Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
119° (2001), Vol. XXV, fasc. 1, pagg. 111-124

M. BAHAJ - O. SIDKI (*)

On the Existence of Solutions of Nonautonomous Neutral Functional Differential Equations (**)

ABSTRACT. — Existence and uniqueness of the solution of a class of nonlinear nonautonomous neutral functional differential equations is proved, in the case the initial value space is $W^{1,1}$. The proof uses the contraction mapping principle.

Sull'esistenza delle soluzioni per certe equazioni differenziali funzionali non autonome

SUNTO. — Usando il principio delle contrazioni, si dimostrano alcuni risultati di esistenza e di unicità per le soluzioni di una certa classe di equazioni differenziali funzionali non autonome e non lineari.

1. - INTRODUCTION

This paper deals with a class of functional differential equation of neutral type with values in a Banach space. A neutral functional differential equation (N.F.D.E) is an equation of the form

$$x'(t) = G(x_t),$$

where G is defined on a subset D of the space of functions from $[-r, 0]$ into X. Here

$$D = W^{1,1}([-r, 0]; X).$$

(*) Indirizzi degli Autori: M. Bahaj, Faculty of Sciences and Technology, Settat, Morocco; O. Sidki: Faculty of Sciences and Technology, Fez Saïss, Fez, Morocco.
(**) Memoria presentata il 18 giugno 2001 da Giorgio Letta, uno dei XL.

A special class of (N.F.D.E) is the class of (retarded type). In this case

$$D = C([-r, 0]; X).$$

Most papers in this domain traited the case $X = \mathbb{R}^n$ (see Hale [6], [7], Webb [12]). The case of infinite dimension has been considered by Dyson and Villella-Bressan [4], Plant [9], and Flaschaka and Leitman [5]. In [9] Plant used the nonlinear semigroup theory to study the (N.F.D.E)

$$x'(t) = G(x_t), \qquad x_0 = \varphi \in C^1([-r, 0]; X), \qquad 0 \leq t \leq T$$

where $G : C^1([-r, 0]; X) \to X$, is Lipschitz continuous. Many authors have used the semigroup approach to neutral equations (see Kunisch [8], Salomon [10]).

In this paper we consider the following nonlinear nonautonomous neutral differential equation:

$$\dot{x}(t) = F(t, x_t), \qquad x_0 = \varphi \in W^{1,1}([-r, 0]; X), \qquad 0 \leq t \leq T \tag{1.1}$$

where $x : [-r, T] \to X$, $0 < r < +\infty$ is the delay and X is a Banach space with norm $|.|$, $F : [0, T] \times W^{1,1}([-r, 0]; X) \to X$ and finally x_t is the history of x at time t defined pointwise by

$$x_t(\theta) = x(t + \theta), \quad \text{for } \theta \in [-r, 0].$$

Throughout this paper we shall assume:

$H_{(F)}$: $F : [0, T] \times W^{1,1}([-r, 0]; X) \to X$ is continuous in (t, φ) and Lipschitz continuous for all $t \in [0, T]$, that is,

$$|F(t, \varphi) - F(t, \psi)| \leq \gamma(t) \|\varphi - \psi\|_{1,1},$$

for some bounded $\gamma(t) \in \mathbb{R}$ and for all $\varphi, \psi \in W^{1,1}$.

At this stage note that the evolution equation associated with (1.1) was studied by J. Dyson and R. Villella-Bressan [4] using the theory of nonlinear operators. They proved the existence and regularity of solutions of (1.1) if F satisfies $H_{(F)}$ and $H'_{(F)}$: There exists a continuous function $h : [0, T] \to X$ which is of bounded variation and a monotone increasing function $L : [0, +\infty) \to [0, +\infty)$ such that:

$$|F(t_1, \varphi) - F(t_2, \varphi)| \leq |h(t_1) - h(t_2)| L(\|\varphi\|_{1,1}),$$

for $0 \leq t_1, t_2 \leq T$ and $\varphi \in W^{1,1}$. The method used in [4] has been to describe an evolution operator starting from the infinitesimal generator, using the Crandall and Pazy theorem [2]. Our approach of these problems is based on a direct method by means of an integral equation. For further details on nonlinear autonomous neutral functional differential equation (see the earlier work [11], and for nonlinear operators see ([3], [9]). We start with the following definition.

