0,\, T_0] \to \mathbb{R}$ such that*

$\theta(t) \to 0$ as $t \to 0$ and the following inequality holds for u_ε and u, defined in (4.1_ε) for μ_ε and μ respectively, with $\|u_\varepsilon - w_\varepsilon u\|_{L^2_m} \leq T_0$,

$$\int_\Omega \alpha(u_\varepsilon - w_\varepsilon u,\, u_\varepsilon - w_\varepsilon u)(dx) + \int_\Omega |u_\varepsilon - w_\varepsilon u|^2 \, d_{\mu_\varepsilon} \leq$$
$$C\theta(\|w_\varepsilon - 1\|_{L^2_m}) .$$

Proof: See Theorem 4.1 in [4]. ■

In order to prove our homogenization problem, in the following we will often use the following proposition.

Proposition 4.8: *Let us assume that $\psi \in D(a, \Omega) \cap L^\infty_m(\Omega)$ and that*

$$\|v_\varepsilon\|_\infty \leq C \quad \text{for some constant } C$$
$$v_\varepsilon \rightharpoonup v \text{ a.e. with respect to } m$$
$$v \in D(a, \Omega) \cap L^\infty_m(\Omega)$$

and the sequence $w_\varepsilon \in D_0(a, \Omega)$ verifies

$$\|w_\varepsilon\|_\infty \leq C \quad \text{and } w_\varepsilon \rightharpoonup w \quad \text{weakly in} \quad D_0(a, \Omega)$$

then

$$\int_\Omega \alpha(w_\varepsilon, \psi)\, v_\varepsilon \, dm \to \int_\Omega \alpha(w, \psi)\, v \, dm .$$

Proof: See Lemma 3.3 in [5]. ■

Let $\phi(x, p)$ with $x \in \Omega$, $p \in \mathbb{R}^+$ be a Carathéodory function. We assume the following hypothesis:

$$\text{(H1)} \qquad \begin{cases} |\phi(x, p_1) - \phi(x, p_2)| \leq K(1 + p_1^{s-\gamma} + p_2^{s-\gamma})\, |p_1 - p_2|^\gamma \\ \text{for any } p_1, p_2 \in \mathbb{R}^+ \text{ with } 0 < \gamma \leq 1 \text{ and } \gamma \leq s < 2 \,; \\ |\phi(x, 0)| \leq c_0 \end{cases}$$

$$\text{(H2)} \qquad \begin{cases} \lambda_0 > 0, V_{\mu_\varepsilon}(\Omega) \cap \{v \in D(a, \Omega) : v - 1 \in D_0(a, \Omega)\} \neq \emptyset \\ \mu_0 \text{ weak positive Kato measure } V_{\mu_0}(\Omega) \cap \{v \in D(a, \Omega) : v - 1 \in D_0(a, \Omega)\} \neq \emptyset \\ \mu_\varepsilon \gamma\text{-converges to } \mu_0, \text{supp } \mu_\varepsilon \subset\subset \Omega . \end{cases}$$

Note that (H1) implies in particular that

(4.6) $$|\phi(x, p)| \leq c_1(1 + |p|^s) \text{ for any } p \in \mathbb{R}^+ \; s < 2$$

so hypothesis (E2) is satisfied. Therefore the existence result of Theorem 3.2 holds in this case. On the other hand, hypothesis (H2) allows us to use the corrector result of Proposition 4.7 and the properties described in Remarks 4.5. We also assume an additional hypothesis on the correctors w_ε defined by (4.5) We assume that

(H3) $$\alpha(w_\varepsilon, w_\varepsilon) \to 0 \text{ a.e. in } \Omega .$$

REMARK 4.9: Hypothesis (H3) holds -for example- in the classical case for a periodically perforated domains with holes of critical size (see [16], [6], [21]). Moreover (H3) holds in the classical case for the correctors considered by Casado-Diaz in [8] and Dal Maso Murat in [13].

