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Schauder Estimates for the Evolution Generalized Stokes Problem in Exterior Domains

ABSTRACT. — We prove the solvability of the initial-boundary value problem (1.1)-(1.2) in the exterior n-dimensional domain in the class of bounded functions that are Hölder continuous together with some their derivatives. We do not make any specific assumptions about the behavior of the solution at infinity, except the boundedness. Equations (1.1) arise in the linearization of equations of motion of some class of non-Newtonian liquids.

1. - INTRODUCTION

The paper is concerned with the initial-boundary value problem

$$\frac{\partial \boldsymbol{v}(x,t)}{\partial t} + \mathcal{A}\left(x,t,\frac{\partial}{\partial x}\right)\boldsymbol{v}(x,t) + \nabla p(x,t) = \boldsymbol{f}(x,t),$$

$$\nabla \cdot \boldsymbol{v}(x,t) = 0, \quad x \in \Omega, \quad t \in (0,T), \tag{1.1}$$

$$\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \qquad \boldsymbol{v}(x,t)|_{x\in S} = \boldsymbol{a}(x,t), \tag{1.2}$$

where Ω is an exterior domain in R^n, $n \geq 2$, with a smooth connected boundary S, $\boldsymbol{v}(x,t) = (v_1(x,t), \dots, v_n(x,t))$ and $p(x,t)$ are unknown functions and $\mathcal{A}\left(x,t,\frac{\partial}{\partial x}\right)$ is a matrix second order elliptic differential operator with real coefficients whose principal part $\mathcal{A}_0\left(x,t,\frac{\partial}{\partial x}\right)$, i.e. the sum of all terms containing only second derivatives, satisfies the condition

$$C^{-1}|\xi|^2|\eta|^2 \leq \mathcal{A}_0(x,t,i\xi)\eta \cdot \eta \leq C|\xi|^2|\eta|^2, \qquad \forall \xi, \eta \in R^n, \tag{1.3}$$

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with a constant C independent of ξ, η, x, t. We assume that $\mathcal{A}$ has a divergence form, i.e.,

$$\mathcal{A}\left(x, t, \frac{\partial}{\partial x}\right) = \nabla \cdot \ell\left(x, t, \frac{\partial}{\partial x}\right)$$

where ℓ is a first order operator:

$$\ell\left(x, t, \frac{\partial}{\partial x}\right) = \left(\sum_{q=1}^{n} \ell_{jk,mq}(x,t)\frac{\partial}{\partial x_q} + \ell_{jk,m}(x,t)\right)_{j,k,m=1,\dots,n},$$

so that $\mathcal{A}_0(x, t, i\xi) = -\sum_{q.m=1}^{n} \ell_{jk,mq}(x,t)\xi_m\xi_q|_{j,k=1,\dots,n}$,

$$\nabla \cdot \ell\left(x, t, \frac{\partial}{\partial x}\right)\boldsymbol{u} = \left(\sum_{m=1}^{n}\frac{\partial}{\partial x_m}\left(\sum_{k,q=1}^{n} \ell_{jk,mq}(x,t)\frac{\partial u_k}{\partial x_q} + \sum_{k=1}^{n} \ell_{jk,m}(x,t)u_k\right)\right)_{j=1,\dots,n}.$$

We study problem (1.1)-(1.2) in anisotropic Hölder spaces. For arbitrary positive non-integral number l: $l = [l] + \alpha$, $\alpha \in (0,1)$, we denote by $C^l(\Omega)$, $C^{l,l/2}(Q_T)$, $C^{l,l/2}(\Sigma_T)$ standard Hölder spaces of functions (or vector fields) given in Ω, $Q_T = \Omega \times (0,T)$ and $\Sigma_T = S \times (0,T)$, respectively. We recall that the norms in $C^l(\Omega)$ and $C^{l,l/2}(Q_T)$ are given by

$$|u|_{C^l(\Omega)} = \sum_{0\le|j|\le[l]} \sup_{\Omega} |D^j u(x)| + [u]_{\Omega}^{(l)},$$

and

$$|u|_{C^{l,l/2}(Q_T)} = \sum_{0\le 2k+|j|\le[l]} \sup_{Q_T} |D_t^k D^j u(x,t)| + [u]_{Q_T}^{(l,l/2)}, \tag{1.4}$$

where $j = (j_1, \dots, j_n)$, $|j| = j_1 + \dots + j_n$, $D^j u(x) = \dfrac{\partial^{|j|}u(x)}{\partial x_1^{j_1}\dots\partial x_n^{j_n}}$,

$$[u]_{\Omega}^{(l)} = \sum_{|j|=[l]} [D^j u]_{\Omega}^{(\alpha)}, \quad [v]_{\Omega}^{(\alpha)} = \sup_{x,y\in\Omega} \frac{|v(x) - v(y)|}{|x-y|^{\alpha}},$$

$$[u]_{Q_T}^{(l,l/2)} = [u]_{x,Q_T}^{(l)} + [u]_{t,Q_T}^{(l/2)},$$

$$[u]_{x,Q_T}^{(l)} = \sup_{t<T} [u(\cdot,t)]_{\Omega}^{(l)}, \quad [u]_{t,Q_T}^{(l/2)} = \sup_{\Omega} [u(x,\cdot)]_{(0,T)}^{(l/2)}.$$

These definitions extend in a standard way to the functions given on S and on Σ_T. In addition, we introduce a mixed semi-norm

$$\langle u \rangle_{Q_T}^{(\alpha,\beta)} = \sup |x-y|^{-\alpha}|t-\tau|^{-\beta}|u(x,t) - u(y,t) - u(x,\tau) + u(y,\tau)|,$$

where $\alpha, \beta \in (0,1)$ and the supremum is taken with respect to $x, y \in \Omega$, $t, \tau \in (0,T)$.

We assume that the coefficients of the operator ℓ belong to $C^{1+\alpha,(1+\alpha)/2}(Q_T)$, hence, the coefficients of $\mathcal{A}$ belong to $C^{a,a/2}(Q_T)$ and the leading coefficients $\ell_{jk,mq}(x.t)$ belong to $C^{1+a,(1+a)/2}(Q_T)$. The known functions $\boldsymbol{f}, \boldsymbol{v}_0, \boldsymbol{a}$ should satisfy the necessary compatibility conditions, in the first line,

$$\nabla \cdot \boldsymbol{v}_0(x) = 0, \quad \boldsymbol{v}_0(x)|_{x\in S} = \boldsymbol{a}(x, 0), \tag{1.5}$$

and one more condition of a non-local character involving $\boldsymbol{a}_t$. Let $p_0(x)$ be a solution of the Neumann problem

$$\nabla^2 p_0(x) = -\nabla \cdot \mathcal{A}\Big(x, 0, \frac{\partial}{\partial x}\Big)\, \boldsymbol{v}_0(x) + \nabla \cdot \boldsymbol{f}(x, 0),$$

$$\frac{\partial p_0(x)}{\partial n}\Big|_{x\in S} = -\,\boldsymbol{n} \cdot \Big(\mathcal{A}\Big(x, 0, \frac{\partial}{\partial x}\Big)\boldsymbol{v}_0(x) - \boldsymbol{f}(x, 0) + \boldsymbol{a}_t(x, 0)\Big) \tag{1.6}$$

where $\boldsymbol{n}(x)$ is the interior with respect to Ω normal to S (if $\boldsymbol{v}$ does not possess the third derivatives and $\boldsymbol{f}$ does not have the first ones, the first equation in (1.6) should be understood in a weak sense). The compatibility condition reads

$$\boldsymbol{a}_t(x, 0) + \mathcal{A}\Big(x, 0, \frac{\partial}{\partial x}\Big)\, \boldsymbol{v}_0(x) + \nabla p_0(x) = \boldsymbol{f}(x, 0), \quad \forall x \in S. \tag{1.7}$$

We often assume that

$$\nabla \cdot \boldsymbol{f}(x, t) = 0, \qquad \boldsymbol{f}(x, t) \cdot \boldsymbol{n}(x)|_{x\in S} = 0 \tag{1.8}$$

or, what is the same thing, $\int_\Omega \boldsymbol{f}(x, t) \cdot \nabla\varphi(x)dx = 0$ for arbitrary smooth $\varphi(x)$. In this case the terms with $\boldsymbol{f}$ in (1.6) drop out.

THEOREM 1.1: *Let $S \in C^{2+a}$, $a \in (0, 1)$, and let the operator $\mathcal{A}$ satisfy the above hypotheses. Assume also that $\boldsymbol{f} \in C^{a,a/2}(Q_T)$, $\boldsymbol{v}_0 \in C^{2+a}(\Omega)$, $\boldsymbol{a} \in C^{2+a,1+a/2}(\Sigma_T)$, and that*

$$\sum_{k=1}^{n} |\mathcal{R}_k(\boldsymbol{a}_t \cdot \boldsymbol{n})|_{C^{a,a/2}(\Sigma_T)} < \infty,$$

where

$$\mathcal{R}_k(b) = -2\partial_k \int_S E(x - y)b(y)dS,$$

E is the fundamental solution of the Laplace equation:

$$E(x) = -\Gamma(n/2)|x|^{2-n}(2\pi^{n/2}(n-2))^{-1}, \ \ \textit{if} \ \ n > 2,$$

$$E(x) = (2\pi)^{-1}\log|x|, \ \ \textit{if} \ \ n = 2;$$

$\partial_k = \dfrac{\partial}{\partial x_k} - n_k\dfrac{\partial}{\partial n}$ *is the kth component of the surface gradient ∇_S on S. Finally, let conditions* (1.5)-(1.8) *be satisfied. Then problem* (1.1), (1.2) *has a unique solution* $(\boldsymbol{v}, p)$,

$\boldsymbol{v} \in C^{2+a,1+a/2}(Q_T)$, $\nabla p \in C^{a,a/2}(Q_T)$, and it satisfies the inequality

$$(1.9) \quad |\boldsymbol{v}|_{C^{2+a,1+a/2}(Q_T)} + |\nabla p|_{C^{a,a/2}(Q_T)} \leq c\Big(|\boldsymbol{f}|_{C^{a,a/2}(Q_T)} + |\boldsymbol{v}_0|_{C^{2+a}(\Omega)} + \\ + |\boldsymbol{a}|_{C^{2+a,1+a/2}(\Sigma_T)} + \sum_{k=1}^{n} |\mathcal{R}_k(\boldsymbol{a}_t \cdot \boldsymbol{n})|_{C^{a,a/2}(\Sigma_T)}\Big).$$

The operators $\mathcal{R}_k$ that can be considered as the Riesz operators on S are continuous in the space $C^a(S)$ but not in $C^{a,a/2}(\Sigma_T)$. However, the estimate (1.9) is coercive in the sense that its second term can be majorized by the first term multiplied by a certain constant.

