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Embedding and Compactness Theorems for Irregular Domains (**)

Abstract. — In this paper we study the multiplication operator defined in a weighted Sobolev space and with values in a suitable Lebesgue space. Some boundedness and compactness results for such operator are obtained.

Teoremi di immersione e compattezza per aperti non regolari

Sunto. — In questo lavoro si studia l'operatore di moltiplicazione definito in uno spazio di Sobolev con peso e a valori in un opportuno spazio di Lebesgue. Si trovano delle condizioni sul fattore di moltiplicazione affinché l'operatore risulti limitato o compatto.

Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$, and $X(\Omega)$ a Sobolev space. Consider the multiplication operator

$$u \to gu \tag{1}$$

defined in $X(\Omega)$ with values in a suitable $L^p(\Omega)$. Several results about boundedness and compactness of the operator (1), under various assumptions on g, are already known (see, for instance, [14], [12], [16], [2], [9], [7], [8], [15], [17], [18], [11], [3], [5]). In

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particular, in [11] the authors studied the multiplication operator

(2) $$u \in W^m_{p_0, p_1}(\Omega, \alpha_0, \alpha_1) \;\rightarrow\; g\partial^r u \in L^p(\Omega),$$

where $m \in \mathbb{N}$, $r \in \{0, 1, \dots, m-1\}$, $p, p_0, p_1 \in [1, +\infty[$, α_0, α_1 are $L^1_{\mathrm{loc}}(\Omega)$ weight functions and $W^m_{p_0, p_1}(\Omega, \alpha_0, \alpha_1)$ is the space of distributions u on Ω such that $\alpha_0^{1/p_0} u \in L^{p_0}(\Omega)$ and $\alpha_1^{1/p_1} \partial^m u \in L^{p_1}(\Omega)$, endowed with a suitable norm. Moreover, g is a function in $L^\sigma_{\mathrm{loc}}(\overline{\Omega})$ for some σ, satisfying the condition

(3) $$\sup_{x \in \Omega} (\beta'(x) \|g\|_{L^\sigma(\Omega \cap B(x, 1))}) < +\infty,$$

where $B(x, 1)$ is the open unit ball of $\mathbb{R}^n$ centered at x and β' is a function related to α_0 and α_1. In this situation, some conditions for the boundedness or the compactness of the operator (2) were given.

In this paper we study the operator (2), under much weaker hypotheses than those assumed in [11]. In fact, fixed a weight function ϱ in $\mathcal{A}(\Omega)$ (see Section 4 for the definition of $\mathcal{A}(\Omega)$) and denoted by S_ϱ the subset of $\partial\Omega$ where ϱ goes to zero, we suppose here that g belongs to $L^\sigma_{\mathrm{loc}}(\overline{\Omega} \setminus S_\varrho)$ and that

(4) $$\sup_{x \in \Omega} (\beta''(x) \|g\|_{L^\sigma(\Omega \cap B(x, \varrho(x)))}) < +\infty,$$

where $B(x, \varrho(x))$ is the open ball of $\mathbb{R}^n$ with center in x and radius $\varrho(x)$, and β'' is a function related to α_0, α_1 and ϱ. We obtain some results on the boundedness and the compactness of the operator (2).

Observe finally that our theorems also generalize the results in [5] (see also [3]), which have been used in the study of a class of second order linear differential equations of elliptic type with singular data (see [6]). In fact, in [5] the operator (1) is defined on a weighted Sobolev space, where the weight functions are suitable powers of a function $\varrho \in \mathcal{A}(\Omega)$, and g belongs to $L^q_{\mathrm{loc}}(\overline{\Omega} \setminus S_\varrho)$ for some q and satisfies a condition like (4) with β'' a suitable power of ϱ and with q instead of σ.

The first part of the paper deals with some function spaces, in which the function g can be chosen, while the last part is devoted to the proofs of our boundedness and compactness theorems for the operator (2).

1. - Notations

In the following, if A is a Lebesgue measurable subset of $\mathbb{R}^n$, $|A|$ denotes its Lebesgue measure, $\mathcal{D}(A)$ is the collection of the restrictions to A of the functions $\zeta \in C_0^\infty(\mathbb{R}^n)$ with *supt* $\zeta \cap \overline{A} \subset A$ and $L^p_{\mathrm{loc}}(A)$ is the space of the functions $f: A \to \mathbb{C}$ such that $\zeta f \in L^p(A)$ for all $\zeta \in \mathcal{D}(A)$, $p \in [1, +\infty]$.

If $f \in L^p(A)$, $p \in [1, +\infty]$, put

$$|f|_{p, A} := \|f\|_{L^p(A)}.$$

If $p \in [1, +\infty[$ and η is a weight function on A (*i.e.* a nonnegative measurable function on A), $L^p(A, \eta)$ is the space of measurable functions $f: A \to \mathbb{C}$ such that $\eta^{1/p} f \in L^p(A)$. For every $f \in L^p(A, \eta)$, define

$$\|f\|_{L^p(A, \eta)} := |\eta^{1/p} f|_{p, A}; \tag{1.1}$$

it is clear that (1.1) is a norm on $L^p(A, \eta)$ if and only if η is almost everywhere positive on A.

We will use the standard multi-index notations

$$\varsigma := (\varsigma_1, \varsigma_2, \dots, \varsigma_n) \in \mathbb{N}_0^n, \quad |\varsigma| := \varsigma_1 + \dots + \varsigma_n, \quad \partial^\varsigma := \partial_{x_1}^{\varsigma_1} \dots \partial_{x_n}^{\varsigma_n},$$

where

$$\partial_{x_i} := \frac{\partial}{\partial x_i}, \qquad i = 1, 2, \dots, n.$$

Moreover, if $m \in \mathbb{N}$, $p_0, p_1 \in [1, +\infty[$ and α_0, α_1 are $L^1_{\text{loc}}(A)$ a.e. positive weight functions on A, we will denote by

- $W^m_{p_0, p_1}(A, \alpha_0, \alpha_1)$ the space of distributions u on A such that $u \in L^{p_0}(A, \alpha_0)$ and $\partial^\mu u \in L^{p_1}(A, \alpha_1)$ with $|\mu| = m$, endowed with the norm

$$\|u\|_{W^m_{p_0, p_1}(A, \alpha_0, \alpha_1)} := \|u\|_{L^{p_0}(A, \alpha_0)} + \sum_{|\mu| = m} \|\partial^\mu u\|_{L^{p_1}(A, \alpha_1)}; \tag{1.2}$$

- $\overset{\circ}{W}{}^m_{p_0, p_1}(A, \alpha_0, \alpha_1)$ the closure of $\mathcal{D}(A)$ in $W^m_{p_0, p_1}(A, \alpha_0, \alpha_1)$.