We are going to prove the following theorem

THEOREM 4.10: *Assume* (H1), (H2), *and* (H3), *and let* $u_\varepsilon \in V^0_{\mu_\varepsilon}(\Omega) \cap L^\infty_m(\Omega)$ *be any sequence of solutions of* (4.1$_\varepsilon$) *with* $\|u_\varepsilon\|_\infty \leq c_0/\lambda_0$. *Up to the extraction of a subsequence we have*

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } D_0(a, \Omega)$$

where u_0 *is a solution of*

(4.7) $$\begin{cases} u_0 \in V^0_{\mu_0}(\Omega) \cap L^\infty_m(\Omega) \\ a_{\lambda_0}(u_0, v) + \int_\Omega u_0 v \, d\mu_0 = \int_\Omega \phi(x, \alpha^{1/2}(u_0, u_0)(x)) \, v \, dm \;\; \forall v \in V^0_{\mu_0}(\Omega) \cap L^\infty_m(\Omega) . \end{cases}$$

PROOF: From now on, in order to simplify the writing, we pose, for any $u \in D(a, \Omega) \cap L^\infty_m(\Omega)$

$$\Phi(u) := \phi(x, \alpha^{1/2}(u, u)(x)) .$$

Step 1: Bounds for the solutions u_ε *and* $\Phi(u_\varepsilon)$.

In view of (4.6), by Theorem 3.2, we know the existence of at least one solution u_ε of (3.1$_\varepsilon$) with $\|u_\varepsilon\|_\infty \leq \frac{c_0}{\lambda_0}$. It also follows from the proof of Theorem 3.2 that

(4.8) $$\|u_\varepsilon\|_{a_1} + \|u_\varepsilon\|_{\mu_\varepsilon} \leq K$$

and the constant depends only on c_1, λ_0 and Ω. We can thus extract a subsequence (still denoted by u_ε) such that $u_\varepsilon \rightharpoonup u_0$ weakly in $D_0(a, \Omega)$, $u_\varepsilon \to u_0$ strongly in $L^p_m(\Omega)$

for any $p < \infty$, weakly* in $L^\infty_m(\Omega)$ and a.e. in Ω. Taking $q = \frac{2}{s} > 1$, we get

$$\int_\Omega |\Phi(u_\varepsilon)|^q \, dm \leq c_1^q \int_\Omega (1 + \alpha(u_\varepsilon, u_\varepsilon)^{s/2})^q \, dm \leq$$

$$const. \int_\Omega (1 + \alpha(u_\varepsilon, u_\varepsilon)) \, dm \leq \text{const}.$$

Extracting a new subsequence, there exists a function Φ_0 such that $\Phi(u_\varepsilon) \rightharpoonup \Phi_0$ weakly in $L^q_m(\Omega)$.

Step 2: A first passage to the limit in (4.1$_\varepsilon$).

In this step and in Step 3, we use the corrector result. Let us consider the sequence of functions w_ε defined by (4.5) and let us mention below the main properties.

(4.9) $$w_\varepsilon \rightharpoonup 1 \quad \text{weakly in } D(a, \Omega),$$

indeed, as we have observed in Remarks 4.5, $z_\varepsilon \rightharpoonup z$ weakly in $D(a, \Omega)$ and $\frac{1}{z} \in D(a, \Omega) \cap L^\infty_m(\Omega)$ then $\frac{z_\varepsilon}{z} \rightharpoonup 1$ weakly in $D(a, \Omega)$. This implies

(4.10) $$w_\varepsilon \to 1 \quad \text{strongly in } L^2_m(\Omega).$$

Moreover, as a simple consequence of what observed just before Proposition 4.7,

(4.11) $$\|w_\varepsilon\|_\infty \leq \frac{1}{\delta}$$

and then by (4.10) and (4.11)

(4.12) $w_\varepsilon \to 1$ strongly in $L^p_m(\Omega)$ for any $1 \leq p < \infty$ and weakly* in $L^\infty_m(\Omega)$.

