Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
123° (2005), Vol. XXIX, fasc. 1, pagg. 455-472

EDOARDO VESENTINI (*)

Periodic and Almost Periodic Linear Isometries of Hardy Spaces

ABSTRACT. — Let T be a strongly continuous semigroup of linear isometries of the Hardy space H^p ($1 \leq p \leq \infty$, $p \neq 2$) of holomorphic functions on the open unit disc of $\mathbb{C}$. The main result of this paper concerns a characterization of the case in which T is almost periodic. As a consequence of a detailed analysis of the behaviour of T, it is shown that T is almost periodic if, and only if, its conformal flow is elliptic. The case in which T is periodic is also investigated.

Isometrie lineari periodiche e quasi periodiche di spazi di Hardy

SUNTO. — Sia T un semigruppo fortemente continuo di isometrie lineari dello spazio di Hardy H^p ($1 \leq p \leq \infty$, $p \neq 2$) delle funzioni olomorfe sul disco unità aperto di $\mathbb{C}$. Un'analisi dettagliata del comportamento di T mostra che T è quasi periodico se, e soltanto se, il suo flusso conforme è ellittico. Viene altresí caratterizzato il caso in cui T è periodico.

The present paper, dedicated to the memory of Luigi Amerio, is devoted to the description of the periodic and almost periodic linear isometries of the Hardy spaces H^p $(1 \leq p \leq \infty, p \neq 2)$.

The Hardy space H^∞ is the space of all bounded holomorphic functions on the open unit disc $\varDelta$ of $\mathbb{C}$; it is a Banach space for the sup-norm.

For $1 \leq p < \infty$, the Hardy space H^p consists of those holomorphic functions f on $\varDelta$ for which the integrals

$$\int_0^{2\pi} |f(re^{i\theta}|^p \, d\theta$$

are bounded as $r \uparrow 1$. Denoting by $\|f\|^p$ the supremum of these integrals, $f \mapsto \|f\|$ is a norm on H^p withy respect to which H^p is a Banach space: a Hilbert space if $p = 2$. In this latter case the theory of linear isometries of H^2 can be linked to the general theory of

(*) Indirizzo dell'Autore: Politecnico di Torino, Dipartimento di Matematica, Corso Duca degli Abruzzi 24, 10129 Torino, Italy.

linear isometries of separable Hilbert spaces by the fact that, if

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

is the power-series expansion of $f \in H^2$, then

$$\|f\|^2 = \sum_{n=0}^{+\infty} |a_n|^2.$$

For $1 \leq p < \infty, p \neq 2$, the linear isometries of H^p into H^p have been fully described by F. Forelli in [6] as weighted composition operators associated to holomorphic maps $\Delta \to \Delta$ represented by non-constant inner functions. Using Forelli's result we will show that a linear isometry $A : H^p \to H^p$ is n-periodic, *i.e.* $A^n = I$ for some integer $n > 1$, if, and only if, there is an infinite set $E \subset \Delta$ with a cluster point in Δ, such that

$$(A^n f)(z) = f(z)$$

for all $f \in H^p$ and all $z \in E$.

Forelli's results were intrumental in the characterization of all strongly continuous semigroups of linear isometries of $T : \mathbb{R}_+ \to \mathcal{L}(H^p)$, which was carried out by E. Berkson in [4] for $1 \leq p < \infty, p \neq 2$, showing that any such semigroup T is a semigroup of weighted composition operators associated with a continuous flow ϕ of holomorphic automorphisms of Δ.

The analysis of the behaviour of the flow ϕ yields a characterization of the cases in which the semigroup T is periodic or almost periodic, leading in particular to the conclusion that T is almost periodic if, and only if, ϕ is elliptic, *i.e.* fixes a point in Δ.

In the final section of this paper, the analysis will be extended to H^∞ following a different approach based on a result by H. P. Lotz, [8], according to which every strongly continuous semigroup of linear operators acting on H^∞ is uniformly continuous.

As a consequence of this result, any strongly continuous semigroup of linear isometries $H^\infty \to H^\infty$ is the restriction to $\mathbb{R}_+$ of a uniformly continuous one-parameter group of isometries of H^∞ onto itself. Surjective isometries $H^\infty \to H^\infty$ were shown (see K. Hoffman, [7]) – as a consequence of a "Banach Stone theorem" – to be weighted composition operators associated with continuous flows of holomorphic automorphisms of Δ. At this point, the extension to H^∞ of the results, concerning periodicity and almost periodicity, established for H^p when $p < \infty$ is easily obtained (*).

1. - Periodic isometries

Let A be a linear isometry of the Banach space H^p ($1 \leq p < \infty; p \neq 2$). According to Theorem 1 of [6], A has a unique representation

$$Af = F \,.\, f \circ \psi \quad \forall\ f \in H^p, \tag{1}$$

(*) Note added in proofs. A forthcoming paper will be devoted to linear isometries of H^∞.

with $F \in \mathrm{Hol}(\Delta)$ and ψ a non-constant inner function. Being

(2) $$F = A1,$$

where 1 is the constant function equal to 1 on Δ, then $F \in H^p$ and $\|F\| = 1$.

As was shown by F. Forelli in [6], F and ψ are related by the equation

(3) $$\int_M |F|^p d\mu = \int_M \frac{1}{P(\psi)} d\mu \quad \forall\, M \in \Sigma(\phi),$$

where: $d\mu$ is the Lebesgue measure on $\partial\Delta$ with $\mu(\partial\Delta) = 1$; Σ is the collection of all μ-measurable sets in $\partial\Delta$; $\Sigma(\psi)$ is the collection of all symmetric differences of sets $U = \psi^{-1}(E)$, with $E \in \Sigma$, and $V = \psi^{-1}(K)$, with $K \in \Sigma$, $\mu(K) = 0$; P is the Poisson kernel

$$P(z) = \frac{1 - |\omega|^2}{|1 - \omega\bar{z}|^2},$$

and

(4) $$\omega = \int \overline{\psi} d\mu.$$

We consider here the particular case in which ψ is a Moebius transformation of Δ,

$$\psi(z) = \frac{az + b}{\bar{b}z + \bar{a}},$$

with $a, b \in \mathbb{C}$, $|a|^2 - |b|^2 = 1$, $z \in \Delta$.