Finally we put

$$|\partial^m u|_{p, A} := \sum_{|\mu| = m} |\partial^\mu u|_{p, A}, \qquad \|\partial^\mu u\|_{L^p(A, \eta)} := \sum_{|\mu| = m} \|\partial^\mu u\|_{L^p(A, \eta)},$$

$$W^m_{p_0, p_1}(A) := W^m_{p_0, p_1}(A, 1, 1), \qquad \overset{\circ}{W}{}^m_{p_0, p_1}(A) := \overset{\circ}{W}{}^m_{p_0, p_1}(A, 1, 1).$$

2. - $\boldsymbol{E}^p_\beta(A)$ spaces

In this section A denotes an open subset of $\mathbb{R}^n$, $n \geq 2$. Consider a family $\mathcal{F} = \{A(x)\}_{x \in A}$ of nonempty open, bounded subsets of $\mathbb{R}^n$ such that

(*a*) $\forall x \in A, \quad x \in A(x) \subset A$;

(*b*) $S := \overline{A} \setminus \bigcup_{x \in A} \overline{A(x)} \neq \emptyset$;

(*c*) $E \subset\subset \overline{A} \setminus S \Rightarrow A_E := \{x \in A : A(x) \cap E \neq \emptyset\} \subset\subset \overline{A} \setminus S$;

(*d*) $\exists c \in \mathbb{R}_+, \exists l : A \to \mathbb{R}_+ : l^{-1} \in L^\infty_{\text{loc}}(\overline{A} \setminus S), \ |A(x)| \leq c\ l(x) \ \ \forall x \in A$.

In Section 4 (see (4.7)) a family $\mathcal{F}$ of this type will be chosen, which will play a central role in our arguments.

We point out that

$$L^p_{\text{loc}}(\overline{A}\setminus S)\subset\bigcap_{x\in A}L^p(A(x)),\qquad p\in[1,+\infty],\tag{2.1}$$

where equality holds, for instance, when the following condition is satisfied:

$$E\subset A,\ \ E\subset\subset\overline{A}\setminus S\Rightarrow\exists k\in\mathbb{N},\ \ \exists x_1,\dots,x_k\in E: E\subset\bigcup_{i=1}^{k}A(x_i).\tag{2.2}$$

Now, for any $p\in[1,+\infty]$ and $\beta: A\to\mathbb{R}_+$ such that

$$\beta\in L^\infty_{\text{loc}}(\overline{A}\setminus S),\tag{e}$$

we define $E^p_\beta(A)$ as the space of functions $g\in L^p_{\text{loc}}(\overline{A}\setminus S)$ such that

$$\|g\|_{E^p_\beta(A)}:=\sup_{x\in A}\left(\frac{\beta(x)}{l^{1/p}(x)}\,|g|_{p,A(x)}\right)<+\infty,\tag{2.3}$$

endowed with the norm defined by (2.3), where we mean that $\dfrac{1}{p}=0$ when $p=+\infty$.

If $\phi: A\to\mathbb{R}_+$ is in $L^\infty_{\text{loc}}(\overline{A}\setminus S)$ and there exists a constant $c\in\mathbb{R}_+$ such that $\beta\leqslant c\phi$, then

$$E^q_\phi(A)\hookrightarrow E^p_\beta(A),\qquad 1\leqslant p\leqslant q\leqslant+\infty.\tag{2.4}$$

Moreover, it turns out that

$$\mathcal{D}(\overline{A}\setminus S)\subset E^p_\beta(A).\tag{2.5}$$

In fact, if $g\in\mathcal{D}(\overline{A}\setminus S)$, then *supt* $g=: K\subset\subset\overline{A}\setminus S$ and so, using $(c)-(e)$, we have

$$\begin{aligned}\|g\|_{E^p_\beta(A)}&=\sup_{x\in A}\left(\frac{\beta(x)}{l^{1/p}(x)}\,|g|_{p,A(x)\cap K}\right)\\&\leqslant\sup_{x\in A_K}\left(\frac{\beta(x)}{l^{1/p}(x)}\,|g|_{\infty,A}\,|A(x)\cap K|^{1/p}\right)\\&\leqslant c^{1/p}\,|g|_{\infty,A}\,|\beta|_{\infty,A_K}.\end{aligned}$$

Therefore the space $\overset{\circ}{E}{}^p_\beta(A):=\overline{C^\infty_0(A)}^{E^p_\beta(A)}$ is well-defined.

In next section two characterizations of $\overset{\circ}{E}{}^p_\beta(A)$ for $p\in[1,+\infty[$ will be given. In order to obtain them, consider a function $\alpha\in C^{0,1}(\overline{A})\cap C^\infty(A)$ such that

$$\alpha(x)\sim\text{dist}\,(x,\partial A)$$

(*i.e.* there exist two positive constants c_1 and c_2 such that, for any $x\in A$, $c_1\alpha(x)\leqslant$

$\leqslant \operatorname{dist}(x, \partial A) \leqslant c_2\,\alpha(x))$, and put

$$A_k := \left\{ x \in A : |x| < k,\ \alpha(x) > \frac{1}{k} \right\}, \qquad k \in \mathbb{N}. \tag{2.6}$$

Moreover, fixed a function $f \in \mathcal{D}([0, +\infty[)$ such that

$$0 \leqslant f \leqslant 1, \qquad f(t) = 1 \text{ if } t < \frac{1}{2}, \qquad f(t) = 0 \text{ if } t > 1,$$

we define

$$\psi_k : x \in \overline{A} \mapsto (1 - f(k\alpha(x)))\, f\left(\frac{|x|}{2k}\right), \qquad k \in \mathbb{N}.$$

It is easy to prove that, for any $k \in \mathbb{N}$, ψ_k belongs to $C_0^\infty(A)$ and

$$0 \leqslant \psi_k \leqslant 1, \qquad \psi_{k|\overline{A_k}} = 1, \qquad \operatorname{supt} \psi_k \subset \overline{A_{2k}}, \qquad \psi_k \psi_{2k} = \psi_k.$$

REMARK 2.1: If $g \in L^p(A)$, for every $k \in \mathbb{N}$ there exists a constant $c(k) \in \mathbb{R}_+$, which does not depend on g, such that:

$$\|\psi_k g\|_{E^p_\beta(A)} \leqslant c(k)\, |g|_{p, A}. \tag{2.8}$$

In fact, setting $K := \operatorname{supt} \psi_k$, if $p = +\infty$, using (c) and (e), we have

$$\begin{aligned} \|\psi_k g\|_{E^\infty_\beta(A)} &= \sup_{x \in A} \left(\beta(x)\, |\psi_k g|_{\infty, A(x) \cap K}\right) \\ &\leqslant \sup_{x \in A_K} \left(\beta(x)\, |g|_{\infty, A(x) \cap K}\right) \\ &\leqslant |\beta|_{\infty, A_K}\, |g|_{\infty, A}; \end{aligned}$$

if $p \in [1, +\infty[$, using (c)-(e), we obtain

$$\begin{aligned} \|\psi_k g\|_{E^p_\beta(A)} &= \sup_{x \in A} \left(\frac{\beta(x)}{l^{1/p}(x)}\, |\psi_k g|_{p, A(x) \cap K} \right) \\ &\leqslant \sup_{x \in A_K} \left(\frac{\beta(x)}{l^{1/p}(x)}\, |g|_{p, A(x) \cap K} \right) \\ &\leqslant \left(\operatorname*{essinf}_{x \in A_K} l(x) \right)^{-1/p} |\beta|_{\infty, A_K}\, |g|_{p, A}. \qquad \blacksquare \end{aligned}$$

3. - Characterizations of $\overset{\circ}{E}{}^p_\beta(A)$

In this section we suppose that $p \in [1, +\infty[$.

Lemma 3.1: *A function g belongs to* $\overset{\circ}{E}{}^p_\beta(A)$ *if and only if* $g \in E^p_\beta(A)$ *and*

$$\lim_{k \to +\infty} \|(1-\psi_k)\, g\|_{E^p_\beta(A)} = 0\,, \tag{3.1}$$

where $(\psi_k)_{k \in \mathbb{N}}$ *is defined by* (2.7).