For the case of a bounded Ω, Theorem 1.1 is proved in [1]. For a classical Stokes problem (when $\mathcal{A} = -\nu I\Delta$, $\nu = const > 0$) it is obtained in [2] both for bounded and exterior domains, under the assumption $\boldsymbol{a} = 0$. Moreover, in [2] a local solution of the nonlinear Navier-Stokes problem is constructed. The Cauchy problem for the Navier-Stokes equations with non-decaying at infinity initial data is solved in [3-5], and the problem in the half-space is treated in [6,7]. The restrictive compatibility conditions (1.7) can be avoided by passing to weighted Hölder spaces with the weights of the form t^γ, $\gamma > 0$, in some terms of the norm (1.4). Parabolic problems in such spaces were studied in [8-10] and in other papers. Coercive estimates of the type (1.9) in weighted Hölder spaces will be obtained in a subsequent publication.

2. - Auxiliary propositions

It is well known that every vector field $\boldsymbol{u}(x)$ can be represented as a sum

$$(2.1) \qquad \boldsymbol{u} = \boldsymbol{u}_1 + \nabla\phi \equiv P_J\,\boldsymbol{u} + P_G\,\boldsymbol{u}$$

where ϕ is a solution of the Neumann problem

$$(2.2) \qquad \Delta\phi(x) = \nabla \cdot \boldsymbol{u}(x), \quad x \in \Omega, \quad \frac{\partial\phi}{\partial n}\Big|_S = \boldsymbol{u}(x)\cdot\boldsymbol{n}(x),$$

and $\boldsymbol{u}_1$ satisfies the conditions

$$\nabla \cdot \boldsymbol{u}_1 = 0, \quad \boldsymbol{u}_1 \cdot \boldsymbol{n}|_S = 0,$$

at least in a weak sense.

If Ω is a bounded domain, then the Weyl projectors P_G and P_J are bounded in $C^a(\Omega)$, i.e.,

$$|P_G\,\boldsymbol{u}|_{C^a(\Omega)} + |P_G\,\boldsymbol{u}|_{C^a(\Omega)} \leq c|\boldsymbol{u}|_{C^a(\Omega)}.$$

For exterior domains, this inequality does not hold, moreover, these projectors, i.e., the solution of problem (2.2) with $\boldsymbol{u} \in C^a(\Omega)$, need an accurate definition. We introduce the space $\widehat{C}^a(\Omega)$ with the norm

$$\|\boldsymbol{u}\|_a = \sup_{\Omega'} |\boldsymbol{u}(x)| + [\boldsymbol{u}]_\Omega^{(a)}$$

where Ω' is a fixed bounded subdomain of Ω. It is clear that the variation of Ω' leads to an equivalent norm, moreover, Ω' can be replaced with S. To be definite, we fix Ω' such that $\mathrm{dist}(\Omega \setminus \bar{\Omega}', S) \geq r_0 > 0$ and $|x-z| \leq r_1$, $\forall x \in \Omega', z \in S$. In the present paper the following proposition is used.

PROPOSITION 2.1: *If* $\boldsymbol{u} = \nabla \cdot U \equiv \left(\sum_{k=1}^{n} \frac{\partial U_{ik}(x)}{\partial x_k} \right)_{i=1,\dots,n}$ *with* $\boldsymbol{u}$, $U_{jk} \in \widehat{C}^{a}(\Omega)$, *then problem* (2.2) *has a solution* $\phi \in \widehat{C}^{a}(\Omega)$ *with* $\nabla\phi \in C^{a}(\Omega)$, *defined up to an additive constant that can be determined, for instance, by the condition*

$$\int_S \phi(x)dS = 0. \tag{2.3}$$

The solution satisfies the inequalities

$$\|\phi\|_a \leq c\|U\|_a, \tag{2.4}$$

$$|\nabla\phi|_{C^a(\Omega)} \leq c(\|U\|_a + \|\boldsymbol{u}\|_a), \tag{2.5}$$

where

$$\|U\|_a \equiv \max_{i,k=1,\dots,n} \|U_{ik}\|_a, \quad \|\boldsymbol{u}\|_a \equiv \max_{i=1,\dots,n} \|u_i\|_a.$$

The norm $\|\boldsymbol{u}\|_a$ in (2.5) can be replaced with $[\boldsymbol{u}]_{\Omega}^{(a)}$, because the identity

$$\boldsymbol{u}(x) = |B_{r_0}|^{-1}\left(\int_{B_{r_0}} (\boldsymbol{u}(x) - \boldsymbol{u}(y))dS_y + \int_{\partial B_{r_0}} (U(y) - U(x))\,\boldsymbol{n}(y)dS_y \right),$$

where $B_{r_0} = \{y \in \Omega : \ |x-y| \leq r_0\}$ implies

$$|\boldsymbol{u}(x)| \leq c\left([\boldsymbol{u}]_{B_{r_0}}^{(a)} + [U]_{B_{r_0}}^{(a)}\right), \quad \forall x \in \Omega.$$

PROOF: Let x_0 be a fixed point of a bounded domain $\Omega^c = R^n \setminus \bar{\Omega}$. We define ϕ as the sum

$$\phi(x) = \phi_1(x) + \phi_2(x) + \phi_0 \tag{2.6}$$

where

$$\phi_1(x) = \sum_{i=1}^{n} \frac{\partial R_i(x)}{\partial x_i},$$

$$R_i(x) = \sum_{k=1}^{n} \int_{\Omega} \left(\frac{\partial E(x-y)}{\partial x_k} - \frac{\partial E(x_0 - y)}{\partial x_{0k}} - \sum_{j=1}^{n} (x_j - x_{0j}) \frac{\partial^2 E(x_0 - y)}{\partial x_{0j} \partial x_{0k}} \right) U_{ik}(y)dy - \tag{2.7}$$

$$- \sum_{k=1}^{n} \int_S E(x-y) U_{ik}(y) n_k(y) dS,$$

ϕ_2 is a solution of the problem

$$\Delta\phi_2(x) = 0, \quad x \in \Omega, \quad \frac{\partial\phi_2}{\partial n} = \boldsymbol{u}\cdot\boldsymbol{n} - \frac{\partial\phi_1}{\partial n}, \quad x \in S, \tag{2.8}$$

vanishing at infinity, and $\phi_0 = -\int_S(\phi_1(x)+\phi_2(x))dS$. Differentiation of (2.7) gives the expression of ϕ_1 in terms of singular integrals (see [11], §8):

$$\phi_1(x) = \frac{1}{n}\sum_{k=1}^n U_{kk}(x) - \sum_{i,k=1}^n \int_S \frac{\partial E(x-y)}{\partial x_i} U_{ik}(y)n_k(y)dS + $$
$$+\sum_{i,k=1}^n \int_\Omega \Big(\frac{\partial^2 E(x-y)}{\partial x_i \partial x_k} - \frac{\partial^2 E(x_0-y)}{\partial x_{0i}\partial x_{0k}}\Big)U_{ik}(y)dy,$$

where the volume integrals are understood as

$$\lim_{\varepsilon\to 0}\int_{\Omega\cap\{|x-y|\geq\varepsilon\}}\Big(\frac{\partial^2 E(x-y)}{\partial x_i \partial x_k} - \frac{\partial^2 E(x_0-y)}{\partial x_{0i}\partial x_{0k}}\Big)U_{ik}(y)dy.$$

It is clear that they are convergent, if $U_{jk}\in \widehat{C}^\alpha(\Omega)$; moreover, $\Delta R_i = \sum_{k=1}^n \frac{\partial U_{ik}}{\partial x_k} = u_i$ and

$$\Delta\phi_1(x) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\Delta R_i = \sum_{i,k=1}^n \frac{\partial^2 U_{ik}}{\partial x_i \partial x_k} = \nabla\cdot\boldsymbol{u}.$$

We extend U_{ik} into Ω^c in such a way that $|U_{ik}|_{C^\alpha(\Omega^c\cup\Omega')} \leq c|U_{ik}|_{C^\alpha(\Omega')}$ and write ϕ_1 in the form

$$\phi_1(x) = \frac{1}{n}\sum_{k=1}^n U_{kk}(x) + \sum_{i,k=1}^n \Big[\int_{R^n}\Big(\frac{\partial^2 E(x-y)}{\partial x_i \partial x_k} - \frac{\partial^2 E(x_0-y)}{\partial x_{0i}\partial x_{0k}}\Big)U_{ik}(y)dy - $$
$$-\int_{\Omega^c}\Big(\frac{\partial^2 E(x-y)}{\partial x_i \partial x_k} - \frac{\partial^2 E(x_0-y)}{\partial x_{0i}\partial x_{0k}}\Big)U_{ik}(y)dy - \int_S E_{x_i}(x-y)U_{ik}(y)n_k(y)dS\Big].$$

According to classical estimates of the volume and surface potentials (see [12,13]),

$$[\phi_1]^{(\alpha)}_\Omega \leq c\sum_{i,k=1}^n\Big([U_{ik}]^{(\alpha)}_{R^n} + |U_{ik}|_{C^\alpha(\Omega^c)}\Big) \leq c\|U\|_\alpha,$$

moreover, for arbitrary $z\in\Omega'$

$$|\phi_1(z)| \leq \frac{1}{n}\sum_{k=1}^n |U_{kk}(z)| + \sum_{i,k=1}^n \Big|\int_\Omega \Big(\frac{\partial^2 E(z-y)}{\partial z_i \partial z_k} - \frac{\partial^2 E(x_0-y)}{\partial x_{0i}\partial x_{0k}}\Big)(U_{ik}(y)-U_{ik}(z))dy\Big| +$$
$$+\sum_{i,k=1}^n |U_{ik}(z)|\Big|\int_\Omega\Big(\frac{\partial^2 E(z-y)}{\partial z_i \partial z_k} - \frac{\partial^2 E(x_0-y)}{\partial x_{0i}\partial x_{0j}}\Big)dy\Big| +$$
$$+\sum_{i,k=1}^n\Big|\int_S E_{x_i}(z-y)U_{ik}(y)n_k(y)dS\Big| \leq c\|U\|_\alpha.$$

Hence,

$$\|\phi_1\|_a \le c\|U\|_a.$$

Now, we estimate $\nabla\phi_1$. It is clear that

$$\phi_1(x) = \lim_{M\to\infty} \phi^{(M)}(x) = \lim_{m\to\infty} \sum_{i=1}^{n} \frac{\partial R_i^{(M)}}{\partial x_i}$$

where

$$R_i^{(M)}(x) = -\int_S E(x-y)U_{ik}(y)n_k(y)dS +$$

$$+\sum_{k=1}^{n}\int_{\Omega}\Big(\frac{\partial E(x-y)}{\partial x_k} - \frac{\partial E(x_0-y)}{\partial x_{0k}} - \sum_{j=1}^{n}(x_j-x_{0j})\frac{\partial^2 E(x_0-y)}{\partial x_{0j}\partial x_{0k}}\Big)U_{ik}(y)\zeta_M(y)dy,$$