DEFINITION 1.1.: [1] *A function* $f:[0,T]\to X$ *belongs to* $W^{1,p}([0,T];X)$ *if and only if there exists a function* $g\in L^p([0,T];X)$ *such that* $f(t)=f(0)+\int_0^t g(s)\,ds$ *for all* $t\in[0,T]$.

For more details on these spaces, we refer the reader to [1].

The main result of this paper is the following theorem:

THEOREM 1.1: *Let F satisfy* $H_{(F)}$, $\varphi\in W^{1,1}$ *and* $E_\varphi=\{x\in W^{1,1}([-r,T];X): x=\varphi$ on $[-r,0]\}$. *Then, the equation* (1.1) *has a unique solution* $\bar{x}\in E_\varphi$, *for all* $T>0$ *and* $\varphi\in W^{1,1}$.

The following mapping will be used in the proof of theorem 1.0.1:

$$(Kx)(t)=\begin{cases}\varphi(0)+\int_0^t F(s,x_s)\,ds & \text{if } t>0\\ \varphi(t) & \text{if } t\in[-r,0].\end{cases}\tag{1.2}$$

Finally we also give an example of integro-differential satisfie $H_{(F)}$. This equation was studied by J. Dyson and R.Villella-Bressan in [4].

2. - PRELIMINARY RESULTS

Denote by $W^{1,1}([-r,0];X)$ the Banach space defined by:

$$W^{1,1}([-r,0];X)=\left\{\begin{array}{c}\varphi\in L^1([-r,0];X),\ \varphi \text{ is absolutely continuous},\\ \dot\varphi \text{ exists a.e, } \dot\varphi\in L^1([-r,0];X) \text{ and}\\ \varphi(\theta)=\varphi(0)+\int_0^\theta\dot\varphi(s)\,ds, \text{ for all } \theta\in[-r,0]\end{array}\right\}.$$

We shall denote the norm in $L^1=L^1([-r,0];X)$ by $\|.\|$ and in $W^{1,1}=W^{1,1}([-r,0];X)$ by $\|.\|_{1,1}$. So,

$$\|\varphi\|_{1,1}=\|\varphi\|+\|\dot\varphi\|.$$

Note that from [1] if $\dim(X)<+\infty$, or X is a reflexive Banach space, then each absolutely continuous function $x:[a, b]\to X$, is a.e. differentiable and

$$x(t) = x(a) + \int_a^t \dot{x}(s)\, ds\,.$$

In $W^{1,1}([-r, 0]; X)$, we define the norm $\|.\|_0$ by:

$$\|\varphi\|_0 = |\varphi(0)| + \int_{-r}^{0} |\dot{\varphi}(\theta)|\, d\theta\,,$$

for all $\varphi \in W^{1,1}$.

In ordre to prove the theorem 1.1 we need to prove the three following lemmas:

Lemma 2.1: *$\|.\|_0$ and $\|.\|_{1,1}$ are two norms equivalents in $W^{1,1}$:*

$$\frac{r}{1+2r}\|.\|_0 \leq \|.\|_{1,1} \leq (1+r)\|.\|_0.$$

Proof: Let $\varphi \in W^{1,1}$, then $\varphi(\theta) = \varphi(0) - \int_\theta^0 \dot{\varphi}(\tau)\, d\tau$, $-r \leq \theta \leq \tau \leq 0$. So: $|\varphi(\theta)| \leq |\varphi(0)| + \int_{-r}^0 |\dot{\varphi}(\tau)|\, d\tau$. By integrating this previous inequality on $[-r, 0]$, we get:

$$\int_{-r}^{0} |\varphi(\tau)|\, d\tau \leq r|\varphi(0)| + r\int_{-r}^{0} |\dot{\varphi}(\tau)|\, d\tau$$

and we add $\int_{-r}^0 |\dot{\varphi}(\tau)|\, d\tau$ both sides of this last inequality we have

$$\int_{-r}^{0} |\dot{\varphi}(\tau)|\, d\tau + \int_{-r}^{0} |\varphi(\tau)|\, d\tau \leq r|\varphi(0)| + (r+1)\int_{-r}^{0} |\varphi(\tau)|\, d\tau\,.$$