Proof: If $g \in \overset{\circ}{E}{}^p_\beta(A)$, for every fixed $\varepsilon > 0$ there exists $g_\varepsilon \in C_0^\infty(A)$ such that

$$\|g - g_\varepsilon\|_{E^p_\beta(A)} \leq \varepsilon\,.$$

Since

$$A \backslash A_k = \left\{ x \in A : |x| \geq k \text{ or } \alpha(x) \leq \frac{1}{k} \right\}, \qquad k \in \mathbb{N}\,,$$

with A_k defined by (2.6), we can find an index $k_\varepsilon \in \mathbb{N}$ such that

$$(A \backslash A_k) \cap \operatorname{supt} g_\varepsilon = \emptyset \qquad \forall k \geq k_\varepsilon\,.$$

Thus, observing that $\psi_{k|\overline{A_k}} = 1$, we obtain

$$(1-\psi_k)\, g_\varepsilon = 0 \qquad \forall k \geq k_\varepsilon$$

and hence

$$\begin{aligned} \|(1-\psi_k)\, g\|_{E^p_\beta(A)} &= \|(1-\psi_k)(g - g_\varepsilon)\|_{E^p_\beta(A)} \\ &\leq \|g - g_\varepsilon\|_{E^p_\beta(A)} \leq \varepsilon \qquad \forall k \geq k_\varepsilon\,, \end{aligned}$$

i.e. (3.1) holds.

Conversely, if $g \in E^p_\beta(A)$ and (3.1) holds, then for every fixed $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\|(1-\psi_{k_\varepsilon})\, g\|_{E^p_\beta(A)} \leq \frac{\varepsilon}{2}\,.$$

Since $g \in L^p_{\text{loc}}(\overline{A} \backslash S)$ and $\psi_{2k_\varepsilon} \in C_0^\infty(A)$, $\psi_{2k_\varepsilon} g$ belongs to $L^p(A)$ and so a function $\phi_\varepsilon \in C_0^\infty(A)$ can be found such that

$$|\psi_{2k_\varepsilon} g - \phi_\varepsilon|_{p,A} \leq \frac{\varepsilon}{2\,c(k_\varepsilon)}\,,$$

where $c(k_\varepsilon)$ is the constant of (2.8) with $k = k_\varepsilon$. Hence, using (2.8) and the properties of ψ_k, we obtain

$$\|g-\psi_{k_\varepsilon}\phi_\varepsilon\|_{E^p_\beta(A)} \leqslant \|(1-\psi_{k_\varepsilon})\, g\|_{E^p_\beta(A)} + \|\psi_{k_\varepsilon} g - \psi_{k_\varepsilon}\phi_\varepsilon\|_{E^p_\beta(A)}$$

$$\leqslant \frac{\varepsilon}{2} + \|\psi_{k_\varepsilon}(\psi_{2k_\varepsilon} g - \phi_\varepsilon)\|_{E^p_\beta(A)}$$

$$\leqslant \frac{\varepsilon}{2} + c(k_\varepsilon)\, |\psi_{2k_\varepsilon} g - \phi_\varepsilon|_{p,A} \leqslant \varepsilon\,.$$

The lemma is proved. ■

Lemma 3.2: *A function g belongs to* $\overset{\circ}{E}{}^p_\beta(A)$ *if and only if* $g\in E^p_\beta(A)$ *and* (3.2) $\forall\varepsilon\in\mathbb{R}_+\ \exists a_\varepsilon\in\mathbb{R}_+,\ b_\varepsilon\in\mathbb{N}$ *such that*

$$\Big(E\in\Sigma(A),\ \sup_{x\in A}|A(x)\cap E\cap A_{b_\varepsilon}| \leqslant a_\varepsilon\Big) \Rightarrow \|g\chi_E\|_{E^p_\beta(A)} \leqslant \varepsilon\,,$$

where $\Sigma(A)$ *is the* σ*-algebra of the Lebesgue measurable subsets of A and* χ_E *is the characteristic function of E.*

Proof: If $g\in \overset{\circ}{E}{}^p_\beta(A)$, for every fixed $\varepsilon>0$ there exists $g_\varepsilon\in C_0^\infty(A)$ such that

$$\|g-g_\varepsilon\|_{E^p_\beta(A)} \leqslant \frac{\varepsilon}{2}\,.$$

Setting $K_\varepsilon := \operatorname{supt} g_\varepsilon$ and $c_\varepsilon := \Big(\operatorname*{essinf}_{x\in A_{K_\varepsilon}} l(x)\Big)^{-1/p} |\beta|_{\infty, A_{K_\varepsilon}} |g_\varepsilon|_{\infty, A}$ (which is well-defined for (c)-(e)), for any $E\in\Sigma(A)$ we have

$$\|g\chi_E\|_{E^p_\beta(A)} \leqslant \frac{\varepsilon}{2} + \|g_\varepsilon\chi_E\|_{E^p_\beta(A)}$$

$$= \frac{\varepsilon}{2} + \sup_{x\in A}\left(\frac{\beta(x)}{l^{1/p}(x)}\, |g_\varepsilon\chi_E|_{p, A(x)\cap K_\varepsilon}\right)$$

$$= \frac{\varepsilon}{2} + \sup_{x\in A_{K_\varepsilon}}\left(\frac{\beta(x)}{l^{1/p}(x)}\, |g_\varepsilon|_{p, A(x)\cap E\cap K_\varepsilon}\right)$$

$$\leqslant \frac{\varepsilon}{2} + c_\varepsilon \sup_{x\in A}|A(x)\cap E\cap K_\varepsilon|^{1/p}.$$

Therefore, choosing $a_\varepsilon := \left(\dfrac{\varepsilon}{2c_\varepsilon}\right)^p$ and $b_\varepsilon\in\mathbb{N}$ such that $A_{b_\varepsilon}\supset K_\varepsilon$, (3.2) follows immediately.

Conversely, assume that $g\in E^p_\beta(A)$ and (3.2) holds. Fixed $\varepsilon>0$, since

$$(A\backslash A_k)\cap A_{b_\varepsilon} = \emptyset \qquad \forall k\geqslant b_\varepsilon,$$

we deduce that

$$\|(1-\psi_k)\, g\|_{E^p_\beta(A)} \leq \|g\chi_{A\setminus A_k}\|_{E^p_\beta(A)} \leq \varepsilon \qquad \forall k \geq b_\varepsilon .$$

Hence (3.1) holds and so $g \in \overset{\circ}{E}{}^p_\beta(A)$ by Lemma 3.1. ■

4. - Weight functions in $\mathcal{A}(A)$

If A is an open subset of $\mathbb{R}^n$, $n \geq 2$, we denote by $\mathcal{A}(A)$ the collection of functions $\delta : A \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,\, y \in A \\ |x-y| < \delta(y)}} \left| \log \frac{\delta(x)}{\delta(y)} \right| < +\infty \tag{4.1}$$

and by $\mathcal{A}_0(A)$ the subset of measurable functions of $\mathcal{A}(A)$.