A first consequence of this choice is that $\Sigma(\psi) = \Sigma$. Furthermore

$$\begin{aligned}
\omega &= \frac{1}{2\pi}\int_0^{2\pi} \overline{\psi(\mathrm{e}^{i\theta})} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{\bar{a}\mathrm{e}^{-i\theta} + \bar{b}}{b\mathrm{e}^{-i\theta} + a} d\theta \\
&= \frac{1}{2\pi a}\int_0^{2\pi} \left(\bar{a}\mathrm{e}^{-i\theta} + \bar{b}\right)\left(1 - \frac{b}{a}\mathrm{e}^{-i\theta} + \left(\frac{b}{a}\mathrm{e}^{-i\theta}\right)^2 + \dots\right) d\theta \\
&= \frac{1}{2\pi a}\int_0^{2\pi} \bar{b}\, d\theta = \frac{\bar{b}}{a};
\end{aligned}$$

$$1 - |\omega|^2 = 1 - |b/a|^2 = \frac{1}{|a|^2};$$

$$P(z) = \frac{1}{|a|^2}\frac{1}{|1 - \frac{b\bar{z}}{\bar{a}}|^2} = \frac{1}{|a - \bar{b}z|^2}.$$

Since

$$\left(\frac{d\psi}{dz}\right)(z) = \frac{1}{\left(\bar{b}z + \bar{a}\right)^2},$$

then,

$$\frac{1}{P(\psi(e^{i\theta}))} = |a - \overline{b}\psi(e^{i\theta})|^2 = \left| a - \overline{b}\frac{ae^{i\theta} + b}{\overline{b}e^{i\theta} + \overline{a}} \right|^2$$

$$= \frac{1}{|\overline{b}e^{i\theta} + \overline{a}|^2} = \left| \left(\frac{d\psi}{dz}\right)(e^{i\theta}) \right|.$$

Hence, (3) reads

$$\int_M |F|^p \, d\mu = \int_M \left|\frac{d\psi}{dz}\right| d\mu \quad \forall\, M \in \Sigma(\psi),$$

implying that

$$|F|^p = \left|\frac{d\psi}{dz}\right| \qquad \text{a.e. on } \partial\Delta.$$

Since $(d\psi/dz)^{1/p}$ is an outer function, there is an inner function G such that

$$F = G\left(\frac{d\psi}{dz}\right)^{1/p}, \tag{5}$$

and the following proposition holds.

Proposition 1: *If the inner function* ψ *in* (1) *is a Moebius transformation of* Δ*, there is an inner function* G *such that*

$$A : H^p \ni f \mapsto G\left(\frac{d\psi}{dz}\right)^{1/p} . f \circ \psi \quad \forall\, f \in H^p. \tag{6}$$

Going back to the general case, suppose now that there exist an integer $n > 1$ and some $z \in \Delta$ such that

$$(A^n f)(z) = f(z) \quad \forall f \in H^p. \tag{7}$$

Since

$$A^p f = F_p \, . f \circ \psi^p \qquad \text{for } p =, 2, \ldots .$$

where

$$F_p = F \, . F \circ \psi \, . \, \cdots \, . F \circ \psi^{p-1},$$

(7) reads now

$$F_n(z) = 1, \tag{8}$$

and $\psi^n(z) = z$.

This latter conclusion implies that either $\psi(z) = z$ or

$$\mathrm{Card}\{z, \psi(z), \dots, \psi^{n-1}(z)\} > 1,$$

in which case, by Proposition 4 of [9], ψ is a holomorphic automorphism of Δ fixing a point $a \in \Delta$.

Hence, if there is an infinite set $E \subset \Delta$ with a cluster point in Δ, such that, if $z \in E$ (7) holds, then ψ is an n-periodic holomorphic automorphism of Δ fixing a point $a \in \Delta$.

Being

$$\psi(a) = a$$

and

$$\frac{1}{1 - |\psi(a)|^2} \left| \left(\frac{d\psi}{dz}\right)(a) \right| = \frac{1}{1 - |(a)|^2},$$

then

$$\left| \left(\frac{d\psi}{dz}\right)(a) \right| = 1. \tag{9}$$

Since

$$1 = F_n(a) = F(a)^n,$$

by (5) and (9) $|G(a)| = 1$, showing that the inner function G is constant:

$$G(z) = \kappa$$

for all $z \in \Delta$ and some $\kappa \in \partial\Delta$, which, by (7), is $\kappa = e^{2\pi i/n}$.

Hence (2) becomes

$$A : H^p \ni f \mapsto e^{2\pi i/n} \left(\frac{d\psi}{dz}\right)^{1/p} . f \circ \phi \;\; \forall \; f \in H^p. \tag{10}$$

Theorem 2 of [6] yields then

THEOREM 1: *If $A \in \mathcal{L}(H^p)$ $(1 \le p < \infty, p \ne 2)$ is an isometry, and if there exists an infinite set $E \subset \Delta$ with a cluster point in Δ such that* (7) *holds for some $n > 1$, all $z \in E$ and all $f \in H^p$, then A is a periodic surjective isometry, with period n, represented by* (10), *where ψ is a periodic Moebius transformation of Δ.*

Vice versa, for any Moebius transformation ψ of Δ, such that $\psi^n = \mathrm{id}$, (2) *defines an n-periodic (surjective) linear isometry of H^p.*

It is worth remarking, at this point, that, if A is periodic, it is necessarily surjective.