Hence,

$$\|\varphi\|_{1,1} \leq (1+r)\, \|\varphi\|_0 \tag{2.1}$$

To get the other inequality in Lemma 2.1 we write, $\varphi(0) = \varphi(\theta) + \int_\theta^0 \dot{\varphi}(\tau)\, d\tau$, for $-r \leq \theta \leq \tau \leq 0$. So: $|\varphi(0)| \leq |\varphi(\theta)| + \int_{-r}^0 |\dot{\varphi}(\tau)|\, d\tau$. Again by integration on $[-r, 0]$:

$$r|\varphi(0)| \leq \int_{-r}^{0} |\varphi(\tau)|\, d\tau + r\int_{-r}^{0} |\dot{\varphi}(\tau)|\, d\tau$$

and we add $r\int_{-r}^{0} |\dot{\varphi}(\tau)|d\tau$ both sides of this last inequality we have

$$r|\varphi(0)| + r\int_{-r}^{0} |\dot{\varphi}(\tau)|d\tau \leq \|\varphi\|_{1,1} + 2r\|\varphi\|_{1,1}.$$

Thus,

$$r\|\varphi\|_0 \leq (1+2r)\|\varphi\|_{1,1} \tag{2.2}$$

It follows, using (2.1) and (2.2), that:

$$\frac{r}{1+2r}\|\varphi\|_0 \leq \|\varphi\|_{1,1} \leq (1+r)\|\varphi\|_0. \qquad \blacksquare$$

LEMMA 2.2: *Let a, b, c be real numbers with $a \leq c \leq b$. If $u \in W^{1,1}([a, c]; X)$ and $v \in W^{1,1}([c, b]; X)$ such that: $u(c) = v(c)$. Then,*

$$w = \begin{cases} u \ on \ [a, c] \\ v \ on \ [c, b] \end{cases}$$

belongs to $W^{1,1}([a, b]; X)$.

PROOF: Let $u \in W^{1,1}([a, c]; X)$. We have: $u(c) = u(a) + \int_a^c \dot{u}(x)\,dx$ and for $\tau \in]c, b]$, $v(\tau) = v(c) + \int_c^\tau \dot{v}(x)\,dx$.

Hence

$$u(c) + v(\tau) = u(a) + v(c) + \int_a^c \dot{u}(x)\,dx + \int_c^\tau \dot{v}(x)\,dx\,.$$

Since $u(c) = v(c)$, we have for $\tau > c$

$$w(\tau) = v(\tau) = u(a) + \int_a^c \dot{u}(x)\,dx + \int_c^\tau \dot{v}(x)\,dx$$

$$= w(a) + \int_a^\tau \dot{w}(x)\,dx\,,$$

and consequently $w \in W^{1,1}([a, b]; X)$. $\blacksquare$

LEMMA 2.3: *For all $\varphi \in W^{1,1}([-r, 0]; X)$, $E_\varphi = E_0 + \{\tilde{\varphi}\}$, where $E_0 = \{x \in W^{1,1}([-r, T]; X): x = 0 \text{ on } [-r, 0]\}$ and $\tilde{\varphi} = \begin{cases} \varphi \text{ on } [-r, 0] \\ \varphi(0) \text{ on } [0, T]. \end{cases}$*

PROOF: For all $x \in E_\varphi$, we have $x = (x - \tilde{\varphi}) + \tilde{\varphi}$ and $(x - \tilde{\varphi})_{|[-r, 0]} = \varphi - \varphi = 0$, and then $(x - \tilde{\varphi}) \in E_0$. $\blacksquare$

3. - Local existence of solutions

Proposition 3.1: *Let $x \in E_\varphi$. Then, the following properties are satisfied*

i) $x_s \in W^{1,1}([-r, 0]; X)$, *for all* $s \in [0, T]$.

ii) The map: $s \in [0, T] \to x_s \in W^{1,1}([-r, 0]; X)$ *is continuous on* $[0, T]$.

iii) $Kx \in E_\varphi$ *and* K *is continuous, Lipschitz on* E_φ *with Lipschitz constant* $\gamma T(T+1)$, *where* $\gamma = \sup_{t \in [0, T]} \gamma(t)$.

Proof: i) The result is a consequence of Lemma 2.2.

ii) Let $s_1, s_2 \in [0, T]$, with $s_2 > s_1$, we have

$$\|x_{s_2} - x_{s_1}\|_0 = |x_{s_2}(0) - x_{s_1}(0)| + \int_{-r}^{0} |\dot{x}_{s_2}(\theta) - \dot{x}_{s_1}(\theta)| d\theta .$$

$|x_{s_2}(0) - x_{s_1}(0)| = |x(s_2) - x(s_1)| \to 0$, as $s_2 \to s_1$.