$\zeta_M(x) = \zeta((x-x_0)/M)$, $\zeta \in C_0^\infty(R^n)$, $\zeta(z) = 1$ for $|z| \le 1$, $\zeta(z) = 0$ for $|z| \ge 2$, $0 \le \zeta(z) \le 1$, $|D^j\zeta_M(x)| \le c(|j|)M^{-|j|}$. For M large enough,

$$\frac{\partial\phi_1^{(M)}(x)}{\partial x_m} = \sum_{i=1}^{n}\frac{\partial^2 R^{(M)}(x)}{\partial x_i\partial x_m} = \sum_{i,k=1}^{n}\frac{\partial^2}{\partial x_m\partial x_i}\int_{\Omega} E(x-y)\frac{\partial}{\partial y_k}(U_{ik}(y)\zeta_M(y))dy =$$

$$= \frac{1}{n}u_m(x) + \sum_{i,k=1}^{n}\int_{\Omega}\frac{\partial^2 E(x-y)}{\partial x_m\partial x_i}\frac{\partial}{\partial y_k}(U_{ik}(y)\zeta_M(y))dy.$$

After simple transformations we obtain

$$\lim_{M\to\infty}\frac{\partial\phi_1^{(M)}(x)}{\partial x_m} = \frac{1}{n}u_m(x) +$$

$$+\sum_{i,k=1}^{n}\Big(\int_{\Omega_1}\frac{\partial^2 E(x-y)}{\partial x_i\partial x_m}(u_i(y)-u_i(x))dy + u_i(x)\lim_{\varepsilon\to 0}\int_{\Omega_{1,\varepsilon}}\frac{\partial^2 E(x-y)}{\partial x_k\partial x_m}dy +$$

$$+\int_{\Omega\setminus\Omega_1}\frac{\partial^3 E(x-y)}{\partial x_k\partial x_m\partial x_i}(U_{ik}(y)-U_{ik}(x))dy -$$

$$-\int_{\partial(\Omega\setminus\Omega_1)}\frac{\partial^2 E(x-y)}{\partial x_m\partial x_i}(U_{ik}(y)-U_{ik}(x))n_k(y)dS\Big) \equiv \frac{\partial\phi_1(x)}{\partial x_m},$$

where $\Omega_1 = \Omega_1(x) = \{y \in \Omega : |x-y| \le r_0\}$ and $\Omega_{1,\varepsilon} = \{y \in \Omega : \varepsilon \le |x-y| \le r_0\}$. The integral $\int_{\Omega_{1,\varepsilon}}\frac{\partial^2 E(x-y)}{\partial x_k\partial x_m}dy$ is uniformly bounded, moreover, it vanishes, if $\mathrm{dist}(x,S) > r_0$, i.e., $x \in \Omega\setminus\Omega'$, hence,

$$\Big|\frac{\partial\phi_1(x)}{\partial x_m}\Big| \le c\big(\|U\|_a + |\boldsymbol{u}|_{C^a(\Omega)}\big) \le c\big(\|U\|_a + \|\boldsymbol{u}\|_a\big).$$

In addition, applying to $\dfrac{\partial \phi_1^{(M)}(x)}{\partial x_m}$ classical estimates of the theory of potentials and passing to the limit, we obtain

$$[\nabla \phi_1]_{\Omega}^{(a)} \leq c\big(\|U\|_a + \|\boldsymbol{u}\|_a\big),$$

so ϕ_1 satisfies (2.4), (2.5).

Let us pass to the estimates of ϕ_2. We observe that

$$\boldsymbol{u} \cdot \boldsymbol{n} - \frac{\partial \phi_1}{\partial n} = \sum_{i=1}^{n} n_i \Delta R_i - \sum_{i,k=1}^{n} n_k \frac{\partial^2 R_i}{\partial x_i \partial x_k} = \sum_{i,k=1}^{n} \left(n_i \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial x_i}\right) \frac{\partial R_i}{\partial x_k} = div_S\, \boldsymbol{M},$$

where $\boldsymbol{M}(x)$ is a tangential vector field on S given by

$$\boldsymbol{M} = \sum_{i,k=1}^{n} (n_i\, \boldsymbol{e}_k - n_k\, \boldsymbol{e}_i) \frac{\partial R_i}{\partial x_k}, \quad \boldsymbol{e}_k = (\delta_{ik})_{i,k=1,\dots,n},$$

and satisfying the inequality

$$|\boldsymbol{M}|_{C^a(S)} \leq c \sum_{i=1}^{n} |\nabla R_i|_{C^a(S)} \leq c\|U\|_a.$$

Hence, $\displaystyle\int_S \left(\boldsymbol{u} \cdot \boldsymbol{n} - \frac{\partial \phi_1}{\partial n}\right) dS = 0$, the solution ϕ_2 of problem (2.8) is well defined, and

$$|\nabla \phi_2|_{C^a(\Omega)} \leq c \left|\boldsymbol{u} \cdot \boldsymbol{n} - \frac{\partial \phi_1}{\partial n}\right|_{C^a(S)} \leq c(\|U\|_a + \|\boldsymbol{u}\|_a).$$

To estimate $\|\phi_2\|_a$, we represent ϕ_2 as a single layer potential

(2.9) $$\phi_2(x) = -2 \int_S E(x-y)\mu(y)dS$$

with μ satisfying the integral equation

(2.10) $$\mu(x) + 2 \int_S \frac{\partial E(x-y)}{\partial n_x} \mu(y)dS = -div_S\, \boldsymbol{M}(x), \quad x \in S.$$

Integrating this equation we find $\int_S \mu(x)dS = 0$. The function

$$\sigma(x) = 2 \int_S \frac{\partial E(x-y)}{\partial n_x} \mu(y)dS$$

satisfies the same kind of equation

$$\sigma(x) + 2 \int_S \frac{\partial E(x-y)}{\partial n_x} \sigma(y)dS = -2 \int_S \frac{\partial E(x-y)}{\partial n_x} div_S\, \boldsymbol{M}(y)dS \equiv -2\sigma_1(x),$$

hence,

$$\sup_S |\sigma(x)| \leq c \sup_S |\sigma_1(x)|.$$

The function σ_1 can be brought to the form

$$\sigma_1(x) = \int_S \nabla_S \frac{\partial E(x-y)}{\partial n_y} \cdot (\boldsymbol{M}(y) - \boldsymbol{M}(x))dS - (n-1)\boldsymbol{M}(x) \cdot \int_S \boldsymbol{n}(y)H(y)\frac{\partial E(x-y)}{\partial n_y} dS,$$

where H is the mean curvature of S. Since

$$\Big|\nabla_S \frac{\partial E(x-y)}{\partial n_y}\Big| \leq c\frac{1}{|x-y|^{n-1}},$$

we obtain

$$|\sigma_1(x)| \leq c|\boldsymbol{M}|_{C^\alpha(S)} \leq c\|U\|_\alpha.$$

Since $\int_S \sigma(x)dS = 0$, the function

$$\phi_2(x) = -2\int_S \nabla_S E(x-y) \cdot \boldsymbol{M}(y)dS + 2\int_S E(x-y)\sigma(y)dS$$

satisfies the inequality

$$|\phi_2|_{C^\alpha(\Omega)} \leq c(|\boldsymbol{M}|_{C^\alpha(S)} + \sup_S |\sigma(x)|) \leq c\|U\|_\alpha,$$

which completes the proof of (2.4).

It is easily seen that the constructed solution of the problem (2.2), (2.3) is unique. Indeed, the solution ϕ_0 of a homogeneous problem is a harmonic function that grows at infinity not faster than $|x|^\alpha$, which implies that it tends to a constant and in fact coincides with this constant. By (2.3), it vanishes. The proposition is proved.

Estimate (2.5) implies

$$|P_G\,\boldsymbol{u}|_{C^\alpha(\Omega)} + |P_J\,\boldsymbol{u}|_{C^\alpha(\Omega)} \leq c(\|U\|_\alpha + \|\boldsymbol{u}\|_\alpha), \tag{2.11}$$

if $\boldsymbol{u} = \nabla \cdot U$; moreover, by interpolation inequality,

$$\begin{aligned} \sup_\Omega |P_G\,\boldsymbol{u}(y)| = \sup_\Omega |\nabla\phi(y)| &\leq c(h^{\alpha/2}[\nabla\phi]^{(\alpha)}_\Omega + h^{-(1-\mu)/2}[\phi]^{(\mu)}_\Omega) \leq \\ &\leq c(h^{\alpha/2}|\boldsymbol{u}|_\alpha + h^{\alpha/2}\|U\|_\alpha + h^{-(1-\mu)/2}\|U\|_\mu), \quad \mu \in (0,1). \end{aligned} \tag{2.12}$$

This makes it possible to estimate the norm $|P_G\,\boldsymbol{u}|_{C^{\alpha,\alpha/2}(Q_T)}$, if $\boldsymbol{u} = \nabla \cdot U$ depends also on $t \in (0,T)$. Let us apply (2.12) to $h^{-\alpha/2}\Delta_t(h)\,\boldsymbol{u}(x,t)$ where

$$\Delta_t(h)u(x,t) = u(x,t+h) - u(x,t),$$

and take supremum with respect to t and h. This leads to

$$[P_G\,\boldsymbol{u}]^{(\alpha/2)}_{t,Q_T} \leq c\Big(\sup_{t<T}[\boldsymbol{u}]^{(\alpha)}_\Omega + \sup_{t<T}\|U\|_\alpha + \langle U\rangle^{(\mu,(1+\alpha-\mu)/2)}_{Q_T} + [U]^{((1+\alpha-\mu)/2)}_{Q_T}\Big),$$

hence,

$$|P_G\nabla \cdot U|_{C^{\alpha,\alpha/2}(Q_T)} \leq c|U|_{C^{1+\alpha,(1+\alpha)/2}(Q_T)}. \tag{2.13}$$

Now, we can prove important estimates of the pressure. We consider at first the function p_0 satisfying (1.6). It is easily seen that under the condition (1.8)

$$\nabla p_0 = -P_G \nabla \cdot \ell\Big(x, 0, \frac{\partial}{\partial x}\Big)\, \boldsymbol{v}_0 + \nabla q_0, \tag{2.14}$$

where q_0 is a harmonic function in Ω satisfying the boundary condition

$$\frac{\partial q_0(x)}{\partial n} = -\,\boldsymbol{a}_t(x,0)\cdot \boldsymbol{n}(x), \quad x \in S.$$

If $n = 2$ and $\int\limits_S \boldsymbol{a}_t \cdot \boldsymbol{n} dS \neq 0$, then q_0 can have a logarithmic growth at infinity, but at any case it satisfies the inequality

$$\|q_0\|_\alpha + |\nabla q_0|_{C^\alpha(\Omega)} \leq c|\boldsymbol{a}_t(\cdot, 0)\cdot \boldsymbol{n}|_{C^\alpha(S)}.$$

Now, applying proposition 2.1, we see that $p_0 \in \widehat{C}^\alpha(\Omega)$, $\nabla p_0 \in C^\alpha(\Omega)$, and

$$\|p_0\|_\alpha + |\nabla p_0|_{C^\alpha(\Omega)} \leq c\big(|\boldsymbol{v}_0|_{C^{2+\alpha}(\Omega)} + |\boldsymbol{a}_t(\cdot,0)\cdot \boldsymbol{n}|_{C^\alpha(S)}\big).$$

Note that this estimate does not guarantee the boundedness of $p_0(x)$ for large $|x|$.