We claim

(4.13) $$\begin{cases} v_\varepsilon \in V^0_{\mu_\varepsilon}(\Omega) \cap L^\infty_m(\Omega), \quad \|v_\varepsilon\|_\infty \leq \text{const.}, \; v_0 \in V^0_{\mu_0}(\Omega) \\ v_\varepsilon \rightharpoonup v_0 \text{ weakly in } D_0(a, \Omega) \Rightarrow \\ \int_\Omega \alpha(v_\varepsilon, w_\varepsilon) \, dm + \int_\Omega v_\varepsilon w_\varepsilon \, d\mu_\varepsilon \to \int_\Omega v_0 \, d\mu_0 \quad \text{as } \varepsilon \to 0. \end{cases}$$

Indeed, by Leibnitz rule and by (4.3) with $v = \frac{v_\varepsilon}{z}$ we have

$$\int_\Omega \alpha(v_\varepsilon, w_\varepsilon) \, dm + \int_\Omega v_\varepsilon w_\varepsilon \, d\mu_\varepsilon =$$

$$\int_\Omega \alpha\left(v_\varepsilon, \frac{z_\varepsilon}{z}\right) dm + \int_\Omega \frac{z_\varepsilon}{z} v_\varepsilon \, d\mu_\varepsilon =$$

$$\int_\Omega \frac{1}{z}\,\alpha(v_\varepsilon, z_\varepsilon)\,dm + \int_\Omega z_\varepsilon\,\alpha\left(v_\varepsilon, \frac{1}{z}\right) dm + \int_\Omega \frac{z_\varepsilon}{z}\,v_\varepsilon\,d\mu_\varepsilon =$$

$$\int_\Omega \alpha\left(\frac{v_\varepsilon}{z}, z_\varepsilon\right) dm - \int_\Omega v_\varepsilon\,\alpha\left(\frac{1}{z}, z_\varepsilon\right) dm + \int_\Omega z_\varepsilon\,\alpha\left(v_\varepsilon, \frac{1}{z}\right) dm + \int_\Omega \frac{z_\varepsilon v_\varepsilon}{z}\,d\mu_\varepsilon =$$

$$\int_\Omega z_\varepsilon\,\alpha\left(v_\varepsilon, \frac{1}{z}\right) dm - \int_\Omega v_\varepsilon\,\alpha\left(\frac{1}{z}, z_\varepsilon - 1\right) dm\,.$$

Let us observe that $\frac{1}{z} \in D(a, \Omega) \cap L^\infty_m(\Omega)$, v_ε, $(z_\varepsilon - 1) \in D_0(a, \Omega) \cap L^\infty_m(\Omega)$. Moreover $v_\varepsilon \rightharpoonup v_0$, $z_\varepsilon - 1 \rightharpoonup z - 1$ weakly in $D_0(a, \Omega)$, thus letting $\varepsilon \to 0$ by Proposition 4.8, by using strong locality of the form, it results that

$$\begin{aligned}
&\int_\Omega \alpha(v_\varepsilon, w_\varepsilon)\,dm + \int_\Omega v_\varepsilon w_\varepsilon\,d\mu_\varepsilon \\
&\to \int_\Omega z\alpha\left(v_0, \frac{1}{z}\right) dm - \int_\Omega v_0\,\alpha\left(\frac{1}{z}, z - 1\right) dm = \\
&- \int_\Omega \frac{1}{z}\,\alpha(v_0, z)\,dm - \int_\Omega v_0\,\alpha\left(\frac{1}{z}, z\right) dm = \\
&- \int_\Omega \alpha\left(\frac{v_0}{z}, z\right) dm = \int_\Omega \frac{v_0}{z}\,z\,d\mu_0 = \int_\Omega v_0\,d\mu_0
\end{aligned}$$

by (4.4) with $v = \frac{v_0}{z}$, so (4.13) is proved.