We consider two cases $1/T \leq r$. Let $0 \leq s_1 < s_2 \leq T$ then $-r \leq -s_2 < -s_1 \leq 0$, so

$$\int_{-r}^{0} |\dot{x}_{s_2}(\theta) - \dot{x}_{s_1}(\theta)| d\theta = \int_{-r}^{-s_2} |\dot{x}_{s_2}(\theta) - \dot{x}_{s_1}(\theta)| d\theta + \int_{-s_2}^{-s_1} |\dot{x}_{s_2}(\theta) - \dot{x}_{s_1}(\theta)| d\theta + \int_{-s_1}^{0} |\dot{x}_{s_2}(\theta) - \dot{x}_{s_1}(\theta)| d\theta .$$

We denote by $\mathcal{C}([-r, 0]; X)$ the space of continuous functions which is dense in $L^1([-r, 0]; X)$, and we put:

$$I_1(s_1, s_2) = \int_{-r}^{-s_2} |\dot{x}_{s_2}(\theta) - \dot{x}_{s_1}(\theta)| d\theta ,$$

$$I_2(s_1, s_2) = \int_{-s_2}^{-s_1} |\dot{x}_{s_2}(\theta) - \dot{x}_{s_1}(\theta)| d\theta ,$$

$$I_3(s_1, s_2) = \int_{-s_1}^{0} |\dot{x}_{s_2}(\theta) - \dot{x}_{s_1}(\theta)| d\theta$$

$I_1(s_1, s_2) = \int_{-r}^{-s_2} |\dot{x}_{s_2}(\theta) - \dot{x}_{s_1}(\theta)| d\theta = \int_{-r}^{-s_2} |\dot{\varphi}(s_2 + \theta) - \dot{\varphi}(s_1 + \theta)| d\theta$, for $\dot{\varphi} \in L^1([-r, 0]; X)$. So, there exists a sequence of functions $\{\psi_n\}_n \subset \mathcal{C}([-r, 0]; X)$ such that $\|\psi_n - \dot{\varphi}\|_{L^1} \to 0$, as $n \to +\infty$.

Then,

$$I_1(s_1, s_2) \leqslant \int_{-r}^{-s_2} |\dot{\varphi}(s_2+\theta) - \psi_n(s_2+\theta)| d\theta + \int_{-r}^{-s_2} |\psi_n(s_2+\theta) - \psi_n(s_1+\theta)| d\theta$$

$$+ \int_{-r}^{-s_2} |\psi_n(s_1+\theta) - \dot{\varphi}(s_1+\theta)| d\theta$$

$$\leqslant 2\|\dot{\varphi} - \psi_n\| + \int_{-r}^{-s_2} |\psi_n(s_2+\theta) - \psi_n(s_1+\theta)| d\theta$$

and by density of $\mathcal{C}$ in L^1, we have $\|\psi_n - \dot{\varphi}\| \to 0$, as $n \to +\infty$. We also have

$$\psi_n \in \mathcal{C}, \quad \text{so} \quad |\psi_n(s_2+\theta) - \psi_n(s_1+\theta)| \to 0, \quad \text{as} \quad s_2 \to s_1.$$

Hence

$$I_1(s_1, s_2) \to 0, \quad \text{as} \quad s_2 \to s_1.$$

For the term I_2, we have

$$I_2(s_1, s_2) = \int_{-s_2}^{-s_1} |\dot{x}_{s_2}(\theta) - \dot{x}_{s_1}(\theta)| d\theta = \int_{-s_2}^{-s_1} |\dot{x}(s_2+\theta) - \dot{\varphi}(s_1+\theta)| d\theta .$$

Since $I_2(s_1, s_2)$ is absolutely continuous with respect to the measure associated with $\dot{x}(s_2+\theta) - \dot{\varphi}(s_1+\theta)$ for the measure of Lebesgue. Hence $I_2(s_1, s_2) \to 0$, as $s_2 \to s_1$. Finally, for the term I_3, we have

$$I_3(s_1, s_2) = \int_{-s_1}^{0} |\dot{x}_{s_2}(\theta) - \dot{x}_{s_1}(\theta)| d\theta = \int_{-s_1}^{0} |\dot{x}(s_2+\theta) - \dot{x}(s_1+\theta)| d\theta .$$

By density of $\mathcal{C}$ in L^1, we show in a similar argument can be used to prove that $I_2(s_1, s_2) \to 0$, as $s_2 \to s_1$. Hence,

$$s \to x_s \quad \text{is continuous on} \quad [0, T].$$

$2/T > r$. Let $s_1, s_2 \in [r, T]$, with $s_1 < s_2$, then $s_2 + \theta > s_1 + \theta \geqslant 0$.