The relation similar to (2.14), i.e.,

$$p = p' + q, \tag{2.15}$$

where $\nabla p' = -P_G \nabla \cdot \ell\Big(x, t, \frac{\partial}{\partial x}\Big)\boldsymbol{v}$ and

$$\Delta q(x,t) = 0, \quad x \in \Omega, \quad \frac{\partial q(x,t)}{\partial n} = -\,\boldsymbol{a}_t(x,t)\cdot \boldsymbol{n}(x), \quad x \in S,$$

holds for arbitrary $t \in (0, T)$.

Proposition 2.2: *Let $\boldsymbol{v} \in C^{2+\alpha, 1+\alpha/2}(Q_T)$ and let (1.8) be satisfied. Then*

$$\langle p\rangle_{Q_T}^{(\rho,\alpha/2)} \leq c\big(|\boldsymbol{v}|_{C^{1+\alpha,(1+\alpha)/2}(Q_T)} + \langle \nabla \boldsymbol{v}\rangle_{Q_T}^{(\rho,\alpha/2)} + [\boldsymbol{a}_t\cdot \boldsymbol{n}]_{t,\Sigma_T}^{(\alpha/2)}\big), \tag{2.16}$$

where $0 < \rho < 1$, and if $\boldsymbol{a}\cdot \boldsymbol{n} = 0$, then

$$\langle p\rangle_{Q_T)}^{(\mu,\alpha_1/2)} \leq c|\boldsymbol{v}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \tag{2.17}$$

with $\alpha_1 + \mu = 1 + \alpha$, $\alpha_1, \mu \in (0,1)$. Finally,

$$|\nabla p|_{C^{\alpha,\alpha/2}(Q_T)} \leq c\Big(|\boldsymbol{v}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + \sum_{k=1}^n [\mathcal{R}_k(\boldsymbol{a}_t\cdot \boldsymbol{n})]_{t,\Sigma_T}^{(\alpha/2)}\Big). \tag{2.18}$$

Proof: We use the relation (2.15), inequalities

$$[p'(\cdot,t)]_\Omega^{(\rho)} \leq \|p'\|_\rho \leq c\|\ell \boldsymbol{v}\|_\rho, \tag{2.19}$$

$$|\nabla p'(\cdot,t)|_{C^\alpha(\Omega)} \leq c\big(|\ell \boldsymbol{v}|_{C^\alpha(\Omega)} + |\mathcal{A}\boldsymbol{v}|_{C^\alpha(\Omega)}\big) \leq c|\boldsymbol{v}(\cdot,t)|_{C^{2+\alpha}(\Omega)} \tag{2.20}$$

and estimates of q:

$$[q(\cdot,t)]_{\Omega}^{(\rho)} \le c \sup_S |\boldsymbol{a}_t(x,t)\cdot \boldsymbol{n}(x)|, \tag{2.21}$$

$$\sup_{\Omega} |\nabla q(x,t)| \le c\Big(\sup_S |\boldsymbol{a}_t(x,t)\cdot \boldsymbol{n}(x)| + \sum_{k=1}^{n} \sup_S |\mathcal{R}_k(\boldsymbol{a}_t(x,t)\cdot \boldsymbol{n}(x))|\Big), \tag{2.22}$$

$$|\nabla q(\cdot,t)|_{C^{a}(\Omega)} \le c|\boldsymbol{a}_t(\cdot,t)\cdot \boldsymbol{n}(\,\cdot\,)|_{C^{a}(S)}.$$

Inequality (2.22) is proved in [1], proposition 2.7, and the proof is valid for exterior domains (in this case the condition $\int_S \boldsymbol{a}\cdot\boldsymbol{n}dS = 0$ is not necessary).

We apply (2.19) and (2.21) to $h^{-a/2}\Delta_t(h)p'(x,t)$ and $h^{-a/2}\Delta_t(h)q(x,t)$, respectively, and take supremum with respect to t and h. This leads to

$$\langle p\rangle_{Q_T}^{(\rho,a/2)} \le \langle p'\rangle_{Q_T}^{(\rho,a/2)} + \langle q\rangle_{Q_T}^{(\rho,a/2)} \le c\Big([\ell\boldsymbol{v}]_{t,Q_T}^{(a/2)} + \langle \ell\boldsymbol{v}\rangle_{Q_T}^{(\rho,a/2)} + [\boldsymbol{a}_t\cdot\boldsymbol{n}]_{t,\Sigma_T}^{(a/2)}\Big) \le$$
$$\le c\Big(|\boldsymbol{v}|_{C^{1+a,(1+a)/2}(Q_T)} + \langle \nabla\boldsymbol{v}\rangle_{Q_T}^{(\rho,a/2)} + [\boldsymbol{a}_t\cdot\boldsymbol{n}]_{t,\Sigma_T}^{(a/2)}\Big),$$

since the coefficients of the operator ℓ belong to $C^{1+a,(1+a)/2}(Q_T)$. If $\boldsymbol{a}\cdot\boldsymbol{n} = 0$, then in the same way we obtain

$$\langle p\rangle_{Q_T}^{(\mu,a_1/2)} \le c\Big([\ell\boldsymbol{v}]_{t,Q_T}^{(a_1/2)} + \langle \ell\boldsymbol{v}\rangle_{Q_T}^{(\mu,a_1/2)}\Big) \le c|\boldsymbol{v}|_{C^{2+a,1+a/2}(Q_T)}.$$

To prove (2.18), we use (2.20) and the interpolation inequality

$$\sup_{\Omega} |\nabla p'(x,t)| \le c\Big(h^{a/2}[\nabla p'(\cdot,t)]_{\Omega}^{(a)} + h^{-(1-\mu)/2}[p'(\cdot,t)]_{\Omega}^{(\mu)}\Big)$$

applied to $h^{-a/2}\Delta_t(h)p'(x,t)$. This gives

$$|\nabla p'|_{C^{a,a/2}(Q_T)} \le c\Big([\nabla p']_{x,\Omega}^{(a)} + \langle p'\rangle_{Q_T}^{(\mu,a_1/2)} + [p']_{x,Q_T}^{(\mu)}\Big) \le c|\boldsymbol{v}|_{C^{2+a,1+a/2}(Q_T)}. \tag{2.23}$$

For ∇q we have the estimate

$$|\nabla q|_{C^{a,a/2}(Q_T)} \le c\Big(|\boldsymbol{a}_t|_{C^{a,a/2}(\Sigma_T)} + [\mathcal{R}(\boldsymbol{a}_t\cdot\boldsymbol{n})]_{t,\Sigma_T}^{(a/2)}\Big).$$

Together with (2.23), it implies (2.18). The proposition is proved.

Now, we assume that the condition $\boldsymbol{u} = \nabla\cdot U$ is not satisfied.

If $\boldsymbol{u}$ is divergence free, then ϕ is a harmonic function representable in the form (2.9) with μ satisfying the equation

$$\mu(x) + 2\int_S \frac{\partial E(x-y)}{\partial n_y}\mu(y)dS = -\,\boldsymbol{u}(x)\cdot\boldsymbol{n}(x), \quad x\in S.$$

Hence,

$$|\nabla\phi|_{C^{a}(\Omega)} \le c|\boldsymbol{u}\cdot\boldsymbol{n}|_{C^{a}(S)},$$

moreover, using proposition 2.7 in [1] it is easy to see that

$$\|\phi\|_a + \sup_{\Omega} |\nabla\phi(x)| \le c\Big(\sup_{S} |\boldsymbol{u}(x)\cdot\boldsymbol{n}(x)| + \sum_{k=1}^{n} \sup_{S} |\mathcal{R}_k(\boldsymbol{u}\cdot\boldsymbol{n})(x)|\Big).$$

For general $\boldsymbol{u} \in \widehat{C}^{\alpha}(\Omega)$, the following proposition holds.

Proposition 2.3: *For arbitrary $\boldsymbol{u} \in \widehat{C}^{\alpha}(\Omega)$ problem* (2.2) *has a solution $\phi(x)$ such that $\nabla\phi \in \widehat{C}^{\alpha}(\Omega)$. It is defined uniquely, up to a linear combination*

$$\Phi_0(x) = \sum_{i=1}^{n} c_i(x_i + \psi_i(x)) + c_0 \tag{2.24}$$

where $c_k = const$ and ψ_i are solutions to the problems

$$\Delta\psi_i(x) = 0, \quad x \in \Omega, \quad \frac{\partial\psi_i}{\partial n} = -n_i(x), \quad x \in S, \quad \lim_{|x|\to\infty} \psi_i(x) = 0.$$

The function ϕ satisfies the inequality

$$\|\nabla\phi\|_a \le c(\|\boldsymbol{u}\|_a + \sum_{k=0}^{n} |c_k|). \tag{2.25}$$

Proof: We define a partial solution of (2.2) by $\Phi(x) = \Phi_1(x) + \Phi_2(x)$, where

$$\Phi_1(x) = \int_{\Omega} \Big(\nabla_x E(x-y) - \nabla_{x_0} E(x_0 - y) - \sum_{i=1}^{n} (x_i - x_{0i}) \frac{\partial \nabla_{x_0} E(x_0 - y)}{\partial x_{0i}} \Big) \cdot \boldsymbol{u}(y) dy,$$

x_0 is the same as in the previous proposition, and $\Phi_2(x)$ is a harmonic function satisfying the boundary condition

$$\frac{\partial\Phi_2(x)}{\partial n} = \boldsymbol{u}(x)\cdot\boldsymbol{n}(x) - \frac{\partial\Phi_1(x)}{\partial n}$$

on S. As above, we have

$$\frac{\partial\Phi_1(x)}{\partial x_m} = \frac{1}{n} u_m(x) + \int_{\Omega} \Big(\frac{\partial\nabla E(x-y)}{\partial x_m} - \frac{\partial \nabla_{x_0} E(x_0 - y)}{\partial x_{0m}} \Big) \cdot \boldsymbol{u}(y) dy$$

where the integral is singular. The functions Φ_1 and Φ_2 satisfy the inequalities

$$\|\nabla\Phi_1\|_a \le c\|\boldsymbol{u}\|_a,$$

$$|\nabla\Phi_2|_{C^{\alpha}(\Omega)} \le c\Big| \boldsymbol{u}\cdot\boldsymbol{n} - \frac{\partial\Phi_1}{\partial n} \Big|_{C^{\alpha}(S)} \le c\|\boldsymbol{u}\|_a,$$

implying (2.25) in the case $c_0 = \ldots = c_n = 0$. It is clear that $\nabla\Phi(x)$ can grow at infinity not faster than $|x|^a$ and Φ not faster than $|x|^{1+a}$.