For $\psi \in V^0_{\mu_0}(\Omega) \cap L^\infty_m(\Omega)$ take ψw_ε which belongs to $V^0_{\mu_\varepsilon}(\Omega) \cap L^\infty_m(\Omega)$, as a test function in (4.1_ε). This yields

$$a_{\lambda_0}(u_\varepsilon, \psi w_\varepsilon) + \int_\Omega u_\varepsilon \psi w_\varepsilon\,d\mu_\varepsilon = \int_\Omega \Phi(u_\varepsilon)\,\psi w_\varepsilon\,dm\,. \tag{4.14}$$

Now w_ε tends to 1 strongly in $L^p_m(\Omega)$ for any $p < \infty$ – see (4.12) – while $\Phi(u_\varepsilon) \rightharpoonup \Phi_0$ weakly in $L^q_m(\Omega)$ with $q = \frac{2}{s} > 1$, thus the right-hand side of (4.14) converges as $\varepsilon \to 0$ to

$$\int_\Omega \Phi_0 \psi\,dm\,.$$

We rewrite the left-hand side of (4.14):

$$\int_\Omega \alpha(u_\varepsilon, \psi w_\varepsilon)\, dm + \lambda_0 \int_\Omega u_\varepsilon \psi w_\varepsilon\, dm + \int_\Omega u_\varepsilon \psi w_\varepsilon\, d\mu_\varepsilon =$$

$$\int_\Omega [\psi \alpha(u_\varepsilon, w_\varepsilon) + w_\varepsilon \alpha(u_\varepsilon, \psi)]\, dm + \lambda_0 \int_\Omega u_\varepsilon \psi w_\varepsilon\, dm + \int_\Omega u_\varepsilon \psi w_\varepsilon\, d\mu_\varepsilon =$$

$$\int_\Omega [\alpha(\psi u_\varepsilon, w_\varepsilon) - u_\varepsilon \alpha(\psi, w_\varepsilon) + w_\varepsilon \alpha(u_\varepsilon, \psi)]\, dm$$

$$+ \lambda_0 \int_\Omega u_\varepsilon \psi w_\varepsilon\, dm + \int_\Omega u_\varepsilon \psi w_\varepsilon\, d\mu_\varepsilon =$$

$$I + II$$

with

$$I = \int_\Omega \alpha(\psi u_\varepsilon, w_\varepsilon)\, dm + \int_\Omega u_\varepsilon \psi w_\varepsilon\, d\mu_\varepsilon$$

and

$$II = \int_\Omega [-u_\varepsilon \alpha(\psi, w_\varepsilon) + w_\varepsilon \alpha(u_\varepsilon, \psi)]\, dm + \lambda_0 \int_\Omega u_\varepsilon \psi w_\varepsilon\, dm$$

$$= \int_\Omega [-u_\varepsilon \alpha(\psi, w_\varepsilon - 1) + w_\varepsilon \alpha(u_\varepsilon, \psi)]\, dm + \lambda_0 \int_\Omega u_\varepsilon \psi w_\varepsilon\, dm\, .$$

Let us pass to the limit as $\varepsilon \to 0$ using (4.13) with $v_\varepsilon = \psi u_\varepsilon$ in I, since $\psi u_\varepsilon \in V^0_{\mu_\varepsilon}(\Omega) \cap L^\infty_m(\Omega)$, $\psi u_0 \in V^0_{\mu_0}(\Omega)$, $\psi u_\varepsilon \rightharpoonup \psi u_0$ weakly in $D_0(a, \Omega)$ and $\|\psi u_\varepsilon\|_\infty < (\|\psi\|_\infty c_0)/(\lambda_0)$,

$$\lim_{\varepsilon \to 0} I = \int_\Omega \psi u_0\, d\mu_0 .$$

By Proposition 4.8, letting $\varepsilon \to 0$ in II, we have

$$\lim_{\varepsilon \to 0} II = \int_\Omega \alpha(u_0, \psi)\, dm + \lambda_0 \int_\Omega u_0 \psi\, dm\, .$$

From Definition 4.1 of the γ-convergence, we finally deduce that u_0 belongs to $V^0_{\mu_0}(\Omega)$

and u_0 satisfies

$$\begin{cases} u_0 \in V^0_{\mu_0}(\Omega) \cap L^\infty_m(\Omega) \\ a_{\lambda_0}(u_0, \psi) + \int\limits_\Omega u_0 \psi \, d\mu_0 = \int\limits_\Omega \Phi_0 \psi \, dm \quad \forall \psi \in V^0_{\mu_0}(\Omega) \cap L^\infty_m(\Omega). \end{cases}$$

Step 3: Correctors results for non-linear problem.