Note that $s_1, s_2 \in [0, r]$ was studied in case 1.

So

$$\int_{-r}^{0} |\dot{x}_{s_2}(\theta) - \dot{x}_{s_1}(\theta)| d\theta = \int_{-r}^{0} |\dot{x}(s_2+\theta) - \dot{x}(s_1+\theta)| d\theta .$$

By density of $\mathcal{C}$ in L^1, we can show in a similar argument in this case that $\int_{-r}^{0} |\dot{x}_{s_2}(\theta) - \dot{x}_{s_1}(\theta)| d\theta$ goes to zero as $s_2 \to s_1$.

iii) Let $x \in E_\varphi$, i.e., $x \in W^{1,1}([0, T]; X)$ and $x = \varphi$ on $[-r, 0]$.

From (1.2) we have $Kx_{|[-r,\,0]} = \varphi$ and $(Kx)(0) = \varphi(0)$. So, for $t \in [0, T]$:

$$(Kx)(t) = (Kx)(0) + \int_0^t F(s, x_s)\, ds = (Kx)(0) + \int_0^t (\dot{K}x)(s)\, ds .$$

Hence, $Kx \in E_\varphi$. Finally, we prove that K is Lipschitz continuous on E_φ. Let $x, y \in E_\varphi$, then $x, y \in W^{1,1}([0, T]; X)$ and $x = y = \varphi$ on $[-r, 0]$.

$$\|Kx - Ky\|_{W^{1,1}([0,\,T];\,X)} = \int_0^T \left| \int_0^t (F(s, x_s) - F(s, y_s))\, ds \right| dt + \int_0^T |F(t, x_t) - F(t, y_t)|\, dt .$$

By the hypothesis $H_{(F)}$ we obtain

$$\text{(3.1)} \qquad \|Kx - Ky\|_{W^{1,1}([0,\,T];\,X)} \leqslant \gamma T \int_0^T \|x_s - y_s\|_{1,1}\, ds + \gamma \int_0^T \|x_s - y_s\|_{1,1}\, ds$$

$$\leqslant \gamma (T+1) \int_0^T \|x_s - y_s\|_{1,1}\, ds .$$

By definition of $W^{1,1}$ and x_s, we estimate $\|x_s - y_s\|_{1,1}$. We consider the following cases.

Case 1: $s \leqslant r$. In this case, we can write

$$\|x_s - y_s\|_{1,1} = \int_{-r}^0 |x_s(\theta) - y_s(\theta)|\, d\theta + \int_{-r}^0 |\dot{x}_s(\theta) - \dot{y}_s(\theta)|\, d\theta =$$

$$\int_{-s}^0 |x(s+\theta) - y(s+\theta)|\, d\theta + \int_{-s}^0 |\dot{x}(s+\theta) - \dot{y}(s+\theta)|\, d\theta ,$$

and by a change of variable $s + \theta = \tau$, we have

$$\|x_s - y_s\|_{1,1} = \int_0^s |x(\tau) - y(\tau)|\, d\tau + \int_0^s |\dot{x}(\tau) - \dot{y}(\tau)|\, d\tau$$

$$\leqslant \|x - y\|_{W^{1,1}([0,\,T];\,X)}$$

Case 2: $s > r$. Then $s + \theta > r + \theta \geqslant 0$ for all θ in $[-r, 0]$.
We can write

$$\|x_s - y_s\|_{1,1} = \int_{-r}^0 |x(s+\theta) - y(s+\theta)|\, d\theta + \int_{-r}^0 |\dot{x}(s+\theta) - \dot{y}(s+\theta)|\, d\theta .$$