Let $\Phi_0(x)$ be a solution of a homogeneous problem (2.2) satisfying this growth condition. Then it behaves at infinity like the first order polynomial and has the form

(2.24). Let us show that the functions $\{1, x_1 + \psi_1(x), ..., x_n + \psi_n(x)\}$ are linearly independent on S. Assume that there exist constants λ_k such that

$$\sum_{i=1}^{n} \lambda_i (x_i + \psi_i(x)) + \lambda_0 = 0, \quad x \in S.$$

Since ψ_i are representable in Ω in the form of single layer potentials $V_i(x) = -2 \int_S E(x-y)\mu_i(y)dS$, and these potentials are continuous, we have

$$\sum_{i=1}^{n} \lambda_i (x_i + V_i(x)) + \lambda_0 = 0, \quad x \in S.$$

The expression in the left hand side is harmonic in $\Omega^c = R^n \setminus \bar{\Omega}$, so it vanishes also in Ω^c. As a consequence, we have

$$\sum_{i=1}^{n} \lambda_i \Big(n_i(x) + \Big(\frac{\partial V_i(x)}{\partial n} \Big)_i \Big) = 0, \quad x \in S,$$

where the normal derivative is calculated from the side of the interior domain Ω^c. On the other hand, we have, by the definition of $\psi_i(x)$, the following relation for the exterior normal derivatives:

$$n_i(x) + \Big(\frac{\partial V_i(x)}{\partial n} \Big)_e = 0, \; i = 1, ..., n.$$

Hence,

$$\sum_{i=1}^{n} \lambda_i \Big(\Big(\frac{\partial V_i(x)}{\partial n} \Big)_i - \Big(\frac{\partial V_i(x)}{\partial n} \Big)_e \Big) = 2 \sum_{i=1}^{n} \lambda_i \mu_i(x) = 0.$$

which implies $\sum_{i=1}^{n} \lambda_i V_i(x) = 0$, $\sum_{i=1}^{n} \lambda_i x_i + \lambda_0 = 0$, and finally $\lambda_k = 0, \; k = 0, ..., n$.

A general solution of problem (2.2) has the form

$$\phi(x) = \Phi(x) + \Phi_0(x)$$

and satisfies (2.25). The constants c_k can be found from the appropriate additional conditions, for instance,

$$\int_S \phi(x)dS = 0, \quad \int_S \phi(x)(x_i + \psi_i(x))dS = 0, \quad i = 1, ..., n.$$

In this case the last term in (2.25) can be omitted. The proposition is proved.

This proposition is not used in the present paper.

If $\boldsymbol{f}$ does not satisfy (1.8) but can be represented in the form (2.1):

$$\boldsymbol{f} = \boldsymbol{f}_1 + \nabla\psi = P_J \boldsymbol{f} + P_G \boldsymbol{f},$$

then we can introduce a new pressure $p_1 = p - \psi$ and replace $\boldsymbol{f}$ with $\boldsymbol{f}_1$ and p with p_1 in (1.1), (1.2) and in the estimate (1.9). It implies the following inequality that is used below:

$$(2.26)\quad |\boldsymbol{v}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_T)} \le c\Big(|\boldsymbol{f}|_{C^{\alpha,\alpha/2}(Q_T)} + |P_G\boldsymbol{f}|_{C^{\alpha,\alpha/2}(Q_T)} + $$
$$+ |\boldsymbol{v}_0|_{C^{2+\alpha}(\Omega)} + |\boldsymbol{a}|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)} + \sum_{k=1}^{n} |\mathcal{R}_k(\boldsymbol{a}_t\cdot\boldsymbol{n})|_{C^{\alpha,\alpha/2}(\Sigma_T)}\Big).$$

3. - The proof of estimate (1.9)

We follow [1], Sec.4, and consider at first problem (1.1), (1.2) with $\boldsymbol{a}=0,\ \boldsymbol{v}_0=0$. We estimate the solution in the neighborhood of arbitrary interior point $x_0\in\Omega$. Let $\zeta(y,t)$ be a cut-off function equal to one in the "parabolic cylinder" $C_{d/2}\equiv \equiv\{x,t: |x-x_0|\le d/2, 0\le t_0-t\le d^2/4\}$ and to zero for $t\le t_0$ outside C_d. We assume that $\mathrm{dist}(x_0,S)>d$. We introduce the functions $\boldsymbol{u}(x,t)=\boldsymbol{v}(x,t)\zeta(x,t)$, $q(x,t)=(p(x,t)-\bar p(t))\zeta(x,t)$ where

$$\bar p(t)=|B_d|^{-1}\int_{B_d} p(x,t)dx,\quad B_d=\{x\in\Omega: |x-x_0|\le d\},$$

and we extend them by zero into the domain $\{t<t_0,\ x\in R^n\}$ outside C_d. The extended functions satisfy the relations

$$\frac{\partial\boldsymbol{u}(x,t)}{\partial t}+\mathcal{A}\Big(x,t,\frac{\partial}{\partial x}\Big)\boldsymbol{u}(x,t)+\nabla q(x,t)=\boldsymbol{f}(x,t)\zeta(x,t)+\boldsymbol{f}'(x,t),$$
$$\nabla\cdot\boldsymbol{u}=\boldsymbol{v}\cdot\nabla\zeta,\quad x\in R^n,\ \ t\le t_0,$$
$$\boldsymbol{u}(x,0)=0,$$

that can be also written as

$$\frac{\partial\boldsymbol{u}(x,t)}{\partial t}+\mathcal{A}_{00}\Big(\frac{\partial}{\partial x}\Big)\boldsymbol{u}(x,t)+\nabla q(x,t)=\boldsymbol{f}(x,t)\zeta(x,t)+\boldsymbol{f}_1(x,t),$$
$$\nabla\cdot\boldsymbol{u}=\boldsymbol{v}\cdot\nabla\zeta,\quad x\in R^n,\quad t\in(0,t_0),$$
$$\boldsymbol{u}(x,0)=0,$$

where

$$\boldsymbol{f}'(x,t)=[\mathcal{A},\zeta]\boldsymbol{v}+(p-\bar p)\nabla\zeta+\boldsymbol{v}\zeta_t,\quad [\mathcal{A},\zeta]\boldsymbol{v}=\mathcal{A}(\zeta\boldsymbol{v})-\zeta\mathcal{A}\boldsymbol{v},$$
$$\boldsymbol{f}_1(x,t)=-\mathcal{A}'\Big(x,t,\frac{\partial}{\partial x}\Big)\boldsymbol{u}+\boldsymbol{f}'(x,t),$$
$$\mathcal{A}_{00}\Big(\frac{\partial}{\partial x}\Big)=\mathcal{A}_0\Big(0,t_0,\frac{\partial}{\partial x}\Big),\quad \mathcal{A}'=\mathcal{A}-\mathcal{A}_{00}.$$

We represent $(\boldsymbol{u}, q)$ in the form

$$\boldsymbol{u} = \boldsymbol{u}^{(1)} + \boldsymbol{u}^{(2)} + \boldsymbol{u}^{(3)}, \quad q = q^{(2)} + q^{(3)},$$

where $\boldsymbol{u}^{(i)}, q^{(i)}$ are defined by

$$\frac{\partial \boldsymbol{u}^{(1)}(x,t)}{\partial t} + \mathcal{A}_{00}\Big(\frac{\partial}{\partial x}\Big)\boldsymbol{u}^{(1)}(x,t) = \boldsymbol{f}(x,t)\zeta(x,t), \quad x \in R^n,\ t \le t_0,$$

$$\boldsymbol{u}^{(1)}(x,0) = 0, \tag{3.1}$$

$$\boldsymbol{u}^{(2)}(x,t) = \nabla \int_{R^n} E(x-y)(\boldsymbol{v}(y,t)\cdot\nabla\zeta(y,t) - \nabla\cdot\boldsymbol{u}^{(1)}(y,t))dy,$$

$$\nabla q^{(2)} = -\,\boldsymbol{u}_t^{(2)},$$

$$\frac{\partial \boldsymbol{u}^{(3)}(x,t)}{\partial t} + \mathcal{A}_{00}\Big(\frac{\partial}{\partial x}\Big)\,\boldsymbol{u}^{(3)}(x,t) + \nabla q^{(3)}(x,t) = \boldsymbol{f}_2(x,t),$$

$$\nabla\cdot\boldsymbol{u}^{(3)} = 0, \qquad \boldsymbol{u}^{(3)}(x,0) = 0$$

with

$$\boldsymbol{f}_2 = \boldsymbol{f}_1 - \mathcal{A}_{00}\Big(\frac{\partial}{\partial x}\Big)\,\boldsymbol{u}^{(2)}.$$

Let us estimate the functions $\boldsymbol{u}^{(i)}$, $i = 1,2,3$. The parabolic Cauchy problem (3.1) has a unique solution $\boldsymbol{u}^{(1)} \in C^{2+\alpha,1+\alpha/2}(\Pi_{t_0})$, $\Pi_{t_0} = R^n \times (0,t_0)$, and

$$[\boldsymbol{u}^{(1)}]^{(2+\alpha,1+\alpha/2)}_{\Pi_{t_0}} \le c[\boldsymbol{f}\zeta]^{(\alpha,\alpha/2)}_{\Pi_{t_0}}$$

(see [14]). Further, according to well-known estimates of the Newtonian potential,

$$[\boldsymbol{u}^{(2)}(\cdot,t)]^{(2+\alpha)}_{R^n} \le c[\boldsymbol{v}\cdot\nabla\zeta - \nabla\cdot\boldsymbol{u}^{(1)}]^{(1+\alpha)}_{R^n} \le c\Big([\boldsymbol{v}\cdot\nabla\zeta]^{(1+\alpha)}_{R^n} + [\boldsymbol{u}^{(1)}]^{(2+\alpha)}_{R^n}\Big).$$

To estimate the Hölder constant of $\boldsymbol{u}_t^{(2)}$ with respect to t, we write $\boldsymbol{u}_t^{(2)}$ in the form

$$\boldsymbol{u}_t^{(2)} = \nabla \int_{R^n} E(x-y)\Big((-\nabla\cdot\ell\boldsymbol{v} - \nabla(p-\bar{p}))\cdot\nabla\zeta + \nabla\cdot\mathcal{A}_{00}\,\boldsymbol{u}^{(1)}\Big)dy =$$

$$= \nabla\int_{R^n}\Big((-\nabla E(x-y)\otimes\nabla\zeta) : \ell\boldsymbol{v} + E(x-y)(\ell\boldsymbol{v} : \nabla\nabla\zeta)\Big)dy -$$

$$-\nabla\int_{R^n}\Big(\nabla E(x-y)\cdot(p-\bar{p})\nabla\zeta - E(x-y)(p-\bar{p})\nabla^2\zeta - \nabla E(x-y)\cdot\mathcal{A}_{00}\,\boldsymbol{u}^{(1)}\Big)dy$$