In the following sections we shall investigate under which conditions strongly continuous semigroups of linear isometries of Hardy spaces H^p are almost periodic or periodic. We will begin by collecting some results on continuous flows of holomorphic automorphisms of Δ, [3], [4].

2. - Continuous flows of holomorphic automorphisms of Δ

Any continuous flow ϕ of holomorphic automorphisms of Δ is defined by a one-parameter subgroup of $SU(1,1)$.

If the holomorphic automorphism ϕ_t is defined by

$$\tilde{\phi}_t = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ \overline{a_{12}(t)} & \overline{a_{11}(t)} \end{pmatrix} \in SU(1,1),$$

i.e.

$$\phi_t(z) = \frac{a_{11}(t)z + a_{12}(t)}{\overline{a_{12}(t)}z + \overline{a_{11}(t)}} \quad (z \in \Delta),$$

then

$$\tilde{\phi}_t = \exp t\Theta,$$

where

$$\Theta = \begin{pmatrix} i\gamma & c \\ \bar{c} & -i\gamma \end{pmatrix}$$

and $\gamma \in \mathbb{R}$, $c \in \mathbb{C}$.

A direct computation yields

$$\begin{aligned}\exp t\Theta =& I + t\Theta + \frac{t^2}{2}\left(|c|^2 - \gamma^2\right)I + \frac{t^3}{3!}\left(|c|^2 - \gamma^2\right)\Theta + \\ &+ \frac{t^4}{4!}\left(|c|^2 - \gamma^2\right)^2 I + \frac{t^5}{5!}\left(|c|^2 - \gamma^2\right)^2 \Theta + \ldots = \\ =&\left(1 + \frac{t^2}{2}\left(|c|^2 - \gamma^2\right) + \frac{t^4}{4!}\left(|c|^2 - \gamma^2\right)^2 + \ldots\right) I + \\ &+ \left(t + \frac{t^3}{3!}\left(|c|^2 - \gamma^2\right) + \frac{t^5}{5!}\left(|c|^2 - \gamma^2\right)^2 + \ldots\right)\Theta.\end{aligned}$$

If $|\gamma| > |c|$, letting $r = \sqrt{\gamma^2 - |c|^2}$, then

$$\exp t\Theta = \begin{pmatrix} \cos(rt) + \frac{i\gamma}{r}\sin(rt) & \frac{c}{r}\sin(rt) \\ \frac{\bar{c}}{r}\sin(rt) & \cos(rt) - \frac{i\gamma}{r}\sin(rt) \end{pmatrix}; \tag{11}$$

if $|\gamma| < |c|$, letting $s = \sqrt{|c|^2 - \gamma^2}$,

$$\exp t\Theta = \begin{pmatrix} \cosh(st) + \frac{i\gamma}{s}\sinh(st) & \frac{c}{s}\sinh(st) \\ \frac{\bar{c}}{s}\sinh(st) & \cosh(st) - \frac{i\gamma}{s}\sinh(st) \end{pmatrix};$$

finally, if $|\gamma| = |c|$, then

$$\exp t\Theta = I + t\Theta = \begin{pmatrix} 1 + it\gamma & tc \\ t\bar{c} & 1 - it\gamma \end{pmatrix}.$$

Hence, if $|\gamma| > |c|$, Θ defines the periodic flow, with period $\tau = 2\pi/r$, of Moebius transformations of Δ

(12) $$\phi_t : z \mapsto \frac{\left(\cos(rt) + i\frac{\gamma}{r}\sin(rt)\right)z + \frac{c}{r}\sin(rt)}{\frac{\bar{c}}{r}\sin(rt)z + \cos(rt) - i\frac{\gamma}{r}\sin(rt)},$$

while, if $|\gamma| < |c|$ or if $|\gamma| = |c|$, the flows defined by Θ are expressed respectively by the Moebius transformations

(13) $$\phi_t : z \mapsto \frac{\left(\cosh(st) + i\frac{\gamma}{s}\sinh(st)\right)z + \frac{c}{s}\sinh(st)}{\frac{\bar{c}}{s}\sinh(st)z + \cosh(st) - i\frac{\gamma}{s}\sinh(st)}$$

$$= \frac{\left(1 + i\frac{\gamma}{s}\tanh(st)\right)z + \frac{c}{s}\tanh(st)}{\frac{\bar{c}}{s}\tanh(st)z + 1 - i\frac{\gamma}{s}\tanh(st)}$$

and

(14) $$\phi_t : z \mapsto \frac{(1 + it\gamma)z + tc}{t\bar{c}z + 1 - it\gamma}.$$

When $|\gamma| > |c|$, the orbit of any $z \in \Delta$ is a circle contained in Δ. If $|\gamma| < |c|$, the orbit of z is an arc whose extremes lie in $\partial\Delta$, while, if $|\gamma| = |c|$, the orbit of z is a circle contained in $\overline{\Delta}$ and tangent to $\partial\Delta$.

By Theorem 6 of [9], if $|\gamma| > |c|$ the periodic flow (12) fixes a unique point a of Δ, which can be determined by looking at the points $z \in \mathbb{C}$ where

(15) $$\frac{d}{dt}\phi_t(z) = 0 \quad \forall\, t \in \mathbb{R}.$$

Since

(16) $$\frac{d}{dt}\phi_t(z) = \frac{-\bar{c}z^2 + 2i\gamma z + c}{\left(\frac{\bar{c}}{r}\sin(rt)z + \cos(rt) - i\frac{\gamma}{r}\sin(rt)\right)^2},$$

the points $z \in \mathbb{C}$ satisfying (15) are the zeros of the polynomial

$$\bar{c}z^2 - 2i\gamma z - c,$$

i.e.,

$$z = i\frac{\gamma \pm r}{\bar{c}}$$

if $c \neq 0$, or $z = 0$ if $c = 0$.

An elementary discussion completes the proof of the following proposition.