One can easily show that $\delta : A \to \mathbb{R}_+$ verifies (4.1) if and only if there exists a constant $\gamma_\delta \in \mathbb{R}_+$ such that

$$\gamma_\delta^{-1}\, \delta(y) \leq \delta(x) \leq \gamma_\delta\, \delta(y) \qquad \forall y \in A,\ \forall x \in A(y, \delta(y)), \tag{4.2}$$

where

$$A(x, r) := A \cap B(x, r), \qquad B(x, r) := \{y \in \mathbb{R}^n \colon |y - x| < r\}, \qquad x \in \mathbb{R}^n,\ r \in \mathbb{R}_+ .$$

For examples and properties of functions in $\mathcal{A}(A)$, we refer to [20] and also to [3], [5], [4]. We just recall (see, for instance, [4]) that for any $\delta \in \mathcal{A}(A)$, if

$$S_\delta := \left\{ z \in \partial A : \lim_{x \to z} \delta(x) = 0 \right\}, \tag{4.3}$$

then

$$\overline{A(x, \delta(x))} \cap S_\delta = \emptyset \qquad \forall x \in A\ ; \tag{4.4}$$

moreover, if $z \in \partial A$, the following equivalence holds:

$$z \in S_\delta \iff \delta(x) \leq |x - z| \quad \forall x \in A\ . \tag{4.5}$$

For all $\delta \in \mathcal{A}_0(A)$ we also have

$$\delta \in L^\infty_{\text{loc}}(\overline{A}), \qquad \delta^{-1} \in L^\infty_{\text{loc}}(\overline{A} \setminus S_\delta)\ . \tag{4.6}$$

Remark 4.1: If $\delta \in \mathcal{A}(A)$, setting

$$A(x) := A(x, \delta(x)), \qquad \forall x \in A, \tag{4.7}$$

and

$$S := \overline{A} \setminus \bigcup_{x \in A} \overline{A(x)}\,, \tag{4.8}$$

one has $S = S_\delta$.

In fact, it follows from (4.4) that $S_\delta \subset S$. Conversely, let $z \in S$ and assume by contradiction that $z \notin S_\delta$. Then (4.5) yields that there exists an element x_0 in A such that $\delta(x_0) > |x_0 - z|$; so $z \in S \cap B(x_0, \delta(x_0))$ and it is easy to show that $z \in \overline{A(x_0)}$, which is not true. ■

REMARK 4.2: If δ is an element of $\mathcal{A}_0(A)$ such that $S_\delta \neq \emptyset$, then the assumptions (a), (b), (c) and (d), introduced in the first part of Section 2, are verified by the family of sets defined by (4.7) and (4.8).

In fact, (a) and (b) are obvious. Concerning (c), for every $E \subset\subset \overline{A} \setminus S$, in this case we have

$$A_E = \{x \in A : \delta(x) > \operatorname{dist}(x, E)\}\,,$$

so that A_E is bounded by Theorem 1.3 in [20], and $\overline{A_E} \subset \overline{A} \setminus S$ by (4.4). Finally, setting

$$l(x) := \delta^n(x), \qquad x \in A\,, \tag{4.9}$$

the condition (d) follows from the second property in (4.6). ■

Consider now the following condition:

h_0) there exist $\delta \in \mathcal{A}_0(A)$ and $\theta_0 \in \left]0, \frac{\pi}{2}\right[$ such that

$$\forall x \in A \;\; \exists C_{\theta_0}(x): \quad \overline{C_{\theta_0}(x, \delta(x))} \subset A\,,$$

where $C_\theta(x)$, $x \in \mathbb{R}^n$, $\theta \in \left]0, \frac{\pi}{2}\right[$, is an open infinite cone with vertex x and aperture θ, and $C_\theta(x, h) := C_\theta(x) \cap B(x, h)$, $h \in \mathbb{R}_+$.

Fixed an open subset A of $\mathbb{R}^n$ verifying the h_0)-property, we denote by $\Gamma(A, \theta, h)$ the family of open cones C with aperture θ, height h and such that $C \subset\subset A$. Let δ and θ_0 be the function and the parameter in the condition h_0), respectively, and let γ_δ be a constant for which (4.2) is verified by the function δ. For every $x \in A$, denote by $G_A(x)$ the union of all cones C in $\Gamma(A, \theta_0, \gamma_\delta^{-1}\delta(x))$ such that $x \in C$. It is easy to show that $G_A(x)$ has the cone property with a cone $C \in \Gamma(A, \theta_1, \lambda_1 \delta(x))$, where θ_1 and λ_1 depend only on n, θ_0 and γ_δ, and that

$$G_A(x) \subset A(x) \qquad \forall x \in A\,,$$

where $A(x)$ is defined by (4.7). Put finally

$$F_A(x) := \{y \in A : x \in G_A(y)\}\,, \qquad x \in A\,.$$

Then clearly

$$y \in F_A(x) \;\Leftrightarrow\; x \in G_A(y), \quad x, y \in A\,;$$

moreover, for any $x \in A$, $F_A(x)$ is measurable and there exist $c_1, c_2 \in \mathbb{R}_+$, depending only on n, θ_0 and γ_δ, such that

$$c_1\,\delta^n(x) \leqslant |F_A(x)| \leqslant c_2\,\delta^n(x) \qquad \forall x \in A \tag{4.10}$$

(see, for instance, Lemma 4.1 in [4]).

5. - Embedding theorems

Let Ω be an open subset of $\mathbb{R}^n$, $n \geqslant 2$, and consider a function $\varrho \in \mathcal{A}_0(\Omega)$ such that $S_\varrho \neq \emptyset$. Moreover, let $\alpha_0, \alpha_1 : \Omega \to \mathbb{R}_+$, $g : \Omega \to \mathbb{C}$ be functions and let m, r, p_0, p_1, p, a, t_0, t_1, τ be parameters satisfying the following conditions:

$$(i_1)\quad \begin{cases} m, r \in \mathbb{N}_0, \qquad r < m, \\ 1 \leqslant p_i \leqslant p < +\infty, \qquad (p_i - 1)^{-1} \leqslant t_i \leqslant +\infty, \qquad i = 0, 1, \\ 1 \leqslant \tau \leqslant +\infty, \qquad \tau > 1 \text{ if } 1 < \dfrac{p_1 t_1}{1 + t_1} = \dfrac{n}{m - r}, \\ \exists a \in \left[\dfrac{r}{m}, 1\right]: \qquad \dfrac{\tau - 1}{p\tau} \geqslant \dfrac{r}{n} + a\left(\dfrac{1 + t_1}{p_1 t_1} - \dfrac{m}{n}\right) + (1 - a)\left(\dfrac{1 + t_0}{p_0 t_0}\right); \end{cases}$$

$$(i_2)\qquad g \in L^{p\tau}_{\mathrm{loc}}(\overline{\Omega} \setminus S_\varrho), \qquad \alpha_i \in L^1_{\mathrm{loc}}(\Omega), \qquad \alpha_i^{-1} \in L^{t_i}_{\mathrm{loc}}(\overline{\Omega} \setminus S_\varrho), \qquad i = 0, 1.$$

Put

$$q := \frac{p\tau}{\tau - 1}, \qquad q_i := \frac{p_i t_i}{1 + t_i} \qquad i = 0, 1, \qquad p\tau =: \sigma\,,$$

where $q = +\infty$ if $\tau = 1$. Write $U^m_{p_0, p_1}(\Omega, \alpha_0, \alpha_1) = \overset{\circ}{W}{}^m_{p_0, p_1}(\Omega, \alpha_0, \alpha_1)$ if Ω has the h_0)-property with $\delta = \varrho$ and $U^m_{p_0, p_1}(\Omega, \alpha_0, \alpha_1) = W^m_{p_0, p_1}(\Omega, \alpha_0, \alpha_1)$ if Ω does not have the h_0)-property.