(since $\boldsymbol{f}$ is divergence free, the sum of the terms containing $\boldsymbol{f}$ vanishes). By inequalities (2.37) and (2.39) in [1],

$$[\boldsymbol{u}_t^{(2)}]_{t,\Pi_{t_0}}^{(\alpha/2)} \le c[\boldsymbol{u}^{(1)}]_{(\Pi_{t_0})}^{(2+\alpha,1+\alpha/2)} + c(d)\Big([\boldsymbol{v}]_{t,C_d}^{(\alpha/2)} + \langle \boldsymbol{v}\rangle_{C_d}^{(\rho,\alpha/2)} + \\ + \sup_{t\in(t_0-d^2,t_0)} |\boldsymbol{v}(\cdot,t)|_{C^{1+\rho}(B_d)} + [\nabla \boldsymbol{v}]_{t,C_d}^{(\alpha/2)} + \langle \nabla\boldsymbol{v}\rangle_{C_d}^{(\rho,\alpha/2)} + \\ + [p-\bar p]_{t,C_d}^{(\alpha/2)} + \langle p\rangle_{C_d}^{(\rho,\alpha/2)} + \sup_{t\in(t_0-d^2,t_0)} |p(\cdot,t)-\bar p(t)|_{C^{\rho}(B_d)}\Big),$$

where ρ is a small positive number. Finally, by estimate (3.9) in [1],

$$[\boldsymbol{u}^{(3)}]_{\Pi_{t_0}}^{(2+\alpha,1+\alpha/2)} + [\nabla q^{(3)}]_{\Pi_{t_0}}^{(\alpha,\alpha/2)} \le c\Big([\boldsymbol{f}_2]_{\Pi_{t_0}}^{(\alpha,\alpha/2)} + [\mathbf{P}_G \boldsymbol{f}_2]_{t,\Pi_{t_0}}^{(\alpha/2)}\Big) \le \\ \le c\Big([\boldsymbol{f}_1]_{\Pi_{t_0}}^{(\alpha,\alpha/2)} + [\mathbf{P}_G \boldsymbol{f}_1]_{t,\Pi_{t_0}}^{(\alpha/2)} + [\boldsymbol{u}^{(2)}]_{\Pi_{t_0}}^{(2+\alpha,1+\alpha/2)}\Big),$$

where $\mathbf{P}_G$ is the Weyl projector in the whole space R^n (see [1]). In its turn,

$$[\boldsymbol{f}_1]_{\Pi_{t_0}}^{(\alpha,\alpha/2)} + [\mathbf{P}_G \boldsymbol{f}_1]_{t,\Pi_{t_0}}^{(\alpha/2)} \le [\boldsymbol{f}']_{\Pi_{t_0}}^{(\alpha,\alpha/2)} + \langle \boldsymbol{f}'\rangle_{\Pi_{t_0}}^{(\rho,\alpha/2)} + cd[\boldsymbol{u}]_{\Pi_{t_0}}^{(2+\alpha,1+\alpha/2)} + \\ + c\Big(\sum_{|j|\le 2} \sup_{\Pi_{t_0}} |D^j \boldsymbol{u}(y,t)| + \langle \mathcal{A}_1 \boldsymbol{u}\rangle_{\Pi_{t_0}}^{(\rho,\alpha/2)} + [\mathcal{A}_1 \boldsymbol{u}]_{t,\Pi_{t_0}}^{(\alpha/2)}\Big),$$

because the leading coefficients of $\mathcal{A}'$ do not exceed cd. Collecting all the above estimates and taking d sufficiently small, we obtain

$$[\boldsymbol{u}]_{\Pi_{t_0}}^{(2+\alpha,1+\alpha/2)} + [\nabla q]_{\Pi_{t_0}}^{(\alpha,\alpha/2)} \le c\Big([\zeta \boldsymbol{f}]_{\Pi_{t_0}}^{(\alpha,\alpha/2)} + M\Big) \tag{3.2}$$

where M is the sum of some lower order norms of p, $\boldsymbol{v}$, $q = \zeta(p-\bar p)$ and $\boldsymbol{u} = \zeta\boldsymbol{v}$ that can be estimated by interpolation inequalities as follows:

$$M \le \varepsilon\Big([\boldsymbol{v}]_{C_d}^{(2+\alpha,1+\alpha/2)} + [\nabla p]_{C_d}^{(\alpha,\alpha/2)}\Big) + \\ + c(\varepsilon)\Big(\sup_{C_d} |\boldsymbol{v}(x,t)| + \sup_{C_d} |p(x,t)-\bar p(t)| + [p-\bar p]_{t,C_d}^{(\alpha/2)}\Big) \le \\ \le \varepsilon\Big([\boldsymbol{v}]_{C_d}^{(2+\alpha,1+\alpha/2)} + [\nabla p]_{C_d}^{(\alpha,\alpha/2)}\Big) + \\ + c(\varepsilon)\Big(\sup_{C_d} |\boldsymbol{v}(x,t)| + \sup_{t<t_0} [p(\cdot,t]_{B_d}^{(\rho)} + \langle p\rangle_{C_d}^{(\rho,\alpha/2)}\Big), \quad \forall\varepsilon\in(0,1).$$

Estimate of the solution of (1.1), (1.2) near the boundary was obtained in [1]. From this estimate and from (3.2) one can deduce, by standard arguments of Schauder's method, the inequality

$$|\boldsymbol{v}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_{t'})} \le c\Big(|\boldsymbol{f}|_{C^{\alpha,\alpha/2}(Q_{t'})} + \sup_{Q_{t'}} |\boldsymbol{v}(x,t)| + \\ + \sup_{Q_{t'}} [p(\cdot,t)]_{\Omega}^{(\rho)} + \langle p\rangle_{Q_{t'}}^{(\rho,\alpha/2)} + d\langle p\rangle_{Q_{t'}}^{(\mu,\alpha_1/2)}\Big)$$

for arbitrary $t' \leq T$. The norms of p can be estimated by (2.16)-(2.18):

$$\sup_{Q_{t'}} [p(\cdot, t)]_{\Omega}^{(\rho)} + \langle p \rangle_{Q_{t'}}^{(\rho, \alpha/2)} + d \langle p \rangle_{Q_{t'}}^{(\mu_1, \alpha_1/2)} \leq$$
$$\leq c\Big(|\boldsymbol{v}|_{C^{1+\alpha,(1+\alpha)/2}(Q_{t'})} + \langle \nabla \boldsymbol{v} \rangle_{Q_{t'}}^{(\rho,\alpha/2)} + d|\boldsymbol{v}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})}\Big).$$

After this, using again the smallness of d and the interpolation inequality, we arrive at

$$|\boldsymbol{v}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_{t'})} \leq c\Big(|\boldsymbol{f}|_{C^{\alpha,\alpha/2}(Q_{t'})} + \sup_{Q_{t'}} |\boldsymbol{v}(x,t)|\Big) \leq$$
$$\leq c\Big(|\boldsymbol{f}|_{C^{\alpha,\alpha/2}(Q_{t'})} + \int_0^{t'} |\boldsymbol{v}_\tau(x,\tau) d\tau|\Big),$$

and, applying the Gronwall lemma, we obtain (1.9) in our case: $\boldsymbol{v}_0 = 0$, $\boldsymbol{a} = 0$.

Now, we turn to a general case. Let $\boldsymbol{a}'(x,t) = b(t)\psi(x)$ where $\psi(x)$ is a solution of the Stokes problem

$$-\Delta\psi(x) + \nabla\pi(x) = 0, \quad \nabla \cdot \psi(x) = 0, \quad x \in \Omega, \quad \psi(x) = \boldsymbol{N}(x), \quad x \in S,$$

vanishing at infinity for $n \geq 3$ and bounded for $n = 2$, with a smooth $\boldsymbol{N}$ satisfying the condition

$$\int_S \boldsymbol{N}(x) \cdot \boldsymbol{n}(x) dS \geq c_1 > 0,$$

and let $b(t)$ be given by

$$b(t) = \Big(\int_S \boldsymbol{N}(x) \cdot \boldsymbol{n}(x) dS \Big)^{-1} \int_S \boldsymbol{a}(x,t) \cdot \boldsymbol{n}(x) dS.$$

Hence, $\int_S (\boldsymbol{a}(x,t) - \boldsymbol{a}'(x,t)) \cdot \boldsymbol{n}(x) dS = 0$. It is clear that $\psi \in C^{2+\alpha}(\Omega)$ and $b \in$ $\in C^{1+\alpha/2}(0,T)$. Further, we define a solenoidal extension $\boldsymbol{v}_0^*(x)$ of the vector field $\boldsymbol{v}_0(x) - b(0)\psi(x)$ into Ω^c such that

$$|\boldsymbol{v}_0^*|_{C^{2+\alpha}(R^n)} \leq c\Big(|\boldsymbol{v}_0|_{C^{2+\alpha}(\Omega)} + |b(0)|\Big)$$

(see [1]) and introduce $\boldsymbol{v}_1(x,t)$ as a solution of the Cauchy problem

$$\boldsymbol{v}_{1t} - \Delta \boldsymbol{v}_1 = 0, \quad \boldsymbol{v}_1(x,0) = \boldsymbol{v}_0^*(x), \quad x \in R^n.$$

It satisfies the equation $\nabla \cdot \boldsymbol{v}_1 = 0$, and the inequality

$$|\boldsymbol{v}_1|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leq c|\boldsymbol{v}_0^*|_{C^{2+\alpha}(R^n)} \leq c\Big(|\boldsymbol{v}_0|_{C^{2+\alpha}(\Omega)} + |b(0)|\Big), \tag{3.3}$$

moreover, for $P_G\, \boldsymbol{v}_{1t} = P_G \Delta \boldsymbol{v}_1$ we have

$$|P_G\, \boldsymbol{v}_{1t}|_{C^{\alpha,\alpha/2}(Q_T)} \leq c|\boldsymbol{v}_1|_{C^{2+\alpha,1+\alpha/2}(Q_T)},$$

by virtue of (2.13). Now, let $\boldsymbol{v}_2$ be a solution of the Stokes problem

$$-\Delta \boldsymbol{v}_2 + \nabla p_2 = 0, \quad \nabla \cdot \boldsymbol{v}_2 = 0, \quad x \in \Omega, \quad \boldsymbol{v}_2 = \boldsymbol{a} - \boldsymbol{v}_1, \quad x \in S,$$

vanishing at infinity for $n \geq 3$ and bounded for $n = 2$. For this vector field the estimate

$$|\boldsymbol{v}_2|_{C^{2+a,1+a/2}(Q_T)} \leq c\Big(|\boldsymbol{a}|_{C^{2+a,1+a/2}(\Sigma_T)} + |\boldsymbol{v}_1|_{C^{2+a,1+a/2}(Q_T)}\Big) \tag{3.4}$$

holds both in bounded and in exterior domain Ω. Moreover, since $\boldsymbol{v}_2$ is divergence free, $\boldsymbol{v}_{2t} \cdot \boldsymbol{n} = \boldsymbol{a}_t \cdot \boldsymbol{n} - \boldsymbol{v}_{1t} \cdot \boldsymbol{n} = \boldsymbol{a}_t \cdot \boldsymbol{n} - \Delta \boldsymbol{v}_1 \cdot \boldsymbol{n}$ and

$$\Delta \boldsymbol{v}_1 \cdot \boldsymbol{n} = \sum_{i,k=1}^{n} \left(n_k \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_k}\right) \frac{\partial v_{1k}}{\partial x_i},$$

we have, by virtue of inequalities (2.58) and (2.40) in [1],

$$|P_G\, \boldsymbol{v}_{2t}|_{C^{a,a/2}(Q_T)} \leq c\Big(|\boldsymbol{a}_t \cdot \boldsymbol{n}|_{C^{a,a/2}(\Sigma_t)} + \sum_{k=1}^{n} [\mathcal{R}_k(\boldsymbol{a}_t \cdot \boldsymbol{n})]^{(a/2)}_{t,\Sigma_T} + |\boldsymbol{v}_1|_{C^{2+a,1+a/2}(Q_T)}\Big).$$