PROPOSITION 2: *The inequality* $|\gamma| > |c|$ *is equivalent to any one of the following conditions:*

ϕ *is periodic;*

ϕ *fixes a (unique) point* $\alpha \in \Delta$.

If $|\gamma| > |c| > 0$, *then*

$$\alpha = i\frac{\gamma + r}{\bar{c}} \qquad \text{if } \gamma < 0$$

or

$$\alpha = i\frac{\gamma - r}{\bar{c}} \qquad \text{if } \gamma > 0.$$

If $c = 0$, ϕ is periodic with period $2\pi/|\gamma|$ and $\alpha = 0$.

If $|\gamma| < |c|$, (13) yields

$$\lim_{t\to-\infty} \phi_t(z) = \frac{\left(1 - i\dfrac{\gamma}{s}\right)z - \dfrac{c}{s}}{-\dfrac{\bar{c}}{s}z + 1 + i\dfrac{\gamma}{s}} = \frac{(s - i\gamma)z - c}{-\bar{c}z + s + i\gamma} = -\frac{c}{s + i\gamma},$$

$$\lim_{t\to+\infty} \phi_t(z) = \frac{\left(1 + i\dfrac{\gamma}{s}\right)z + \dfrac{c}{s}}{\dfrac{\bar{c}}{s}z + 1 - \dfrac{\gamma}{s}} = \frac{(s + i\gamma)z + c}{\bar{c}z + s - i\gamma} = \frac{c}{s - i\gamma}$$

for all points $z \in \Delta$ (because

$$\det\begin{pmatrix} s - i\gamma & -c \\ -\bar{c} & s + i\gamma \end{pmatrix} = \det\begin{pmatrix} s + i\gamma & c \\ \bar{c} & s - i\gamma \end{pmatrix} = 0).$$

Hence:

PROPOSITION 3: *If* $|\gamma| < |c|$, *the points*

$$\alpha = -\frac{c}{s + i\gamma} \in \partial\Delta \quad \text{and} \quad \beta = \frac{c}{s - i\gamma} \in \partial\Delta \tag{17}$$

are the Wolff points of the semiflows $\mathbb{R}_+ \ni t \mapsto \phi_{-t}$ *and* $\mathbb{R}_+ \ni t \mapsto \phi_t$, *where* ϕ_t *is given by* (13).

For any $t \in \mathbb{R}$, *they are the unique fixed points of the continuous extension of* ϕ_t *to* $\bar{\Delta}$.

If $\gamma = 0$, the Wolff points are $-e^{i \arg c}$ and $e^{i \arg c}$.

Finally, (14) yields

PROPOSITION 4: *If* $|\gamma| = |c|\ (\neq 0)$, *the point* $i\dfrac{c}{\gamma}$ *is the Wolff point of the two semiflows* $\mathbb{R}_+ \ni t \mapsto \phi_{-t}$ *and* $\mathbb{R}_+ \ni t \mapsto \phi_t$, *where* ϕ_t *is given by* (14).

It is also the unique fixed point of the continuous extension of ϕ_t *to* $\overline{\Delta}$.

As a consequence of some of these facts, the following corollary holds.

COROLLARY 1: *If the orbit of some* $z \in \Delta$ *by the flow* ϕ *is relatively compact in* Δ, *then the set*

$$\mathrm{Fix}(\phi) = \{z \in \Delta : \phi_t(z) = z\ \forall\, t \in \mathbb{R}\}$$

is non-empty (and consists of a unique point).

Vice versa, if $\mathrm{Fix}(\phi) \neq \emptyset$, *then the orbit of any* $z \in \Delta$ *is relatively compact in* Δ.

It will be useful in the following to compute an expression, in terms of α and β, of (13) when $|\gamma| < |c|$.

First of all, (13) yields

$$\phi_t(z) = \frac{[(s + i\gamma)e^{2st} + s - i\gamma]z + c(e^{2st} - 1)}{\overline{c}(e^{2st} - 1)z + (s - i\gamma)e^{2st} + s + i\gamma}.$$

Since

$$s + i\gamma = -\frac{c}{\alpha},$$

and therefore

$$s - i\gamma = -\frac{\overline{c}}{\overline{\alpha}} = -\overline{c}\alpha,$$

then

$$c = (s - i\gamma)\beta = -\overline{c}\alpha\beta.$$

Furthermore,

$$s + i\gamma = \overline{s - i\gamma} = \frac{\overline{c}}{\overline{\beta}} = \overline{c}\beta.$$

Then

$$\phi_t(z) = \frac{\overline{c}(\beta e^{2st} - \alpha)z + c(e^{2st} - 1)}{\overline{c}(e^{2st} - 1)z + \overline{c}(\beta - \alpha e^{2st})} = \frac{(\beta e^{2st} - \alpha)z - (e^{2st} - 1)\alpha\beta}{(e^{2st} - 1)z + \beta - \alpha e^{2st}}. \tag{18}$$

Hence,

$$\left(\frac{d\phi_t}{dz}\right)(z) = \left(\frac{e^{st}(\alpha - \beta)}{(1 - e^{2st})z + e^{2st}\alpha - \beta}\right)^2. \tag{19}$$

REMARK: A direct computation shows that, setting

$$\sigma_{\alpha,\beta}(z) = \frac{z-\alpha}{z-\beta},$$

(18) can be written

$$\phi_t(z) = \sigma_{\alpha,\beta}{}^{-1}\left(\mathrm{e}^{2st}\sigma_{\alpha,\beta}(z)\right),$$

which is the expression of ϕ_t given by Theorem (1.6) in [3].

If $|\gamma| < |c|$, $|\gamma| = |c|$, $|\gamma| > |c|$, the flow ϕ is called, respectively, *elliptic, parabolic, hyperbolic.*

3. - PERIODIC SEMIGROUPS OF ISOMETRIES OF H^p

Let $T : \mathbb{R}_+ \to \mathcal{L}(H^p)$ be a strongly continuous semigroup of linear isometries of H^p into itself, and let $X : \mathcal{D}(X) \subset \mathcal{L}(H^p) \to \mathcal{L}(H^p)$ be its infinitesimal generator.