A change of variable gives

$$\|x_s - y_s\|_{1,1} = \int_{s-r}^{s} |x(\tau) - y(\tau)|\,d\tau + \int_{s-r}^{s} |\dot{x}(\tau) - \dot{y}(\tau)|\,d\tau\,.$$

Since $[s-r, s] \subset [0, T]$ we have

$$\begin{aligned}\|x_s - y_s\|_{1,1} &\leq \int_{0}^{s} |x(\tau) - y(\tau)|\,d\tau + \int_{0}^{s} |\dot{x}(\tau) - \dot{y}(\tau)|\,d\tau \\ &\leq \int_{0}^{T} |x(\tau) - y(\tau)|\,d\tau + \int_{0}^{T} |\dot{x}(\tau) - \dot{y}(\tau)|\,d\tau \\ &= \|x - y\|_{W^{1,1}([0,T];X)}\,.\end{aligned}$$

Finally from all these estimates, we deduce that

$$\|x_s - y_s\|_{1,1} \leq \int_{0}^{s} |x(\tau) - y(\tau)|\,d\tau + \int_{0}^{s} |\dot{x}(\tau) - \dot{y}(\tau)|\,d\tau \tag{3.2}$$

$$\leq \|x - y\|_{W^{1,1}([0,T];X)}\,. \tag{3.3}$$

It follows from (3.1) and (3.3), that

$$\begin{aligned}\|Kx - Ky\|_{W^{1,1}([0,T];X)} &\leq \gamma(T+1)\int_{0}^{T} \|x - y\|_{W^{1,1}([0,T];X)}\,ds \\ &= \gamma T(T+1)\,\|x - y\|_{W^{1,1}([0,T];X)}\,.\end{aligned}$$

The proof is completed. ■

The following theorem is a immediate consequence of the following fact: If $T < \dfrac{-\gamma + \sqrt{4\gamma + \gamma^2}}{2\gamma}$, K is strict contraction from E_φ into E_φ and by the Banach fixed point theorem, there exists $\bar{x} \in E_\varphi$, such that $K\bar{x} = \bar{x}$. Thus $\bar{x}$ is a solution of (1.1) for all $t \in \left[0, \dfrac{-\gamma + \sqrt{4\gamma + \gamma^2}}{2\gamma}\right[$.

THEOREM 3.1: *Let F satisfy $H_{(F)}$. Then,* (1.1) *has one solution $\bar{x} \in E_\varphi$, for all $\varphi \in W^{1,1}([-r, 0]; X)$. $\bar{x}$ is defined on $[0, T]$ with $T < \dfrac{-\gamma + \sqrt{4\gamma + \gamma^2}}{2\gamma}$.*

4. - Global existence of solutions

Denote by

$$\|f\|_{(0,\,a)} = |f(0)| + \int_0^a |\dot f(s)|\,ds\,.$$

Proposition 4.1: *For all $n \geqslant 1$ and $x, y \in E_0$, we have*

$$\|K^n x - K^n y\|_{(0,\,T)} \leqslant \frac{\gamma^n (1+T)^{2n}}{2^n.\, n!} \|x-y\|_{(0,\,T)}\,.$$

Proof: If $x \in E_0$, $(Kx)(t) = \begin{cases} \int_0^t F(s, x_s)\,ds & \text{if } t \in [0, T] \\ 0 & \text{if } t \in [-r, 0] \end{cases}$

So, $Kx \in E_0$. Since $x \in E_0$, then

$$\|x\|_{(0,\,T)} = \int_0^T |\dot x(s)|\,ds\,.$$

Let $x, y \in E_0$, and by $H_{(F)}$ we have

$$\begin{aligned} &\|Kx - Ky\|_{(0,\,T)} \\ &= \int_0^T \left| \frac{d}{dt}(Kx)(t) - \frac{d}{dt}(Ky)(t) \right| dt \\ &= \int_0^T |F(t, x_t) - F(t, y_t)|\,dt \\ &\leqslant \gamma \int_0^T \|x_t - y_t\|_{1,\,1}\,dt \end{aligned}$$

and from (3.2) we have

$$\|x_t - y_t\|_{1,\,1} \leqslant \int_0^t |x(\tau) - y(\tau)|\,d\tau + \|x-y\|_{(0,\,t)}.$$