In the following we shall denote by E an open subset of Ω, $E \subset\subset \overline{\Omega} \setminus S_\varrho$, having the cone property when Ω has the h_0)-property with $\delta = \varrho$, while $E := E_0 \cap \Omega$, where E_0 is a bounded open subset of $\mathbb{R}^n$ with the cone property, when Ω does not have the h_0)-property. Further we denote by θ and h the aperture and the height of the characteristic cone of E or E_0, respectively. Put finally

$$H_i := b_i\,|g|_{\sigma, E}\,|\alpha_i^{-1}|^{1/p_i}_{t_i, E}, \qquad i = 0, 1,$$

where

$$b_0 := h^{-r+n(\frac{1}{q}-\frac{1}{q_0})}, \qquad b_1 := h^{m-r+n(\frac{1}{q}-\frac{1}{q_1})}.$$

LEMMA 5.1: *If i_1) and i_2) hold, then for every $u \in U^m_{p_0,p_1}(\Omega, \alpha_0, \alpha_1)$ we have $g\partial^\mu u \in L^p(E)$ when $|\mu| = r$ and*

$$(5.1) \qquad |g\partial^\mu u|_{p,E} \leq c(H_0 \|u\|_{L^{p_0}(E,\alpha_0)} + H_0^{1-a} H_1^a \|u\|^{1-a}_{L^{p_0}(E,\alpha_0)} \|\partial^m u\|^a_{L^{p_1}(E,\alpha_1)}),$$

where $c \in \mathbb{R}_+$ depends only on m, r, q, q_0, q_1, a and θ.

PROOF: Fixed $u \in U^m_{p_0,p_1}(\Omega, \alpha_0, \alpha_1)$, since $E \subset\subset \overline{\Omega} \setminus S_\varrho$ and $\alpha_i^{-1} \in L^{t_i}_{\mathrm{loc}}(\overline{\Omega} \setminus S_\varrho)$, $i = 0, 1$, it follows that $u_{|E} \in W^m_{q_0,q_1}(E)$ and

$$(5.2) \qquad |\partial^m u|_{q_1,E} \leq |\alpha_1^{-1}|^{1/p_1}_{t_1,E} \|\partial^m u\|_{L^{p_1}(E,\alpha_1)},$$

$$(5.3) \qquad |u|_{q_0,E} \leq |\alpha_0^{-1}|^{1/p_0}_{t_0,E} \|u\|_{L^{p_0}(E,\alpha_0)}.$$

If E has the cone property, let

$$\chi : x \in E \mapsto \frac{x}{h}, \qquad E^\star := \chi(E), \qquad u^\star := u \circ \chi^{-1} : y \in E^\star \mapsto u(hy).$$

We point out that:

1) $E^\star$ has the cone property with characteristic cone having aperture θ and unitary height;

2) $u^\star \in W^m_{q_0,q_1}(E^\star)$ and $|\partial^\varsigma u^\star|_{t,E^\star} = h^{|\varsigma| - n/t} |\partial^\varsigma u|_{t,E}$.

From well-known embedding theorems for bounded open subset of $\mathbb{R}^n$ endowed with the cone property (see, for instance, [10], [13]), we deduce that $\partial^\mu u^\star \in L^q(E^\star)$ when $|\mu| = r$ and

$$(5.4) \qquad |\partial^\mu u^\star|_{q,E^\star} \leq c(|u^\star|_{q_0,E^\star} + |\partial^m u^\star|^a_{q_1,E^\star} |u^\star|^{1-a}_{q_0,E^\star}),$$

where $c \in \mathbb{R}_+$ depends only on m, r, q, q_0, q_1, a and θ.

By the previous observations and from (5.4) it follows that

$$(5.5) \qquad |\partial^\mu u|_{q,E} \leq c(h^{-r+n(\frac{1}{q}-\frac{1}{q_0})} |u|_{q_0,E} + h^{(m-\frac{n}{q_1})a - \frac{n}{q_0}(1-a) - r + \frac{n}{q}} |\partial^m u|^a_{q_1,E} |u|^{1-a}_{q_0,E}).$$

On the other hand, if $E := E_0 \cap \Omega$ and u_0 is the zero extension of u outside Ω, we can define:

$$\chi_1 : x \in E_0 \mapsto \frac{x}{h}, \qquad E_0^\star := \chi_1(E_0), \qquad u_0^\star := u_0 \circ \chi_1^{-1} : y \in E_0^\star \mapsto u_0(hy).$$

Then, arguing as before, we can show that (5.4) holds with $E_0^\star$ and $u_0^\star$ instead of $E^\star$ and $u^\star$, respectively; it follows that (5.5) is true with E_0 and u_0 instead of E and u, respectively, so that (5.5) still holds with $E := E_0 \cap \Omega$.
Therefore, by using (5.2), (5.3) and (5.5), we have

$$(5.6)\qquad |g\partial^\mu u|_{p,E} \le |g|_{\sigma,E}\, |\partial^\mu u|_{q,E} \le c(h^{-r+n(\frac{1}{q}-\frac{1}{q_0})}\, |g|_{\sigma,E}\, |\alpha_0^{-1}|_{t_0,E}^{1/p_0} \|u\|_{L^{p_0}(E,\alpha_0)} +$$

$$+ h^{(m-\frac{n}{q_1})a - \frac{n}{q_0}(1-a) - r + \frac{n}{q}}\, |g|_{\sigma,E}\, |\alpha_1^{-1}|_{t_1,E}^{a/p_1}\, |\alpha_0^{-1}|_{t_0,E}^{(1-a)/p_0} \|\partial^m u\|^a_{L^{p_1}(E,\alpha_1)} \|u\|^{1-a}_{L^{p_0}(E,\alpha_0)})$$

and this completes the proof. ■

Define now the functions $\beta_0, \beta_1 \colon \Omega \to \mathbb{R}_+$ as follows:

$$\beta_0(x) := (\varrho(x))^{-r+n(\frac{1}{q}-\frac{1}{q_0}+\frac{1}{\sigma})}\, |\alpha_0^{-1}|_{t_0,\Omega(x)}^{1/p_0}, \qquad x \in \Omega,$$

$$\beta_1(x) := (\varrho(x))^{m-r+n(\frac{1}{q}-\frac{1}{q_1}+\frac{1}{\sigma})}\, |\alpha_1^{-1}|_{t_1,\Omega(x)}^{1/p_1}, \qquad x \in \Omega\,.$$

In the sequel we consider the following alternative assumptions:

i_3) Ω has the h_0)-property with $\delta = \varrho$;

i_3') Ω does not have the h_0)-property and there exists a function $\tilde\varrho \in \mathcal{A}_0(\mathbb{R}^n \setminus S_\varrho)$ such that $\tilde\varrho_{|\Omega} \equiv \varrho$.