According to Theorem (1.6) of [4], there are a unique continuous flow ϕ of Moebius transformations of Δ and a unique family $\{\rho_t : t \in \mathbb{R}_+\}$ of inner functions ρ_t such that

$$(20)\qquad (T(t)f)(z) = \rho_t(z)\left[\left(\frac{d}{dz}\phi_t\right)(z)\right]^{1/p} f\left(\phi_t(z)\right)\ \forall t \in \mathbb{R}_+,\ \forall z \in \Delta,\ \forall f \in H^p.$$

Furthermore,

$$\rho_{t_1+t_2}(z) = \rho_{t_1}(z)\,.\,\rho_{t_2}\left(\phi_{t_1}(z)\right)\ \forall\, t_1, t_2 \in \mathbb{R}_+,\ \forall\, z \in \Delta,$$

and $t \mapsto \rho_t$ is a continuous map of $\mathbb{R}_+$ into H^1.

Vice versa, given ϕ and $t \mapsto \rho_t$ as above, satisfying the latter two conditions, the map $\mathbb{R}_+ \ni t \mapsto T(t)$ expressed by (20) is a strongly continuous semigroup of isometries of H^p. The flow ϕ is called, the *conformal flow* or the *conformal group*, [4], of the semigroup T.

If the flow ϕ is periodic with period $\tau > 0$ - and therefore is elliptic, [9] - then, by Theorem 4.1 of [4], the role of the family $\{\rho_t : t \in \mathbb{R}_+\}$ is played by the family $\{\mathrm{e}^{i\delta t} : t \in \mathbb{R}_+\}$ for some fixed $\delta \in \mathbb{R}$, so that (20) is replaced by

$$(21)\qquad (T(t)f)(z) = \mathrm{e}^{i\delta t}\left[\left(\frac{d}{dz}\phi_t\right)(z)\right]^{1/p} f\left(\phi_t(z)\right),\ \forall z \in \Delta,\ \forall f \in H^p$$

and for all $t \in \mathbb{R}_+$.

Vice versa, given any real constant δ and any non-trivial periodic continuous flow ϕ of holomorphic automorphisms of Δ, the map $\mathbb{R} \ni t \mapsto T(t)$ defined by (21) is a strongly continuous group of (surjective) isometries of H^p. For any $t \in \mathbb{R}$, ϕ_t is represented by (12), with $r = 2\pi/\tau$, and the fixed point a is given by Proposition 2. Furthermore

$$(22)\qquad \left(\frac{d}{dz}\phi_t\right)(z) = \frac{1}{\left(\frac{\bar{c}}{r}\sin(rt)z + \cos(rt) - i\frac{\gamma}{r}\sin(rt)\right)^2}.$$

Hence,

$$\left(\frac{d}{dz}\phi_t\right)(a) = e^{\mp\frac{4\pi i}{\tau}t}, \tag{23}$$

according as $a = i\dfrac{\gamma \pm r}{\bar{c}}$.

We will now describe the infinitesimal generator X of the group T expressed by (21).
As a consequence of (22),

$$\frac{\partial^2}{\partial t\,\partial z}(\phi_t)(z) = -2\frac{\bar{c}\cos(rt)z - r\sin(rt) - i\gamma\cos(rt)}{\left(\frac{\bar{c}}{r}\sin(rt)z + \cos(rt) - i\frac{\gamma}{r}\sin(rt)\right)^3},$$

and therefore

$$\lim_{t\to 0}\frac{1}{t}\left\{e^{i\delta t}\left[\left(\frac{d}{dz}\phi_t\right)(z)\right]^{1/p} - 1\right\} = \tag{24}$$
$$= \left(\frac{d}{dt}\left\{e^{i\delta t}\left[\left(\frac{d}{dz}\phi_t\right)(z)\right]^{1/p}\right\}\right)_{t=0} = i\delta + 2\frac{i\gamma - \bar{c}z}{p}.$$

For $f \in H^p$, $t \neq 0$ and $z \in \Delta$,

$$\frac{1}{t}(T(t)f - f)(z) = \frac{1}{t}\left\{e^{i\delta t}\left[\left(\frac{d}{dz}\phi_t\right)(z)\right]^{1/p} f(\phi_t(z)) - f(z)\right\}$$
$$= \frac{1}{t}\left\{e^{i\delta t}\left[\left(\frac{d}{dz}\phi_t\right)(z)\right]^{1/p} - 1\right\} f(\phi_t(z)) + \frac{1}{t}\left[f(\phi_t(z)) - f(z)\right].$$

By (16),

$$\lim_{t\to 0}\frac{1}{t}\left[f(\phi_t(z)) - f(z)\right] = \left(\frac{df}{dz}(\phi_t(z))\right)_{t=0} \times$$
$$\times\left(\left(\frac{d}{dt}\phi_t\right)(z)\right)_{t=0} = \left(-\bar{c}z^2 + 2i\gamma z + c\right)\frac{df}{dz}.$$

Since the modulus of the restriction to $\partial\Delta$ of the polynomial $z \mapsto -\bar{c}z^2 + 2i\gamma z + c$ is bounded away from zero, then, by (24),

$$f \in \mathcal{D}(X) \Longrightarrow \frac{df}{dz} \in H^p,$$

and therefore

$$\mathcal{D}(X) = \left\{f \in H^p : \frac{df}{dz} \in H^p\right\}. \tag{25}$$

Furthermore

$$X = X_0 + i\delta I, \tag{26}$$

where X_0 is the infinitesimal generator of the periodic group T_0 of composition operators $H^p \ni f \mapsto T_0(t)f = f \circ \phi_t$, with period τ, domain $\mathcal{D}(X_0) = \mathcal{D}(X)$, and

$$(X_0 f)(z) = 2\frac{i\gamma - \bar{c}z}{p} f(z) + \left(-\bar{c}z^2 + 2i\gamma z + c\right)\frac{df}{dz}(z)$$

for all $f \in \mathcal{D}(X)$ and all $z \in \Delta$.