Let us take $v = u_\varepsilon$ as a test function in (4.1_ε). We pass easily to the limit since u_ε converges strongly to u_0 in $L^p_m(\Omega)$ for any $1 \leq p < \infty$. We obtain

$$(4.15) \qquad \begin{cases} \int\limits_\Omega \alpha(u_\varepsilon, u_\varepsilon) \, dm + \int\limits_\Omega u_\varepsilon^2 \, d\mu_\varepsilon = \int\limits_\Omega \Phi(u_\varepsilon) \, u_\varepsilon \, dm - \lambda_0 \int\limits_\Omega u_\varepsilon^2 \, dm \\ \to \int\limits_\Omega \Phi_0 u_0 \, dm - \lambda_0 \int\limits_\Omega u_0^2 \, dm = \int\limits_\Omega \alpha(u_0, u_0) \, dm + \int\limits_\Omega u_0^2 \, d\mu_0. \end{cases}$$

We now claim that for any $\psi \in V^0_{\mu_0}(\Omega) \cap L^\infty_m(\Omega)$

$$(4.16) \qquad \begin{cases} \int\limits_\Omega \alpha(u_\varepsilon - w_\varepsilon \psi, u_\varepsilon - w_\varepsilon \psi) \, dm + \int\limits_\Omega (u_\varepsilon - w_\varepsilon \psi)^2 \, d\mu_\varepsilon \\ \to \int\limits_\Omega \alpha(u_0 - \psi, u_0 - \psi) \, dm + \int\limits_\Omega (u_0 - \psi)^2 \, d\mu_0 \\ \text{as } \varepsilon \to 0. \end{cases}$$

To prove this claim, we write the left-hand side of (4.16) as

$$\int\limits_\Omega [\alpha(u_\varepsilon, u_\varepsilon) + \alpha(w_\varepsilon, \psi^2 w_\varepsilon) - 2\alpha(w_\varepsilon, \psi u_\varepsilon) + w_\varepsilon^2 \alpha(\psi, \psi) - 2 w_\varepsilon \alpha(u_\varepsilon, \psi)] \, dm$$

$$+2 \int\limits_\Omega u_\varepsilon \alpha(w_\varepsilon, \psi) \, dm + \int\limits_\Omega u_\varepsilon^2 \, d\mu_\varepsilon + \int\limits_\Omega \psi^2 w_\varepsilon^2 \, d\mu_\varepsilon - 2 \int\limits_\Omega \psi w_\varepsilon u_\varepsilon \, d\mu_\varepsilon =$$

$$\left[\int\limits_\Omega \alpha(u_\varepsilon, u_\varepsilon) \, dm + \int\limits_\Omega u_\varepsilon^2 \, d\mu_\varepsilon \right] + \left[\int\limits_\Omega \alpha(w_\varepsilon, \psi^2 w_\varepsilon) \, dm + \int\limits_\Omega (\psi^2 w_\varepsilon) \, w_\varepsilon \, d\mu_\varepsilon \right]$$

$$-2 \left[\int\limits_\Omega \alpha(w_\varepsilon, \psi u_\varepsilon) \, dm + \int\limits_\Omega \psi w_\varepsilon u_\varepsilon \, d\mu_\varepsilon \right]$$

$$+ \left[\int\limits_\Omega [w_\varepsilon^2 \alpha(\psi, \psi) - 2 w_\varepsilon \alpha(u_\varepsilon, \psi)] \, dm + 2 \int\limits_\Omega u_\varepsilon \alpha(w_\varepsilon, \psi) \, dm \right]$$