Write

$$\gamma = \gamma_\varrho, \qquad \tilde\gamma = \gamma_{\tilde\varrho}.$$

We can now prove the following

Theorem 5.1: *If i_1), i_2) hold and, in addition, either i_3) or i_3') is satisfied, then for every $g \in E^\sigma_{\beta_0}(\Omega) \cap E^\sigma_{\beta_1}(\Omega)$ and any $u \in U^m_{p_0,p_1}(\Omega, \alpha_0, \alpha_1)$ we have that $g\partial^\mu u \in L^p(\Omega)$ when $|\mu| = r$ and*

$$(5.8)\qquad |g\partial^\mu u|_{p,\Omega} \le c(\|g\|_{E^\sigma_{\beta_0}(\Omega)} \|u\|_{L^{p_0}(\Omega,\alpha_0)} +$$

$$+ \|g\|^a_{E^\sigma_{\beta_1}(\Omega)} \|g\|^{1-a}_{E^\sigma_{\beta_0}(\Omega)} \|\partial^m u\|^a_{L^{p_1}(\Omega,\alpha_1)} \|u\|^{1-a}_{L^{p_0}(\Omega,\alpha_0)}),$$

where $c \in \mathbb{R}_+$ depends only on $n, \gamma, \theta_0, m, r, q, q_0, q_1, a$ when i_3) holds, while it depends only on $n, \tilde\gamma, m, r, q, q_0, q_1, a$ if i_3') is verified.

Proof: Assume that i_1), i_2) and i_3) hold.
Given $g \in E^\sigma_{\beta_0}(\Omega) \cap E^\sigma_{\beta_1}(\Omega)$, $u \in W^m_{p_0,p_1}(\Omega, \alpha_0, \alpha_1)$ and $\mu \in \mathbb{N}_0^n$ such that $|\mu| = r$,

from (4.10) it follows:

$$(5.9)\qquad \int_\Omega |g|^p\,|\partial^\mu u|^p\,dx = \int_\Omega |g|^p\,|\partial^\mu u|^p \left(\int_{F_\Omega(x)} dy\right)\left(\int_{F_\Omega(x)} dy\right)^{-1} dx$$

$$\leq c_1 \int_\Omega \varrho^{-n}(x)\,|g(x)|^p\,|\partial^\mu u(x)|^p\,dx \int_\Omega \chi_{F_\Omega(x)}(y)\,dy$$

$$= c_1 \int_\Omega dy \int_\Omega \varrho^{-n}(x)\,|g(x)|^p\,|\partial^\mu u(x)|^p \chi_{F_\Omega(x)}(y)\,dx$$

$$= c_1 \int_\Omega dy \int_{G_\Omega(y)} \varrho^{-n}(x)\,|g(x)|^p\,|\partial^\mu u(x)|^p\,dx$$

$$\leq c_2 \int_\Omega \varrho^{-n}(y)\,dy \int_{G_\Omega(y)} |g(x)|^p\,|\partial^\mu u(x)|^p\,dx,$$

where $c_1, c_2 \in \mathbb{R}_+$ depend only on n, γ and θ_0. From (5.9) and Lemma 5.1 we have:

$$(5.10)\qquad \int_\Omega |g|^p\,|\partial^\mu u|^p\,dx \leq c_3 \Bigg(\int_\Omega (\varrho(y))^{(-r+\frac{n}{q}-\frac{n}{q_0})p}\,|g|^p_{\sigma,\,G_\Omega(y)}\,|\alpha_0^{-1}|^{p/p_0}_{t_0,\,G_\Omega(y)} \cdot$$

$$\cdot\,\varrho^{-n}(y)\left(\int_{G_\Omega(y)} \alpha_0\,|u|^{p_0}\,dx\right)^{p/p_0} dy\,+$$

$$+\int_\Omega (\varrho(y))^{(m-r+\frac{n}{q}-\frac{n}{q_1})ap}\,(\varrho(y))^{(-r+\frac{n}{q}-\frac{n}{q_0})(1-a)p}\cdot$$

$$\cdot\,|g|^{ap}_{\sigma,\,G_\Omega(y)}\,|\alpha_1^{-1}|^{ap/p_1}_{t_1,\,G_\Omega(y)}\,|g|^{(1-a)p}_{\sigma,\,G_\Omega(y)}\,|\alpha_0^{-1}|^{\frac{(1-a)p}{p_0}}_{t_0,\,G_\Omega(y)}\,\varrho^{-n}(y)\cdot$$

$$\cdot\left(\int_{G_\Omega(y)} \alpha_1\,|\partial^m u|^{p_1}\,dx\right)^{ap/p_1}\left(\int_{G_\Omega(y)} \alpha_0\,|u|^{p_0}\,dx\right)^{\frac{(1-a)p}{p_0}} dy\Bigg),$$

where $c_3 \in \mathbb{R}_+$ depend only on $n, \gamma, \theta_0, m, r, q, q_0, q_1$ and a.

Observe now that, from Lemma 1.2 in [19], one has:

$$(5.11)\qquad \int_\Omega \varrho^{-n}(y)\left(\int_{G_\Omega(y)} \alpha_0\,|u|^{p_0}\,dx\right)^{p/p_0} dy \leq c_4 \left(\int_\Omega \alpha_0\,|u|^{p_0}\,dx\right)^{p/p_0}$$

and

$$(5.12)\qquad \int_\Omega \varrho^{-na}(y)\left(\int_{G_\Omega(y)} \alpha_1\,|\partial^m u|^{p_1}\,dx\right)^{ap/p_1} \varrho^{-n(1-a)}(y)\left(\int_{G_\Omega(y)} \alpha_0\,|u|^{p_0}\,dx\right)^{\frac{(1-a)p}{p_0}} dy$$

$$\leqslant \left(\int_\Omega \varrho^{-n}(y)\left(\int_{G_\Omega(y)} \alpha_1\,|\partial^m u|^{p_1}\,dx\right)^{p/p_1} dy\right)^{a}\left(\int_\Omega \varrho^{-n}(y)\left(\int_{G_\Omega(y)} \alpha_0\,|u|^{p_0}\,dx\right)^{p/p_0} dy\right)^{1-a}$$

$$\leqslant c_5\left(\int_\Omega \alpha_1\,|\partial^m u|^{p_1}\,dx\right)^{ap/p_1}\left(\int_\Omega \alpha_0\,|u|^{p_0}\,dx\right)^{\frac{(1-a)p}{p_0}},$$

where $c_4,\ c_5 \in \mathbb{R}_+$ depend only on n, γ and θ_0. By using (5.10), (5.11), (5.12), we easily deduce (5. 8) and the required result.

Suppose now that i_1), i_2) and i_3') hold. It is easy to prove that

$$\Omega_0 := \bigcup_{x\in\Omega} B\left(x,\ \frac{1}{2}\varrho(x)\right),$$

has the h_0)-property with $\delta = \frac{1}{2\tilde{\gamma}}\,\underset{\circ}{\tilde{\varrho}}$ and for any $\theta_0 \in \left]0,\ \frac{\pi}{2}\right[$.

Given $g \in E^\sigma_{\beta_0}(\Omega) \cap E^\sigma_{\beta_1}(\Omega)$, $u \in \overset{\circ}{W}{}^m_{p_0,\,p_1}(\Omega,\ \alpha_0,\ \alpha_1)$ and $\mu \in \mathbb{N}_0^n$ such that $|\mu| = r$, the same arguments used to obtain (5.9) yield:

$$(5.13)\qquad \int_\Omega |g|^p\,|\partial^\mu u|^p\,dx = \int_{\Omega_0} |g_0|^p\,|\partial^\mu u_0|^p$$

$$\leqslant c_6 \int_{\Omega_0} \tilde{\varrho}^{-n}(y)\,dy \int_{G_{\Omega_0}(y)} |g_0(x)|^p\,|\partial^\mu u_0(x)|^p\,dx,$$

where $c_6 \in \mathbb{R}_+$ depend only on n and $\tilde{\gamma}$, and f_0 denotes the zero extension outside Ω of a function f defined on Ω.