We will now establish a condition under which the semigroup T is periodic.

Let $z_0 \in \Delta$ and $\tau > 0$ be such that

$$(T(\tau)f)(z_0) = f(z_0) \;\; \forall f \in H^p, \tag{27}$$

and such that for any $t \in (0, \tau)$

$$(T(t)f)(z_0) \neq f(z_0) \quad \text{for some } f \in H^p. \tag{28}$$

Choosing $f = 1$, (20) yields

$$(T(t)\,1)(z) = \rho_t(z)\left[\left(\frac{d}{dz}\phi_t\right)(z)\right]^{1/p},$$

and therefore, by (27),

$$\rho_\tau(z_0)\left[\left(\frac{d}{dz}\phi_\tau\right)(z_0)\right]^{1/p} = 1. \tag{29}$$

Hence,

$$f\left(\phi_\tau(z_0)\right) = f(z_0) \;\; \forall f \in H^p,$$

i.e.,

$$\phi_\tau(z_0) = z_0.$$

Thus, [9], ϕ is a periodic flow with period τ/n for some positive integer n, and therefore

$$\phi_\tau(z) = z \;\; \forall z \in \Delta \tag{30}$$

and

$$\phi_t(a) = a \tag{31}$$

for all $t \in \mathbb{R}$ and some $a \in \Delta$.

The condition (28) - holding for any $t \in (0, \tau)$ and, given t, for some $f \in H^p$ - implies that $n = 1$, *i.e.* that ϕ has period τ.

Since $\rho_t(z) = e^{i\delta t}$ for some $\delta \in \mathbb{R}$, (29) reads now

$$e^{i\delta\tau} = 1,$$

that is to say,

$$\delta = \frac{2\pi m}{\tau} \tag{32}$$

for some positive integer m.

In conclusion, the following Proposition holds.

Proposition 5: *If there are* $z_0 \in \Delta$ *and* $\tau > 0$ *satisfying* (27) *and* (28), *the strongly continuous semigroup T of linear isometries* $H^p \to H^p$ $(1 \le p < \infty, p \neq 2)$ *is the restriction to* $\mathbb{R}_+$ *of a strongly continuous group of surjective linear isometries of* H^p *which is periodic with period* τ, *is represented by* (21), *i.e.,*

$$(T(t)f)(z) = e^{\frac{2\pi m i}{\tau}t}\left[\left(\frac{d}{dz}\phi_t\right)(z)\right]^{1/p} f(\phi_t(z)) \tag{33}$$

for all $t \in \mathbb{R}_+, z \in \Delta, f \in H^p$, *where* ϕ *is a continuous flow of holomorphic automorphisms of* Δ, *periodic with period* τ, *and m is a positive integer.*

Corollary 2: *Any strongly continuous group T of linear isometries of* H^p $(1 \le p < \infty, p \neq 2)$, *periodic with period* $\tau > 0$, *is represented by* (33) *where* ϕ *is a continuous flow of holomorphic automorphisms of* Δ, *periodic with period* τ, *and m is a positive integer.*

4. - Almost periodic semigroups of isometries of H^p

Let the continuous semiflow ϕ be elliptic, let $\tau > 0$ be its period and let the group T be expressed by (21) with $\delta \in \mathbb{R}$. By (26), X is a perturbation of X_0 by the operator $i\delta I$. Thus X and X_0 have the same eigenvectors, and both the spectrum and the point-spectrum of X are obtained by translating by $i\delta$ respectively the spectrum and the point-spectrum of X_0. Since ϕ is periodic with period τ - and therefore also T_0 is periodic with period τ (and vice versa) - by Theorem 3.1 of [1] the spectrum $\sigma(X_0)$ of X_0 consists entirely of eigenvalues $in\tau/2\pi$ for some $n \in \mathbb{Z}$, and the corresponding eigenvectors span a dense linear subspace of H^p. Thus, by Theorem 9 of [2], T is almost periodic, and the following proposition holds.

Proposition 6: *If the conformal flow of the semigroup T of linear isometries of* H^p $(1 \le p < \infty, p \neq 2)$ *is elliptic, T is (the restriction to* $\mathbb{R}_+$ *of) an almost periodic group.*

Remark: This Proposition can be proved without appealing to [1] and [2], replacing T by a group $\tilde{T}$ whose conformal flow fixes 0, that is to say, by the group

$$t \mapsto \tilde{T}(t) = S^{-1} \circ T(t) \circ S,$$

where S is defined by

$$(Sf)(z) = \left(\left(\frac{d\psi}{dz}\right)(z)\right)^{1/p} f(\psi(z)) \quad (f \in H^p, z \in \Delta),$$

and

$$\psi : \Delta \ni z \mapsto \frac{z - a}{1 - \bar{a}z}.$$

Since the conformal flow of $\tilde{T}$ fixes 0, a direct inspection of (12) and (21) shows that there are $\mu, \upsilon \in \mathbb{R}$ such that

$$(\tilde{T}(t)f)(z) = e^{i\mu t} f(e^{i\upsilon t} z) \tag{34}$$

for all $f \in H^p$, $t \in \mathbb{R}$ and $z \in \Delta$ showing that the function $t \mapsto (\tilde{T}(t)f)$ is almost periodic. Thus, also T is almost periodic, and this alternate proof of Proposition 6 is complete.

We will consider now the cases in which there are no fixed points of ϕ in Δ.