$$I + II + III + IV.$$

$$I = \left[\int_\Omega \alpha(u_\varepsilon, u_\varepsilon)\, dm + \int_\Omega u_\varepsilon^2 \, d\mu_\varepsilon \right]$$

$$\rightarrow \left[\int_\Omega \alpha(u_0, u_0)\, dm + \int_\Omega u_0^2 \, d\mu_0 \right]$$

as $\varepsilon \rightarrow 0$ by (4.15),

$$II = \left[\int_\Omega \alpha(w_\varepsilon, \psi^2 w_\varepsilon)\, dm + \int_\Omega (\psi^2 w_\varepsilon)\, w_\varepsilon \, d\mu_\varepsilon \right] \rightarrow \int_\Omega \psi^2 \, d\mu_0$$

as $\varepsilon \rightarrow 0$ by (4.13) with $v_\varepsilon = \psi^2 w_\varepsilon$.

$$III = -2 \left[\int_\Omega \alpha(w_\varepsilon, \psi u_\varepsilon)\, dm + \int_\Omega (\psi u_\varepsilon)\, w_\varepsilon \, d\mu_\varepsilon \right] \rightarrow -2 \int_\Omega \psi u_0 \, d\mu_0$$

as $\varepsilon \rightarrow 0$ by (4.13) with $v_\varepsilon = \psi u_\varepsilon$. Moreover, by Proposition 4.8

$$IV =$$

$$\int_\Omega [w_\varepsilon^2 \alpha(\psi, \psi)\, dm - 2 w_\varepsilon \alpha(u_\varepsilon, \psi)]\, dm + 2 \int_\Omega u_\varepsilon \alpha(w_\varepsilon, \psi)\, dm =$$

$$\int_\Omega [w_\varepsilon^2 \alpha(\psi, \psi)\, dm - 2 w_\varepsilon \alpha(u_\varepsilon, \psi)]\, dm + 2 \int_\Omega u_\varepsilon \alpha(w_\varepsilon - 1, \psi)\, dm$$

$$\rightarrow \int_\Omega [\alpha(\psi, \psi) - 2\alpha(u_0, \psi)]\, dm\,.$$

Therefore

$$\lim_{\varepsilon \rightarrow 0} I + II + III + IV =$$

$$\int_\Omega \alpha(u_0, u_0)\, dm + \int_\Omega u_0^2 \, d\mu_0 + \int_\Omega \psi^2 \, d\mu_0$$

$$-2 \int_\Omega \psi u_0 \, d\mu_0 + \int_\Omega \alpha(\psi, \psi)\, dm - 2 \int_\Omega \alpha(u_0, \psi)\, dm =$$

$$\int_\Omega \alpha(u_0 - \psi, u_0 - \psi)\, dm + \int_\Omega (u_0 - \psi)^2 \, d\mu_0,$$

that is (4.16) holds.

Taking now $\psi = u_0$ in (4.16) we obtain that

$$u_\varepsilon - w_\varepsilon u_0 \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0 \quad \text{strongly in} \quad D_0(a, \Omega)\,. \tag{4.17}$$

Step 4: Identifying Φ_0 *as* $\Phi(u_0) = \phi(x, \alpha^{1/2}(u_0, u_0))$.

Firstly let us make some remarks. Let us consider the inequality (3.7) when v is replaced by νv with $\nu = (\alpha^{1/2}(u, u)(x))/(\alpha^{1/2}(v, v)(x))$. The function $\alpha(u, v)$ is continuous on $\Omega - E$ where $m(E) = 0$. Let $x \in \Omega - E$ fixed. Then

$$|\alpha(u, v)(\cdot)| \leqslant \alpha^{1/2}(u, u)(\cdot)\, \alpha^{1/2}(v, v)(\cdot)\,. \tag{4.18}$$