From (5.13) and Lemma 5.1 we deduce:

$$(5.14)\qquad \int_\Omega |g|^p\,|\partial^\mu u|^p\,dx \leqslant c_7\left(\int_{\Omega_0} (\tilde{\varrho}(y))^{(-r+\frac{n}{q}-\frac{n}{q_0})p}\,|g|^p_{\sigma,\,\Omega\cap B(y,\,\frac{1}{2\tilde{\gamma}}\tilde{\varrho}(y))}\cdot\right.$$

$$\cdot|\alpha_0^{-1}|^{p/p_0}_{t_0,\,\Omega\cap B(y,\,\frac{1}{2\tilde{\gamma}}\tilde{\varrho}(y))}\,\tilde{\varrho}^{-n}(y)\left(\int_{G_{\Omega_0}(y)} (\alpha_0)_0\,|u_0|^{p_0}\,dx\right)^{p/p_0} dy\ +$$

$$+\int_{\Omega_0} (\tilde{\varrho}(y))^{(m-r+\frac{n}{q}-\frac{n}{q_1})ap}\,(\tilde{\varrho}(y))^{(-r+\frac{n}{q}-\frac{n}{q_0})(1-a)p}\,|g|^{ap}_{\sigma,\,\Omega\cap B(y,\,\frac{1}{2\tilde{\gamma}}\tilde{\varrho}(y))}\cdot$$

$$\cdot|\alpha_1^{-1}|^{ap/p_1}_{t_1,\,\Omega\cap B(y,\,\frac{1}{2\tilde\gamma}\tilde\varrho(y))}\,|g|^{(1-a)p}_{\sigma,\,\Omega\cap B(y,\,\frac{1}{2\tilde\gamma}\tilde\varrho(y))}\,|\alpha_0^{-1}|^{\frac{(1-a)p}{p_0}}_{t_0,\,\Omega\cap B(y,\,\frac{1}{2\tilde\gamma}\tilde\varrho(y))}\,\tilde\varrho^{-n}(y)\cdot$$

$$\cdot\left(\int_{G_{\Omega_0}(y)}(\alpha_1)_0\,|\partial^m u_0|^{p_1}\,dx\right)^{ap/p_1}\left(\int_{G_{\Omega_0}(y)}(\alpha_0)_0\,|u_0|^{p_0}\,dx\right)^{\frac{(1-a)p}{p_0}}dy\Bigg),$$

where $c_7\in\mathbb{R}_+$ depend only on n, $\tilde\gamma$, m, r, q, q_0, q_1 and a.

Observe now that, if $y\in\Omega_0\backslash\Omega$, there exists $x\in\Omega$ such that $y\in B\left(y,\ \frac{\tilde\varrho(x)}{2}\right)$. Hence it is easy to show that

$$\Omega\cap B\left(y,\ \frac{1}{2\,\tilde\gamma}\,\tilde\varrho(y)\right)\subset\Omega\cap B(x,\tilde\varrho(x))$$

and so

$$\sup_{y\in\Omega_0}(\tilde\varrho(y))^{-r+\frac{n}{q}-\frac{n}{q_0}}\,|\alpha_0^{-1}|^{1/p_0}_{t_0,\,\Omega\cap B(y,\,\frac{1}{2\tilde\gamma}\tilde\varrho(y))}\,|g|_{\sigma,\,\Omega\cap B(y,\,\frac{1}{2\tilde\gamma}\tilde\varrho(y))}\leqslant$$
$$\leqslant c_8\sup_{x\in\Omega}(\varrho(x))^{-r+\frac{n}{q}-\frac{n}{q_0}}\,|\alpha_0^{-1}|^{1/p_0}_{t_0,\,\Omega(x)}\,|g|_{\sigma,\,\Omega(x)},\tag{5.15}$$

$$\sup_{y\in\Omega_0}(\tilde\varrho(y))^{m-r+\frac{n}{q}-\frac{n}{q_1}}\,|\alpha_1^{-1}|^{1/p_1}_{t_1,\,\Omega\cap B(y,\,\frac{1}{2\tilde\gamma}\tilde\varrho(y))}\,|g|_{\sigma,\,\Omega\cap B(y,\,\frac{1}{2\tilde\gamma}\tilde\varrho(y))}\leqslant$$
$$\leqslant c_9\sup_{x\in\Omega}(\varrho(x))^{m-r+\frac{n}{q}-\frac{n}{q_1}}\,|\alpha_1^{-1}|^{1/p_1}_{t_1,\,\Omega(x)}\,|g|_{\sigma,\,\Omega(x)},\tag{5.16}$$

where c_8, $c_9\in\mathbb{R}_+$ depend only on $\tilde\gamma$, m, r, q, q_0 and q_1.

Using (5.14), (5.15) and (5.16), the same argument of the first case gives the required result. ■

6. - Compactness results

For every $k\in\mathbb{N}$ we denote by Ω'_k an open subset of $\mathbb{R}^n$ having the cone property such that

$$\operatorname{supt}\psi_k\subset\Omega'_k\subset\subset\Omega\,,$$

where the functions ψ_k are the ones defined in Section 2.

Lemma 6.1: *If i_1), i_2) are satisfied and*

$$\frac{1}{p}>\frac{r}{n}+\frac{1}{q_1}-\frac{m}{n}\,,\tag{6.1}$$

then, for every fixed $\varepsilon\in\mathbb{R}_+$ and $k\in\mathbb{N}$, there exists $c(\varepsilon,k)\in\mathbb{R}_+$ such that for $|\mu|=r$

the following holds:

(6.2) $$|\psi_k g \partial^\mu u|_{p,\Omega} \leq \varepsilon |\partial^m u|_{q_1,\Omega'_k} + c(\varepsilon, k)\, |u|_{q_0,\Omega'_k} \qquad \forall u \in W^m_{p_0,p_1}(\Omega, \alpha_0, \alpha_1)\,.$$

Proof: Given $u \in W^m_{p_0,p_1}(\Omega, \alpha_0, \alpha_1)$ and $k \in \mathbb{N}$, clearly

(6.3) $$|\psi_k g \partial^\mu u|_{p,\Omega} \leq |g \partial^\mu u|_{p,\Omega'_k},$$

where Ω'_k is the open set defined above. Using (5.5) with $E = \Omega'_k$, we have

(6.4) $$|g\partial^\mu u|_{p,\Omega'_k} \leq |g|_{\sigma,\Omega'_k}\, |\partial^\mu u|_{q,\Omega'_k} \leq c(k)(|\partial^m u|^a_{q_1,\Omega'_k}\, |u|^{1-a}_{q_0,\Omega'_k} + |u|_{q_0,\Omega'_k})\,,$$

where $c(k) \in \mathbb{R}_+$ does not depend on u.