If the continuous flow ϕ of Moebius transformations of Δ is represented by (14) - in which case the continuous extension of ϕ_t to $\overline{\Delta}$ fixes the (unique) point $a = ic/\gamma \in \partial\Delta$ for all $t \in \mathbb{R}$ - then ([4], Theorem 5.1, pp. 408-410) there are constants $\delta \in \mathbb{R}$, $\mu \in \mathbb{R}_+$, $k \in \mathbb{C}$ such that

$$(T(t)f)(z) = e^{t[i\delta - ik\mu t - \mu(a+z)/(a-z)]} \left[\left(\frac{d}{dz} \phi_t \right)(z) \right]^{1/p} f(\phi_t(z)),$$

i.e.,

$$(T(t)f)(z) = \frac{e^{t[i\delta - ikt\mu - \mu(a+z)/(a-z)]}}{(t\bar{c}z + 1 - it\gamma)^{2/p}} f\left(\frac{(1 + it\gamma)z + tc}{t\bar{c}z + 1 - it\gamma} \right) \tag{35}$$

for all $t \in \mathbb{R}_+$, $z \in \Delta$, $f \in H^p$.

For $z = 0$ and $f = 1$ (35) yields

$$(T(t)1)(0) - 1 = \frac{e^{t[\mu(t\Im k - 1) + i(\delta - t\mu\Re k)]}}{(1 - it\gamma)^{2/p}} - 1, \tag{36}$$

and therefore

$$|(T(t)1)(0) - 1| \geq \left| \frac{e^{\mu(t\Im k - 1)t}}{(1 + t^2\gamma^2)^{1/p}} - 1 \right|.$$

Since

$$\lim_{t \to +\infty} \left| \frac{e^{\mu(t\Im k - 1)t}}{(1 + t^2\gamma^2)^{1/p}} - 1 \right| = +\infty$$

if $\Im k > 0$, $\mu > 0$, and

$$\lim_{t \to +\infty} \left| \frac{e^{\mu(t\Im k - 1)t}}{(1 + t^2\gamma^2)^{1/p}} - 1 \right| = 1$$

if $\Im k \leq 0$ and $\mu \geq 0$, there exists some $l > 0$ such that

$$|(T(t)1)(0) - 1| \geq \frac{1}{2}$$

whenever $t > l$.

Hence (see, *e.g.*, [5], p. 36),

$$2^{1/p}|T(t)1 - 1| \geq |(T(t)1)(0) - 1| \geq \frac{1}{2}$$

for all $t > l$, proving thereby

LEMMA 1: *If the strongly continuous semigroup T of linear isometries of H^p $(1 \le p < \infty, p \neq 2)$ is expressed by (35), where the flow ϕ of Moebius transformations of Δ fixes one point in $\partial\Delta$, then T is not almost periodic.*

We will reach the same conclusion in the case in which the flow ϕ appearing in (20) fixes two distinct points α and β of $\partial\Delta$. By Theorem 5.11 of [4] the action of $T(t)$ on any $f \in H^p$ is given by the equation

$$(T(t)f)(z) = \mathrm{e}^{R(t,z)}\left(\left(\frac{d}{dz}\phi_t\right)(z)\right)^{1/p} f\big(\phi_t(z)\big), \tag{37}$$

where:

$\phi_t(z)$ and $\dfrac{d\phi_t}{dt}$ are given by (18) and (19);

$$R(t,z) = i\delta t - \mu(\overline{\alpha}\beta - 1)\frac{z-\alpha}{z-\beta}\big(\mathrm{e}^{2st} - 1\big) + \upsilon\frac{z-\beta}{(\overline{\alpha}\beta - 1)(z-\alpha)}\big(\mathrm{e}^{-2st} - 1\big),$$

and $\delta \in \mathbb{R}, \mu \in \mathbb{R}_+, \upsilon \in \mathbb{R}_+$ are constants.

We will show that the semigroup T is not almost periodic.

Choosing $f = 1$ and $z = 0$, then

$$(T(t)1)(0) = \exp\left(i\delta t - \mu(\overline{\alpha}\beta - 1)\frac{\alpha}{\beta}\big(\mathrm{e}^{2st} - 1\big) + \upsilon\frac{\beta}{\overline{\alpha}\beta - 1}\big(\mathrm{e}^{-2st} - 1\big)\right) \times \left(\frac{\mathrm{e}^{st}(\alpha-\beta)}{\mathrm{e}^{2st}\alpha - \beta}\right)^{\frac{2}{p}}.$$

Therefore, setting

$$\alpha = \mathrm{e}^{ia}, \quad \beta = \mathrm{e}^{ib}$$

with $a, b \in \mathbb{R}$, $a \not\equiv b \mod 2\pi$, so that

$$\Re\left((\overline{\alpha}\beta - 1)\frac{\alpha}{\beta}\right) = 1 - \cos(a-b)$$

and

$$\Re\left(\frac{\beta}{\overline{\alpha}\beta - 1}\right) = \frac{\cos a - \cos b}{2(1-\cos(a-b))},$$

we have

$$\begin{aligned}|(T(t)1)(0)| &= \exp\big(\mu(1-\cos(a-b))\big(1 - \mathrm{e}^{2st}\big)\\ &= \upsilon\big(\mathrm{e}^{-2st} - 1\big)\frac{\cos a - \cos b}{2(1-\cos(a-b))}\left|\frac{\mathrm{e}^{st}(\alpha-\beta)}{\mathrm{e}^{2st}\alpha - \beta}\right|^{\frac{2}{p}}.\end{aligned}$$

Since

$$\lim_{t\to+\infty} |(T(t)1)(0)| = 0,$$

and therefore

$$\lim_{t\to+\infty} \big|\,|(T(t)1)(0)| - 1\big| = 1,$$

there is some $l > 0$ such that, if $t > l$, then

$$|(T(t)1)(0) - 1| \geq \frac{1}{2}.$$

In conclusion, if $t > l$,

$$2^{\frac{1}{p}}|T(t)1 - 1| \geq |(T(t)1)(0) - 1| \geq \frac{1}{2}.$$

That proves

LEMMA 2: *If the strongly continuous semigroup T of linear isometries of H^p ($1 \leq p < \infty, p \neq 2$) is expressed by* (37), *where the flow ϕ of Moebius transformations of Δ fixes two distinct points in $\partial\Delta$, then T is not almost periodic.*

As a consequence of Proposition 6, and of Lemmas 1 and 2 the following theorem holds.