It implies

$$\begin{aligned} &(\alpha^{1/2}(u, u)(\cdot) - \alpha^{1/2}(v, v)(\cdot))^2 = \\ &\alpha(u, u)(\cdot) + \alpha(v, v)(\cdot) - 2\alpha^{1/2}(u, u)(\cdot)\, \alpha^{1/2}(v, v)(\cdot) \\ &\leqslant \alpha(u, u)(\cdot) + \alpha(v, v)(\cdot) - 2\alpha(u, v)(\cdot) = \alpha(u - v, u - v)(\cdot)\,. \end{aligned} \tag{4.19}$$

In the proof of this step we use hypothesis (H1) with $p_1 = \alpha^{1/2}(u_\varepsilon, u_\varepsilon)$ and $p_2 = \alpha^{1/2}(w_\varepsilon u_0, w_\varepsilon u_0)$. By (4.19) we have

$$\begin{aligned} &|\Phi(u_\varepsilon) - \Phi(w_\varepsilon u_0)| \leqslant \\ &K(1 + \alpha(u_\varepsilon, u_\varepsilon)^{(s-\gamma)/2} + \alpha(w_\varepsilon u_0, w_\varepsilon u_0)^{(s-\gamma)/2})\, \alpha(u_\varepsilon - w_\varepsilon u_0, u_\varepsilon - w_\varepsilon u_0)^{\gamma/2}. \end{aligned}$$

It can be easily proved that any term in the right-hand side converges to zero strongly in $L^1_m(\Omega)$. Let us consider, for example, the last term in the case $s > \gamma$. It is enough to apply Hölder's inequality with $p = \dfrac{2}{s-\gamma}$ and $p' = \dfrac{2}{2-s+\gamma}$ to get

$$\begin{aligned} &\int_\Omega \alpha(u_0 w_\varepsilon, u_0 w_\varepsilon)^{(s-\gamma)/2}\, \alpha(u_\varepsilon - u_0 w_\varepsilon, u_\varepsilon - u_0 w_\varepsilon)^{\gamma/2}\, dm \\ &\leqslant \left(\int_\Omega \alpha(u_0 w_\varepsilon, u_0 w_\varepsilon)\, dm\right)^{(s-\gamma)/2} \left(\int_\Omega \alpha(u_\varepsilon - u_0 w_\varepsilon, u_\varepsilon - u_0 w_\varepsilon)^{\gamma/(2-s+\gamma)}\, dm\right)^{(2-s+\gamma)/2}. \end{aligned}$$

The condition $s < 2$ implies that $\dfrac{\gamma}{2-s+\gamma} < 1$, and the result follows from (4.17). Then we have proved that

$$\Phi(u_\varepsilon) - \Phi(w_\varepsilon u_0) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{strongly in} \quad L^1_m(\Omega) \tag{4.20}$$

that is

$$\phi(x, \alpha^{1/2}(u_\varepsilon, u_\varepsilon)) - \phi(x, \alpha^{1/2}(w_\varepsilon u_0, w_\varepsilon u_0)) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{strongly in } L^1_m(\Omega)\,.$$

Let us now prove that

$$\Phi(w_\varepsilon u_0) \to \Phi(u_0) \quad \text{as} \quad \varepsilon \to 0 \quad \text{strongly in} \quad L^1_m(\Omega)\,. \tag{4.21}$$

Let us observe that, by the Leibnitz rule,

$$\alpha(w_\varepsilon u_0, w_\varepsilon u_0) = u_0^2\, \alpha(w_\varepsilon, w_\varepsilon) + w_\varepsilon^2\, \alpha(u_0, u_0) + 2 u_0 w_\varepsilon\, \alpha(w_\varepsilon, u_0)\,. \tag{4.22}$$