Since

$$|x(\tau) - y(\tau)| \leqslant |x(0) - y(0)| + \int_0^\tau |\dot x(s) - \dot y(s)|\,ds\,,$$

then, for $\tau \in [0, t]$, we have

$$\|x_t - y_t\|_{1,1} \leqslant \int_0^t \left[\int_0^\tau |\dot{x}(s) - \dot{y}(s)|\, ds \right] d\tau + \|x - y\|_{(0,t)} \tag{4.2}$$

$$\leqslant \int_0^t \left[\int_0^t |\dot{x}(s) - \dot{y}(s)|\, ds \right] d\tau + \|x - y\|_{(0,t)}$$

$$\leqslant \int_0^t \|x - y\|_{(0,t)}\, d\tau + \|x - y\|_{(0,t)}$$

$$= (1 + t)\|x - y\|_{(0,t)}.$$

And from (4.1) and (4.2), we obtain

$$\|Kx - Ky\|_{(0,T)} \leqslant \gamma \int_0^T (1 + t)\|x - y\|_{(0,t)}\, dt \tag{4.3}$$

$$\leqslant \gamma \|x - y\|_{(0,T)} \frac{(1 + T)^2}{2}.$$

Then by the inequality (4.3), we have,

$$\|K^2 x - K^2 y\|_{(0,T)} \leqslant \gamma \int_0^T (1 + s)\|Kx - Ky\|_{(0,s)}\, ds$$

$$\leqslant \gamma^2 \int_0^T \left[(1 + s) \int_0^s (1 + t)\|x - y\|_{(0,t)}\, dt \right] ds$$

$$\leqslant \gamma^2 \|x - y\|_{(0,T)} \int_0^T (1 + s) \left[\frac{(1 + s)^2}{2} - \frac{1}{2} \right] ds$$

$$\leqslant \gamma^2 \|x - y\|_{(0,T)} \int_0^T \frac{(1 + s)^3}{2}\, ds$$

$$\leqslant \gamma^2 \|x - y\|_{(0,T)} \left[\frac{(1 + T)^4}{8} - \frac{1}{8} \right]$$

$$\leqslant \gamma^2 \frac{(1 + T)^4}{8} \|x - y\|_{(0,T)}$$

Thus, we prove easily by induction that

$$\|K^n x - K^n y\|_{(0,\,T)} \leqslant \frac{\gamma^n (1+T)^{2n}}{2^n .\, n!} \|x - y\|_{(0,\,T)} \qquad \blacksquare$$

We are now prepared to prove the main theorem of this paper

PROOF OF THEOREM 1.1: On E_0 we define a mapping K_0 by

$$(K_0 x^0)(t) = \begin{cases} \int_0^t F(s,\, x_s^0 + \tilde{\varphi}_s) & \text{if } t \in [0,\, T] \\ 0 & \text{if } t \in [-r,\, 0] \end{cases} \qquad \text{for all } x^0 \in E_0 .$$

It is easy to verify that K_0 maps E_0 into itself. Using the same arguments as in the proof of proposition 4.1, we get that

$$\|K_0^n x^0 - K_0^n y^0\|_{(0,\,T)} \leqslant \frac{\gamma^n (1+T)^{2n}}{2^n .\, n!} \|x^0 - y^0\|_{(0,\,T)}$$

for all $x^0, y^0 \in E_0$.

Then, for all $T > 0$, there exists an integer $N > 0$, such that for all $n \geqslant N$, we have $\frac{\gamma^n (1+T)^{2n}}{2^n .\, n!} < 1$. So, K_0^n is a strict contraction from E_0 into E_0 and therefore there exists one $\bar{x}^0 \in E_0$ such that $K_0^n \bar{x}^0 = \bar{x}^0$. So $\bar{x}^0$ is one fixed point of K in E_0. Now lemma 2.3 gives the existence of $\bar{x} \in E_\varphi$ such that $\bar{x}(t) = \bar{x}^0(t) + \varphi(0)$, for all $t \in [0,\, T]$. Recall that from (1.2) we have

$$(Kx)(t) = (K_0 x^0)(t) + \varphi(0),$$

where $x(t) = x^0(t) + \varphi(0)$. Then consequently,

$$(K\bar{x})(t) = (K_0 \bar{x}^0)(t) + \varphi(0) = \bar{x}^0(t) + \varphi(0) = \bar{x}(t).$$

Thus $\bar{x}$ is fixed point of K in E_φ which completes the proof of theorem 1.1.