It is easily seen that $\boldsymbol{v}_2(x,0) = b(0)\psi(x)$; hence, the difference $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}_1 - \boldsymbol{v}_2$ is a solution of the problem

$$\boldsymbol{w}_t + \mathcal{A}\Big(x,t,\frac{\partial}{\partial x}\Big)\boldsymbol{w} + \nabla p = \boldsymbol{b}, \qquad \nabla \cdot \boldsymbol{w} = 0, \quad \boldsymbol{w}(x,0) = 0, \qquad \boldsymbol{w}|_{x\in S} = 0 \tag{3.5}$$

with

$$\begin{aligned}\boldsymbol{b}(x,t) = &\boldsymbol{f}(x,t) - \frac{\partial \boldsymbol{v}_1}{\partial t} - \mathcal{A}\Big(x,t,\frac{\partial}{\partial x}\Big)\,\boldsymbol{v}_1 - \frac{\partial \boldsymbol{v}_2}{\partial t} - \mathcal{A}\Big(x,t,\frac{\partial}{\partial x}\Big)\,\boldsymbol{v}_2 = \\ &= \boldsymbol{f}(x,t) - \frac{\partial \boldsymbol{v}_1}{\partial t} - \nabla \cdot \ell\Big(x,t,\frac{\partial}{\partial x}\Big)\boldsymbol{v}_1 - \frac{\partial \boldsymbol{v}_2}{\partial t} - \nabla \cdot \ell\Big(x,t,\frac{\partial}{\partial x}\Big)\boldsymbol{v}_2\end{aligned}$$

satisfying the inequality

$$\begin{aligned}|\boldsymbol{b}|_{C^{a,a/2}(Q_T)} + |P_G\, \boldsymbol{b}|_{C^{a,a/2}(Q_T)} \leq c\Big(&|\boldsymbol{f}|_{C^{a,a/2}(Q_T)} + |\boldsymbol{v}_1|_{C^{2+a,1+a/2}(Q_T)} + \\ &+ |\boldsymbol{v}_2|_{C^{2+a,1+a/2}(Q_T)} + |\boldsymbol{a}|_{C^{2+a,1+a/2}(\Sigma_T)} + \sum_{k=1}^{n} [\mathcal{R}_k(\boldsymbol{a}_t \cdot \boldsymbol{n})]^{(a/2)}_{t,\Sigma_T}\Big).\end{aligned}$$

In addition, for $x \in S$ we have $\boldsymbol{v}_{1t}(x,0) + \boldsymbol{v}_{2t}(x,0) = \boldsymbol{a}_t(x,0)$, and, as a consequence,

$$\boldsymbol{b}(x,0) = \boldsymbol{f}(x,0) - \boldsymbol{a}_t((x,0) - \mathcal{A}\Big(x,0,\frac{\partial}{\partial x}\Big)\boldsymbol{v}_0,$$

from which it follows that the compatibility condition (1.7) is satisfied in problem (3.5). Hence, by (2.26),

$$\begin{aligned}|\boldsymbol{w}|_{C^{2+a,1+a/2}(Q_T)} + |\nabla p|_{C^{a,a/2}(Q_T)} &\leq c\Big(|\boldsymbol{b}|_{C^{a,a/2}(Q_T)} + |P_G\, \boldsymbol{b}|_{C^{a,a/2}(Q_T)}\Big) \leq \\ &\leq c\Big(|\boldsymbol{f}|_{C^{a,a/2}(Q_T)} + |\boldsymbol{v}_0|_{C^{2+a}(\Omega)} + |\,\boldsymbol{a}|_{C^{2+a,1+a/2}(\Sigma_T)} + \sum_{k=1}^{n} [\mathcal{R}_k(\boldsymbol{a}_t \cdot \boldsymbol{n})]^{(a/2)}_{t,\Sigma_T}\Big).\end{aligned}$$

Together with (3.3)-(3.4), this estimate implies (1.9).

4. - On the solvability of problem (1.1), (1.2).

We repeat here the arguments in [1], Sec. 5, with small modifications. At first we consider the case $\boldsymbol{a} = 0,\ \boldsymbol{v}_0 = 0$, moreover, we assume that

$$\boldsymbol{f}(x,0) = 0, \quad x \in \Omega, \tag{4.1}$$

and that the coefficients of the operator $\mathcal{A} = \nabla \cdot \ell$ are independent of t. Thus, we deal with the problem

$$\frac{\partial \boldsymbol{v}(x,t)}{\partial t} + \nabla \cdot \ell\Big(x, \frac{\partial}{\partial x}\Big)\boldsymbol{v}(x,t) + \nabla p(x,t) = \boldsymbol{f}(x,t),$$

$$\nabla \cdot \boldsymbol{v} = 0, \quad x \in \Omega, \quad t \in (0,T), \tag{4.2}$$

$$\boldsymbol{v}(x,0) = 0, \qquad \boldsymbol{v}(x,t)|_{x\in S} = 0 \tag{4.3}$$

with $\boldsymbol{f} \in C^{a,a/2}(Q_T)$ satisfying (1.8), (4.1).

We approximate $\boldsymbol{f}$ with divergence free vector fields $\boldsymbol{f}^R \in C^{a,a/2}(Q_T) \cap L_2(Q_T)$ with $\boldsymbol{f}^R \cdot \boldsymbol{n}|_{x\in S} = 0$, $R \gg 1$. They are defined by

$$\boldsymbol{f}^R(x,t) = \boldsymbol{f}(x,t)\zeta_R(x) + \boldsymbol{f}_R(x,t),$$

where $\zeta_R(x) = \zeta(x/R)$ is a standard cut-off function equal to one for $|x| \le R$ and to zero for $|x| \ge 2R$ and $\boldsymbol{f}_R$ is a solution of the problem

$$\nabla \cdot \boldsymbol{f}_R = -\boldsymbol{f} \cdot \nabla\zeta_R(x), \quad x \in K_R \equiv \{R \le |x| \le 2R\}, \quad \boldsymbol{f}_R|_{x\in\partial K_R} = 0$$

constructed in [15]. For $x \in R^n \setminus K_R$, we set $\boldsymbol{f}_R = 0$. It is known that $\boldsymbol{f}_R$ satisfies some estimates in the Hölder norms, in particular,

$$[\boldsymbol{f}_R]^{(a)}_{R^n} \le c|\boldsymbol{f}\zeta|_{C^a(K_R)}, \quad \sup_{K_R}|\boldsymbol{f}_R(x,t)| \le c\sup_{K_R}|\boldsymbol{f}(x,t)|$$

with constants independent of R (see [16, 17]). Hence,

$$|\boldsymbol{f}^R|_{C^{a,a/2}(Q_T)} \le c|\boldsymbol{f}|_{C^{a,a/2}(Q_T)}.$$

The problem

$$\frac{\partial \boldsymbol{v}^R(x,t)}{\partial t} + \mathcal{A}\Big(x, \frac{\partial}{\partial x}\Big)\boldsymbol{v}^R(x,t) + \nabla p^R(x,t) = \boldsymbol{f}^R(x,t),$$

$$\nabla \cdot \boldsymbol{v}^R = 0, \quad x \in \Omega, \quad t \in (0,T),$$

$$\boldsymbol{v}^R(x,0) = 0, \qquad \boldsymbol{v}^R(x,t)|_{x\in S} = 0 \tag{4.4}$$

has a unique generalized solution $\boldsymbol{v}^R \in L_2(Q_T)$ with $\boldsymbol{v}^R_{x_j}, \boldsymbol{v}^R_t \in L_2(Q_T)$ satisfying conditions (4.3) and the integral identity

$$\int_0^T\!\!\int_\Omega \boldsymbol{v}^R_t(x,t)\cdot\eta(x,t)\,dxdt - \int_0^T\!\!\int_\Omega \ell\Big(x,\frac{\partial}{\partial x}\Big)\,\boldsymbol{v}^R : \nabla\eta\,dxdt = \int_0^T\!\!\int_\Omega \boldsymbol{f}^R(x,t)\cdot\eta(x,t)\,dxdt$$

for arbitrary solenoidal $\eta \in L_2(Q_T)$ with $\eta_{x_j} \in L_2(Q_T)$, $\eta|_{x\in S} = 0$. The existence of the unique weak solution and the estimate

$$(4.5)\qquad \int_0^T\int_\Omega (|\boldsymbol{v}_t^R(x,t)|^2 + |\nabla \boldsymbol{v}^R(x,t)|^2 + |\boldsymbol{v}^R(x,t)|^2)dxdt \le c\int_0^T\int_\Omega |\boldsymbol{f}^R|^2 dxdt$$

is proved by Galerkin's method in a quite standard way on the basis of the Gårding inequality

$$-\int_\Omega \ell \boldsymbol{v}^R : \nabla \boldsymbol{v}^R dx \ge c_1\int_\Omega |\nabla \boldsymbol{v}^R(x,t)|^2 dx - c_2\int_\Omega |\boldsymbol{v}^R(x,t)|^2 dx,$$

that is a consequence of (1.3). Now, we can consider $\boldsymbol{v}^R$ as a weak solution of an elliptic problem

$$\mathcal{A}_0\Big(x,\frac{\partial}{\partial x}\Big)\boldsymbol{v}^R + \nabla p^R = \boldsymbol{f}^R - \boldsymbol{v}_t^R,\quad \nabla\cdot \boldsymbol{v}^R = 0,\quad x\in\Omega,\quad \boldsymbol{v}^R|_S = 0$$

depending on the parameter $t\in(0,T)$. By the regularity theorem for such problems (see [18]), there exist the second derivatives $\boldsymbol{v}^R_{x_jx_k}\in L_2(Q_T)$ whose norm can be estimated by the L_2- norms of $\boldsymbol{f}^R$, $\boldsymbol{v}^R$ and of the first derivatives of $\boldsymbol{v}^R$. Moreover, there exist the pressure function $p^R\in L_2(Q_T)$ with $\nabla p^R\in L_2(Q_T)$ such that (4.2) holds in a strong sense, and, along with (4.5), inequality

$$\sum_{|j|=2}|D^j\boldsymbol{v}^R|^2_{L_2(Q_T)} + |\nabla p^R|^2_{L_2(Q_T)} \le c|\boldsymbol{f}^R|^2_{L_2(Q_T)}$$

is satisfied.