THEOREM 2: *The semigroup T of linear isometries of H^p is almost periodic if, and only if, the conformal flow of T fixes a point of Δ. In that case, T is the restriction to $\mathbb{R}_+$ of an almost periodic group of surjective isometries of H^p.*

5. - SEMIGROUPS OF LINEAR ISOMETRIES OF H^∞

So far we have always considered Hardy spaces H^p with p finite (and $p \neq 2$). We will now examine briefly the case of strongly continuous semigroups of linear isometries acting on the Hardy space H^∞ of all bounded holomorphic functions on Δ, endowed with the sup-norm.

Let $T : \mathbb{R}_+ \to \mathcal{L}(H^\infty)$ be a strongly continuous semigroup of linear isometries of H^∞ into itself. According to a result by H.P. Lotz, [8], T, as any strongly continuous semigroup of linear operators acting on H^∞, is uniformly continuous. Hence, it is the restriction to $\mathbb{R}_+$ of a uniformly continuous group on $\mathbb{R}$, which will be denoted by the same symbol T.

Since for any $t < 0$ and any $f \in H^\infty$

$$\|f\| = \|T(-t)\,T(t)f\| = \|T(t)f\|,$$

$T : \mathbb{R} \to H^\infty$ is a group of surjective isometries. According to [7] (Corollary, p. 147), there are a function $c : \mathbb{R} \to \partial\Delta$ and a family $\{\phi_t : t \in \mathbb{R}\}$ of holomorphic automorphisms of Δ such that

$$T(t)f = c(t) f \circ \phi_t \;\; \forall \;\; t \in \mathbb{R} \;\; f \in H^\infty, \tag{38}$$

Being $c(t) = T(t)1$, c is a continuous homomorphism of $\mathbb{R}$ into $\partial\Delta$. Therefore there is $\delta \in \mathbb{R}$ such that

$$c(t) = e^{i\delta t} \;\; \forall \;\; t \in \mathbb{R}. \tag{39}$$

Furthermore

$$\phi_{s+t} = \phi_s \circ \phi_t \ \ \forall \ \ s, t \in \mathbb{R}$$

and the map $t \to \phi_t(z)$ is continuous; thus, ϕ is a continuous flow of holomorphic automorphisms of Δ.

If T_0 is the group of composition operators in H^∞ defined by ϕ, that is to say: $T_0(t)f = f \circ \phi_t$ for all $t \in \mathbb{R}$ and all $f \in H^\infty$, then

$$T(t) = \mathrm{e}^{i\delta t} T_0(t)$$

and the infinitesimal generators X and X_0 of T and T_0 are bounded linear operators on H^∞ related by (26).

Suppose first that the flow ϕ is elliptic. Arguing as in the proof of Proposition 6, we reach the conclusion that, if ϕ is elliptic, the group T is almost periodic.

In the other two cases, *i.e.*, if there are no points in Δ which are fixed by ϕ, there exist one or two points of $\partial\Delta$ which are Wolff points of the semiflows $\mathbb{R} \ni t \mapsto \phi_{-t}$ and $\mathbb{R} \ni t \mapsto \phi_t$. Thus, denoting by $\imath$ the "coordinate function" $\imath(z) = z$, (38) and 39) imply that there is $k > 0$ such that

$$\|T(t)\imath\| > \frac{1}{2} \qquad \text{whenever} \ \ |t| > k,$$

proving thereby that the group T is not almost periodic.

In conclusion, the following theorem holds which extends to H^∞ what we have seen for H^p when $p \neq 2$ is finite and ≥ 1.

THEOREM 3: *A strongly continuous semigroup T of linear isometries of H^∞ into itself is the restriction to $\mathbb{R}_+$ of a strongly continuous group of surjective isometries. It is almost periodic if, and only if, the conformal flow ϕ associated to T by* (38) *is elliptic, i.e. fixes a point of Δ.*

Moreover, T is periodic if, and only if, ϕ fixes a point of Δ, c is periodic and its period is a rational multiple of the period of ϕ.

REFERENCES

[1] H. BART, *Periodic strongly continuous semigroups*, Ann. Mat. Pura Appl., (4) **115** (1977), 311-318.

[2] H. BART - S. GOLDBERG, *Characterizations of almost periodic strongly continuous groups and semigroups*, Math. Ann., **236** (1978), 105-116.

[3] E. BERKSON - R. KAUFMAN - H. PORTA, *Moebius transformations of the disc and one-parameter groups of isometries of H^p*, Trans. Amer. Math. Soc., **199** (1974), 223-239.

[4] E. BERKSON, *One-parameter semigroups of isometries into H^p*, Pacific J. Math., **86** (1980).

[5] P. L. Duren, *Theory of H^p-spaces*, Academic Press, New York and London, 1970.

[6] F. FORELLI, *The isometries of H^p*, Canad. J. Math., **16** (1964), 721-728.

[7] K. HOFFMAN, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.
[8] H. P. LOTZ, *Semigroups on L^∞ and H^∞*, in R. Nagel (ed.), *One-parameter semigroups of positive operators*, Lect. Notes in Math., vol. 1184, Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1986, 54-59.
[9] E. VESENTINI, *Periodic points and non-wandering points of continuous dynamical systems*, Advances in Math., **134** (1998), 308-327.

Direttore responsabile: Prof. A. BALLIO - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5 - Bologna