If $a < 1$, from (6.3) and (6.4), we easily deduce (6.2), while if i_1) is verified only for $a = 1$, we have

$$\frac{\tau - 1}{p\tau} = \frac{r}{n} + \frac{1}{q_1} - \frac{m}{n}\,;$$

hence it follows from (6.1) that $\tau < +\infty$, and so, for any $\lambda \in \mathbb{R}_+$, there exists $g_\lambda \in L^\infty(\Omega'_k)$ such that

$$|g - g_\lambda|_{\sigma,\Omega'_k} \leq \lambda\,.$$

Now clearly

(6.5) $$|g\partial^\mu u|_{p,\Omega'_k} \leq |(g - g_\lambda)\, \partial^\mu u|_{p,\Omega'_k} + |g_\lambda|_{\infty,\Omega'_k}\, |\partial^\mu u|_{p,\Omega'_k};$$

using (5.5) with $E = \Omega'_k$, we have:

(6.6) $$|(g - g_\lambda)\, \partial^\mu u|_{p,\Omega'_k} \leq |g - g_\lambda|_{\sigma,\Omega'_k}\, |\partial^\mu u|_{q,\Omega'_k} \leq \lambda c(k) \|u\|_{W^m_{q_0,q_1}(\Omega'_k)},$$

with $c(k) \in \mathbb{R}_+$ independent of u and λ.

On the other hand, (6.1) implies that there exists $a_1 \in \left[\frac{r}{m}, 1\right[$ such that

$$\frac{1}{p} \geq \frac{r}{n} + a_1\left(\frac{1}{q_1} - \frac{m}{n}\right) + (1 - a_1)\,\frac{1}{q_0}\,;$$

thus, by known results (see for instance [13]), it follows that for every $\eta \in \mathbb{R}_+$ there exists $c(\eta, k) \in \mathbb{R}_+$, independent of u, such that

(6.7) $$|\partial^\mu u|_{p,\Omega'_k} \leq \eta\, |\partial^m u|_{q_1,\Omega'_k} + c(\eta, k)\, |u|_{q_0,\Omega'_k}.$$

Therefore from (6.3), (6.5)-(6.7) we deduce that (6.2) holds. ■

THEOREM 6.1: *In the hypotheses of Theorem* 5.1, *if*

$$(6.8)\qquad \begin{cases} g \in \overset{\circ}{E}{}^{\sigma}_{\beta_0}(\Omega) & \text{when } a < 1\,, \\ g \in \overset{\circ}{E}{}^{\sigma}_{\beta_0}(\Omega) \cap \overset{\circ}{E}{}^{\sigma}_{\beta_1}(\Omega) & \text{when } a = 1\,, \end{cases}$$

and (6.1) *holds, then for every* $\varepsilon \in \mathbb{R}_+$ *there exist* $c(\varepsilon) \in \mathbb{R}_+$ *and an open set* $\Omega_\varepsilon \subset\subset \Omega$ *such that for* $|\mu| = r$ *we have*:

$$(6.9)\quad |g\partial^\mu u|_{p,\,\Omega} \leq \varepsilon \|u\|_{W^m_{p_0,\,p_1}(\Omega,\,\alpha_0,\,\alpha_1)} + c(\varepsilon)\,|u|_{q_0,\,\Omega_\varepsilon} \qquad \forall u \in U^m_{p_0,\,p_1}(\Omega,\,\alpha_0,\,\alpha_1)\,.$$

PROOF: Given $u \in U^m_{p_0,\,p_1}(\Omega,\,\alpha_0,\,\alpha_1)$ and $\mu \in \mathbb{N}_0^n$ with $|\mu| = r$, clearly

$$(6.10)\qquad |g\partial^\mu u|_{p,\,\Omega} \leq |\psi_k g \partial^\mu u|_{p,\,\Omega} + |(1-\psi_k)\, g\partial^\mu u|_{p,\,\Omega}\,,$$

where the functions ψ_k are the ones defined in Section 2.

Moreover, by Lemma 6.1 we have that for any $\varepsilon \in \mathbb{R}_+$ and $k \in \mathbb{N}$ there exists $c(\varepsilon,\,k) \in \mathbb{R}_+$, independent of u, such that

$$(6.11)\qquad |\psi_k g\partial^\mu u|_{p,\,\Omega} \leq \varepsilon \|\partial^m u\|_{L^{p_1}(\Omega'_k,\,\alpha_1)} + c(\varepsilon,\,k)\,|u|_{q_0,\,\Omega'_k}.$$

On the other hand, by Theorem 5.1 the following holds:

$$(6.12)\qquad |(1-\psi_k)\, g\partial^\mu u|_{p,\,\Omega} \leq c(\|(1-\psi_k)\, g\|_{E^\sigma_{\beta_0}(\Omega)} \|u\|_{L^{p_0}(\Omega,\,\alpha_0)} + \|(1-\psi_k)\, g\|^a_{E^\sigma_{\beta_1}(\Omega)} \cdot$$
$$\cdot \|(1-\psi_k)\, g\|^{1-a}_{E^\sigma_{\beta_0}(\Omega)} \|\partial^m u\|^a_{L^{p_1}(\Omega,\,\alpha_1)} \|u\|^{1-a}_{L^{p_0}(\Omega,\,\alpha_0)})\,,$$

with $c \in \mathbb{R}_+$ independent of u and k.

Let now $a < 1$. It follows from (6.12) that for any $\varepsilon \in \mathbb{R}_+$ there exists $c_1(\varepsilon) \in \mathbb{R}_+$, independent of u and k, such that

$$(6.13)\qquad |(1-\psi_k)\, g\partial^\mu u|_{p,\Omega} \leq \varepsilon \|(1-\psi_k)\, g\|_{E^\sigma_{\beta_1}(\Omega)} \|\partial^m u\|_{L^{p_1}(\Omega,\alpha_1)} +$$
$$+ c_1(\varepsilon)\|(1-\psi_k)\, g\|_{E^\sigma_{\beta_0}(\Omega)} \|u\|_{L^{p_0}(\Omega,\,\alpha_0)}.$$

By Lemma 3.1, fixed $\varepsilon \in \mathbb{R}_+$, there exists $k_\varepsilon \in \mathbb{N}$ such that

$$(6.14)\qquad \|(1-\psi_{k_\varepsilon})\, g\|_{E^\sigma_{\beta_0}(\Omega)} \leq \varepsilon\,.$$

The result can now be deduced from (6.10), (6.11), (6.13) and (6.14).

Consider the case $a = 1$. By Lemma 3.1, fixed $\varepsilon \in \mathbb{R}_+$ there exists $k_\varepsilon \in \mathbb{N}$ such that

$$(6.15)\qquad \|(1-\psi_{k_\varepsilon})\, g\|_{E^\sigma_{\beta_0}(\Omega)} \leq \varepsilon\,, \qquad \|(1-\psi_{k_\varepsilon})\, g\|_{E^\sigma_{\beta_1}(\Omega)} \leq \varepsilon\,.$$

Then, from (6.10)-(6.12) and (6.15) we deduce the result. ■

THEOREM 6.2: *In the hypotheses of Theorem* 6.1 *and if*

$$\frac{1}{q_1} - \frac{m}{n} < \frac{1}{q_0}, \tag{6.16}$$

then for $|\mu| = r$ *the operator*

$$u \in U^m_{p_0, p_1}(\Omega, \alpha_0, \alpha_1) \mapsto g\partial^\mu u \in L^p(\Omega) \tag{6.17}$$

is compact.

PROOF: It follows from Theorem 6.1 arguing as was done in [11] in order to deduce Theorem 3.3 from Theorem 3.2. ■

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