5. - AN EXAMPLE

In this section we discuss an interesting example of theorem. We apply our results to the integro-differential equation

$$(5.1) \qquad \begin{cases} \dot{x}(t) = \int_{t-r}^{t} K_1(t,\, \tau,\, x(\tau))\, d\tau + \int_{t-r}^{t} K_2(t,\, \tau,\, \dot{x}(\tau))\, d\tau & \text{if } t \in [0,\, T] \\ x(t) = \varphi(t) & \text{if } t \in [-r,\, 0] \end{cases}$$

where $\varphi \in W^{1,1}$ and $K_i \colon [0, T] \times [-r,\, T] \times X \to X$, satisfy the following hypothesis

(H) There are bounded functions $\gamma_1, \gamma_2 : [0, T] \to \mathbb{R}$ such that for all $t \in [0, T]$,

$\tau \in [-r, T]$ and $x_1, x_2 \in X$: $|K_i(t, \tau, x_1) - K_i(t, \tau, x_2)| \leq \gamma_i(t) |x_1 - x_2|$, $i = 1, 2$.

Define $F : [0, T] \times W^{1,1} \to X$ by

$$F(t, \varphi) = \int_{t-r}^{t} K_1(t, \tau, \varphi(\tau - t))\, d\tau + \int_{t-r}^{t} K_2(t, \tau, \varphi(\tau - t))\, d\tau, \tag{5.2}$$

for all $t \in [0, T]$, $\varphi \in W^{1,1}$.

To prove $H_{(F)}$, we have, for all $t \in [0, T]$, from (H) that

$$\begin{aligned} |F(t, \varphi) - F(t, \psi)| &\leq \int_{t-r}^{t} \gamma_1(t) |\varphi(\tau - t) - \psi(\tau - t)| d\tau \\ &+ \int_{t-r}^{t} \gamma_2(t) |\dot{\varphi}(\tau - t) - \dot{\psi}(\tau - t)| d\tau \\ &\leq \max\{\gamma_1(t), \gamma_2(t)\} \|\varphi - \psi\|_{1,1}. \end{aligned}$$

Thus theorem 1.1 applies to equation (5.1) with F as in (5.2).

REFERENCES

[1] H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions sur les espaces de Hilbert*, North Holland, Mathematics Studies (1973).

[2] M. G. Crandall - A. Pazy, *Nonlinear evolution equations in Banach spaces*, Israel J. Math. (1975), 57-94.

[3] J. Dyson - R. Villella-Bressan, *Functional Differential Equations and Nonlinear Evolution Operators*, Proc.Roy.Edinburgh Sect. A 75 (1976), 223-234.

[4] J. Dyson - R. Villella-Bressan, *A Semigroup Approach to Nonlinear Nonautonomous Neutral Functional Differential Equations*, Journal of differential equations, New-York, London. Vol. 59, N. 2 (1985), 206-228.

[5] H. FLaschaka - M. J. Leitman, *On semigroup of nonlinear operators and the solution of the functional differential equation $x'(t) = F(x_t)$*, J. Math. Anal. Appl., 49 (1975), 649-658.

[6] J. Hale, *Functional differential equations*, Applied Mathematics Series, Vol. 3, Springer Verlag, New-York (1971).

[7] J. Hale, *Theory of functional differential equations*, Springer-Verlag, New York, Vol. 3, A.M.S, (1977).

[8] K. Kunisch, *Neutral differential equations in L^p spaces and averaging approximations*, J. Nonlinear Anal. TMA 3 (1979), 419-448.

[9] A. T. Plant, *Nonlinear Semigroups Generated by Neutral Functional Differential Equations*, Fluid Mechanics Research Instute, Report 72, University of Essex, 1976.

[10] D. Salomon, *Neutral Functional differential equations and semigroups of operators*, Dynamische Systeme, Report 80, Universitat Bremen (1982).
[11] O. Sidki, *Une approche par la théorie des semigroupes non linéaires de la résolution d'une classe d'équations différentielles fonctionnelles de type neutre, Application à une équation de dynamique de population*, Thèse. N d'ordre 221, 1994, Université de Pau.
[12] G. F. Webb, *Autonomous nonlinear functional differential equations and nonlinear semigroups*, Math. Anal. Appl., 46 (1972), 1-12.

Direttore responsabile: Prof. A. Ballio - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5 - Bologna