Next, we show that $\boldsymbol{v}^R$ and p^R belong to the appropriate Hölder spaces. To this end, we approximate $\boldsymbol{f}^R$ with the vector fields

$$\boldsymbol{f}^R_\varepsilon(x,t) = \omega_\varepsilon(t)\star \boldsymbol{f}^R(x,t-2\varepsilon) = \int_{|\tau|\le\varepsilon}\omega_\varepsilon(\tau)\,\boldsymbol{f}^R(x,t-\tau-2\varepsilon)d\tau,$$

where ω_ε is a standard mollifying kernel and $\boldsymbol{f}^R(x,t)=0$ for $t<0$. The corresponding solutions of problem (4.2), (4.3) have the same structure:

$$\boldsymbol{v}^R_\varepsilon(x,t) = \omega_\varepsilon(t)\star\boldsymbol{v}^R(x,t-2\varepsilon),\quad p^R_\varepsilon(x,t) = \omega_\varepsilon(t)\star p^R(x,t-2\varepsilon),$$

$\boldsymbol{v}^R(x,t)=0$, $p^R(x,t)=0$ for $t<0$. It is clear that $\boldsymbol{f}^R_\varepsilon$, $\boldsymbol{v}^R_\varepsilon$, p_ε are infinitely differentiable with respect to t, in particular, $\boldsymbol{v}^R_\varepsilon$, $\boldsymbol{v}^R_{\varepsilon t}\in W^2_2(\Omega)$ for arbitrary fixed $t\in(0,T)$. By the embedding theorem, $\boldsymbol{v}^R_{\varepsilon t}\in L_q(\Omega)$ with $1/q\le 1/2-2/n$. By the regularity theorem for elliptic problems, $\boldsymbol{v}^R_\varepsilon\in W^2_q(\Omega)$, $\nabla p^R_\varepsilon\in L_q(\Omega)$. Since $D^m_t\boldsymbol{v}^R_\varepsilon\in L_2(\Omega)$ for arbitrary m, it follows that $\boldsymbol{v}^R_{\varepsilon t}\in L_{q_1}(\Omega)$ with $1/q_1<1/q-2/n\le 1/2-4/n$, hence, $\boldsymbol{v}^R_\varepsilon\in W^2_{q_1}(\Omega)$, $\nabla p^R_\varepsilon\in L_{q_1}(\Omega)$. Repeating these arguments, we increase the exponent q indefinitely, until we can conclude that $\boldsymbol{v}^R_{\varepsilon t}\in C^a(\Omega)$ and, as a consequence, $\boldsymbol{v}^R_\varepsilon\in C^{2+a}(\Omega)$, $\nabla p^R_\varepsilon\in C^a(\Omega)$. By (1.9),

$$|\boldsymbol{v}^R_\varepsilon|_{C^{2+a,1+a/2}(Q_T)} + |\nabla p^R_\varepsilon|_{C^{a,a/2}(Q_T)} \le c|\boldsymbol{f}^R_\varepsilon|_{C^{a,a/2}(Q_T)}.$$

Now, let $\varepsilon \to 0$. Since $\boldsymbol{f}^R(x,0)=0$, the zero extension of $\boldsymbol{f}^R$ into the domain $\{t<0\}$ belongs to the same class $C^{\alpha,\alpha/2}$. As $\varepsilon \to 0$, the vector fields $\boldsymbol{f}^R_\varepsilon$ approximate $\boldsymbol{f}^R$ in $C^{\alpha',\alpha'/2}(Q_T)$ with arbitrary $\alpha'<\alpha$ and the norms $|\boldsymbol{f}^R|_{C^{\alpha,\alpha/2}(Q_T)}$ remain uniformly bounded. Hence, $\boldsymbol{v}^R_\varepsilon \to \boldsymbol{v}^R$ in $C^{2+\alpha',1+\alpha'/2}(Q_T)$, $\nabla p^R_\varepsilon \to \nabla p^R$ in $C^{\alpha',\alpha'/2}(Q_T)$, and, moreover, $(\boldsymbol{v}^R, p^R)$, $\boldsymbol{v}^R \in C^{2+\alpha,1+\alpha/2}(Q_T)$, $\nabla p^R \in C^{\alpha,\alpha/2}(Q_T)$, is a solution of (4.2), (4.3). Finally, passing to the limit as $R\to\infty$, we obtain the solution $\boldsymbol{v}\in C^{2+\alpha,1+\alpha/2}(Q_T)$, $\nabla p \in C^{\alpha,\alpha/2}(Q_T)$ of problem (1.1), (1.2).

Let us prove the solvability of this problem with the operator $\mathcal{A}$ of a general form. It follows from (4.1) that

$$\boldsymbol{v}(x,0)=\boldsymbol{v}_t(x,0)=0, \quad x\in\Omega.$$

For arbitrary $\boldsymbol{v}\in C^{2+\alpha,1+\alpha/2}(Q_T)$ satisfying these conditions we have

$$\begin{aligned}
&\Big|\mathcal{A}\Big(x,t,\frac{\partial}{\partial x}\Big)\boldsymbol{v} - \mathcal{A}\Big(x,0,\frac{\partial}{\partial x}\Big)\boldsymbol{v}\Big|_{C^{\alpha,\alpha/2}(Q_{t'})} \le \\
&\quad \le c\Big(\max_{km,jq}\sup_{Q_{t'}}|\ell_{km,jq}(x,t)-\ell_{km,jq}(x,0)|\,|\boldsymbol{v}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})} + \\
&\quad + \max_{k,m,j,q}|\ell_{km,jq}|_{C^{\alpha,\alpha/2}(Q_{t'})}\sum_{|j|=2}\sup_{Q_{t'}}|D^j\boldsymbol{v}(x,t)| + |\boldsymbol{v}|_{C^{1+\alpha,1/2+\alpha/2}(Q_{t'})}\Big) \le c(t')^{\alpha/2}|\boldsymbol{v}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})},
\end{aligned}$$

$$\Big|P_G\nabla\cdot\Big(\ell\Big(x,t\frac{\partial}{\partial x}\boldsymbol{v}\Big)-\ell\Big(x,0,\frac{\partial}{\partial x}\Big)\boldsymbol{v}\Big)\Big|_{C^{\alpha,\alpha/2}(Q_{t'})} \le c(t')^{\alpha/2}|\boldsymbol{v}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})}.$$

which guarantees the solvability of problem (1.1), (1.2) in a small time interval. From this it follows that the solution exists for $t\in(0,T_0)$.

It remains to remove restriction (4.1). Let us consider problem (1.1), (1.2) with $\boldsymbol{v}_0=0$, $\boldsymbol{a}=0$ and with arbitrary $\boldsymbol{f}\in C^{\alpha,\alpha/2}(Q_T)$ satisfying only (1.8) and the compatibility condition (1.7), i.e., $\boldsymbol{f}(x,0)|_S=0$. We construct a divergence free extension of $\boldsymbol{f}$, $\boldsymbol{f}^*$, into the whole space R^n with a compact support, such that

$$|\boldsymbol{f}^*(\cdot,t)|_{C^\alpha(R^n)} \le c|\boldsymbol{f}(\cdot,t)|_{C^\alpha(\Omega)}, \quad \sup_{x\in R^n}|\boldsymbol{f}^*(x,t)| \le c\sup_{x\in\Omega}|\boldsymbol{f}(x,t)|,$$

and, as a consequence,

$$|\boldsymbol{f}^*|_{C^{\alpha,\alpha/2}(\Pi_T)} \le c|\boldsymbol{f}|_{C^{\alpha,\alpha/2}(Q_T)}$$

(see the details in [1]). Let $\boldsymbol{v}^*(x,t)$ be a solution of the Cauchy problem

$$\boldsymbol{v}^*_t(x,t) - \Delta\boldsymbol{v}^*(x,t) + \nabla p^*(x,t) = \boldsymbol{f}^*(x,t),$$

$$\nabla\cdot\boldsymbol{v}^* = 0, \quad x\in R^n, \quad t\in(0,T),$$

$$\boldsymbol{v}^*(x,0)=0,$$

i.e.

$$\boldsymbol{v}^*(x,t) = \int_0^t\int_{R^n}\Gamma(x-y,t-\tau)\,\boldsymbol{f}^*(y,\tau)\,dyd\tau,$$

where Γ is the fundamental solution of the heat equation, and $p^* = 0$. Further, we define $\boldsymbol{w}$ as a solution of the Stokes problem in Ω:

$$-\Delta \boldsymbol{w} + \nabla r = 0, \quad \nabla \cdot \boldsymbol{w} = 0, \quad \boldsymbol{w}|_S = -\boldsymbol{v}^*|_S.$$

Since $\boldsymbol{v}_t^*(x,0) = 0$ for $x \in S$, we have $\boldsymbol{w}(x,0) = \boldsymbol{w}_t(x,0) = 0$, $x \in \Omega$.

Both $\boldsymbol{v}^*$ and $\boldsymbol{w}$ belong to $C^{2+a,1+a/2}(Q_T)$, moreover, $P_G \boldsymbol{v}_t^* = P_G \Delta \boldsymbol{v}^* \in C^{a,a/2}(Q_T)$, and, as we have seen above,

$$|P_G \boldsymbol{w}_t|_{C^{a,a/2}(Q_T)} \leq c\Big(|\boldsymbol{v}_t^* \cdot \boldsymbol{n}|_{C^{a,a/2}(\Sigma_T)} + \sum_{k=1}^n [\mathcal{R}_k(\boldsymbol{v}_t^* \cdot \boldsymbol{n})]_{t,\Sigma_T}^{(a/2)}\Big).$$

Since $\boldsymbol{f} \cdot \boldsymbol{n}|_{x \in S} = 0$, we have $\mathcal{R}_k(\boldsymbol{v}_t^* \cdot \boldsymbol{n}) = \mathcal{R}_k(\Delta \boldsymbol{v}^* \cdot \boldsymbol{n})$. The function $\Delta \boldsymbol{v}^* \cdot \boldsymbol{n}$ can be written in the form

$$\Delta \boldsymbol{v}^* \cdot \boldsymbol{n} = \sum_{j,k=1}^n \Big(n_j \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial x_j}\Big) \frac{\partial v_j^*}{\partial x_k} = \mathrm{div}_S\, \boldsymbol{\Phi}$$

with

$$\boldsymbol{\Phi} = \sum_{j,k=1}^n (n_j \boldsymbol{e}_k - n_k \boldsymbol{e}_j) \frac{\partial v_j^*}{\partial x_k},$$

hence, by inequality (2.40) in [1],

$$[\mathcal{R}_k(\boldsymbol{v}_t^* \cdot \boldsymbol{n})]_{t,\Sigma_T}^{(a/2)} \leq c|\boldsymbol{v}^*|_{C^{2+a,1+a/2}(Q_T)}.$$

For the difference $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}^* - \boldsymbol{w}$ we obtain the problem

$$\boldsymbol{u}_t + \mathcal{A}\boldsymbol{u} + \nabla p = \boldsymbol{h}, \quad \nabla \cdot \boldsymbol{u} = 0, \qquad \boldsymbol{u}(x,0) = 0, \quad \boldsymbol{u}|_{x \in S} = 0,$$

where

$$\boldsymbol{h} = -\boldsymbol{w}_t - \mathcal{A}\boldsymbol{w} - \mathcal{A}\boldsymbol{v}^* - \Delta \boldsymbol{v}^* \in C^{a,a/2}(Q_T), \quad P_G \boldsymbol{h} \in C^{a,a/2}(Q_T).$$

In addition, $\boldsymbol{h}(x,0) = 0$. As was shown above, this problem has a solution $(\boldsymbol{u}, p)$, $\boldsymbol{u} \in C^{2+a,1+a/2}(Q_T)$, $\nabla p \in C^{a,a/2}(Q_T)$. The proof of Theorem 1.1 is now complete.

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