Let us consider (4.18) with $u = u_0$ and $v = w_\varepsilon$. By (H3) $\alpha(w_\varepsilon, u_0)$ converges to 0 a.e. in Ω. Thus, letting ε to zero in (4.22), using again (H3) and since $w_\varepsilon \to 1$ a.e. in Ω we have that $\alpha(w_\varepsilon u_0, w_\varepsilon u_0) \to \alpha(u_0, u_0)$ a.e. in Ω. By the continuity of ϕ with respect to p we have

$$\phi(x, \alpha^{1/2}(w_\varepsilon u_0, w_\varepsilon u_0)) \to \phi(x, \alpha^{1/2}(u_0, u_0)) \quad \text{as} \quad \varepsilon \to 0 \quad \text{a.e. in } \Omega,$$

that is

$$\Phi(w_\varepsilon u_0) \to \Phi(u_0) \quad \text{a.e. in } \Omega.$$

On the other hand, by (4.6) and by (4.20)

$$\int_\Omega |\Phi(w_\varepsilon u_0)|\, dm$$

$$\leqslant c_1 \int_\Omega \left[1 + \alpha^{s/2}(w_\varepsilon u_0, w_\varepsilon u_0)\right] dm$$

$$\leqslant \text{const.} \int_\Omega [1 + |u_0|^s \alpha^{s/2}(w_\varepsilon, w_\varepsilon) + |w_\varepsilon|^s \alpha^{s/2}(u_0, u_0) + |w_\varepsilon u_0|^{s/2} |\alpha^{s/2}(w_\varepsilon, u_0)|]\, dm.$$

The limit of the right hand side is zero when $m(\Omega)$ tends to zero. Indeed we know that $w_\varepsilon, u_0 \in L^\infty_m(\Omega)$ (then also their product belongs to the same space), and since $w_\varepsilon \rightharpoonup 1$ weakly in $D(a, \Omega)$ then w_ε is bounded in $D(a, \Omega)$:

$$\int_\Omega \alpha(w_\varepsilon, w_\varepsilon)\, dm \leqslant \text{const.}$$

Then, for any subset $E \subset \Omega$,

$$\lim_{m(E)\to 0} \int_E |u_0|^s \alpha^{s/2}(w_\varepsilon, w_\varepsilon)\, dm \leqslant \lim_{m(E)\to 0} \|u_0^s\|_\infty\, m(E)^{(2-s)/2} \int_\Omega \alpha(w_\varepsilon, w_\varepsilon)\, dm = 0$$

and

$$\lim_{m(E)\to 0} \int_E |w_\varepsilon|^s \alpha^{s/2}(u_0, u_0) \leqslant \lim_{m(E)\to 0} \|w_\varepsilon^s\|_\infty \int_E \alpha^{s/2}(u_0, u_0)\, dm = 0,$$

by the equicontinuity of the Lebesgue integral. Moreover, since

$$\int_\Omega |\alpha(u_0, w_\varepsilon)| \leqslant \frac{1}{2} \int_\Omega \alpha(u_0, u_0) + \frac{1}{2} \int_\Omega \alpha(w_\varepsilon, w_\varepsilon) \leqslant \text{const.},$$

thus

$$\lim_{m(E)\to 0} \int_E |u_0 w_\varepsilon|^{s/2} \alpha^{s/2}(w_\varepsilon, u_0)\, dm \leqslant \lim_{m(E)\to 0} \|(u_0 w_\varepsilon)^{s/2}\|_\infty\, m(E)^{(2-s)/2} \int_\Omega |\alpha(w_\varepsilon, u_0)|\, dm = 0.$$

All these computations implies that, for any subset $E \subset \Omega$,

$$\lim_{m(E)\to 0} \int_E |\Phi(w_\varepsilon u_0)|\, dm = 0 .$$

The proof of (4.21) is then achieved using Vitali's convergence Theorem. From (4.20) and (4.21) we obtain that

$$\Phi(u_\varepsilon) = \phi(x, \alpha^{1/2}(u_\varepsilon, u_\varepsilon)) \to \phi(x, \alpha^{1/2}(u_0, u_0)) = \Phi(u_0) \quad \text{as} \quad \varepsilon \to 0 \quad \text{strongly in} \quad L^1_m(\Omega),$$

which proves that

$$\Phi_0 = \phi(x,\, \alpha^{1/2}(u_0,\, u_0)) = \Phi(u_0) .$$

Using this result in the limit problem for u_0 at the end of Step 2, we complete the proof of the Theorem. ■

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