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Sharp Embeddings for Classes of Weights and Applications

ABSTRACT. — Sharp inequalities for G_r-constants when r varies in the exact range $[q, q+\epsilon[$ so that, a weight in the G_q-class of Gehring also belongs to G_r, are derived from previous work [12]. Analogous inequalities are deduced for the A_p-class of Muckenhoupt ([29]) and another more general system of weight classes ([35]). Applications related to the BMO space of John-Nirenberg are given.

Inclusioni ottimali per classi di pesi ed applicazioni

SUNTO. — A partire da un precedente lavoro [12] si individuano le migliori G_r-costanti per pesi appartenenti ad una data classe G_q di Gehring al variare di r nel rango esatto $[q, q+\epsilon[$. Analoghe questioni vengono trattate per le classi A_p di Muckenhoupt ed altri sistemi di classi di pesi ([29], [35]) in una dimensione. Alcune applicazioni connesse allo spazio BMO di John-Nirenberg.

1. - INTRODUCTION

A non negative measurable function ω on the space $\mathbb{R}^n$ satisfies the A_p-condition, $1 < p < \infty$ (for short: $\omega \in A_p$) if there exists a constant $A \geq 1$ such that, for any cube $Q \subset \mathbb{R}^n$ with edges parallel to the coordinate axes, one has

$$(1.1) \qquad \fint_Q \omega(x)dx \left(\fint_Q \omega^{\frac{-1}{p-1}}(x)dx \right)^{p-1} \leq A$$

where $\fint_Q v = \frac{1}{|Q|}\int_Q v$ denotes the mean value of v over Q. We call the smallest constant A the A_p-norm of ω and denote it by $A_p(\omega)$. The A_p-class was introduced in 1972 by B.

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Muckenhoupt [34] in connection with boundedness properties of the Hardy-Littlewood maximal operator M defined on the weighted space $L^p_{loc}(\mathbb{R}^n, \omega dx)$ by

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f(y)|\, dy. \tag{1.2}$$

An unexpected openess property holds for the A_p-classes, namely, if $\omega \in A_p$ then automatically $\omega \in A_s$ for some $s < p$.

Almost simultaneously (1973) another important class of weights was singled out by F.W. Gehring [18], the G_q-class, $1 < q < \infty$, in connection with the integrability properties of the gradient of quasiconformal mappings. Namely, a non negative weight v on $\mathbb{R}^n$ satisfies the G_q-condition (for short: $v \in G_q$) if there exists a constant $G \geq 1$ such that for all cubes $Q \subset \mathbb{R}^n$ as above

$$\left(\fint_Q v^q \right)^{\frac{1}{q}} \leq G \fint_Q v \tag{1.3}$$

We refer to (1.3) as a "reverse" Hölder inequality, and mention its "openess" property as well; namely if $v \in G_q$, then automatically $v \in G_r$ for some $r > q$.

The problem to determine exactly the supremum of exponents $r > q$ such that $v \in G_q \Rightarrow v \in G_r$ as a function of q and G_q-norm of v was first solved in [36], [12] for $n = 1$ and v a non increasing function. In [29] it was shown that in the case $n = 1$ it is enough to consider only monotone weights to find the extremal values of r, and the A_p-case was solved as well. See also [35]. In Section 2 we will report precisely on this elegant subject, which recently has been completely settled, in [41].

For $p = 1$ condition A_p has the form A_1:

$$\fint_Q \omega \leq A \ \operatorname*{ess\,inf}_{x \in Q} \omega(x) \tag{1.4}$$

while for $p = \infty$ we adopt for A_∞ the definition

$$\sup_Q \left\{ \fint_Q \omega \cdot \exp\left(\fint_Q \lg \frac{1}{\omega} \right) \right\} = A_\infty(\omega) < \infty \tag{1.5}$$

In the same spirit, we define $v \in G_\infty$ (see [15], [32]) by the condition

$$\operatorname*{ess\,sup}_Q \ v \leq B \fint_Q v \tag{1.6}$$

and $v \in G_1$ by

$$\sup_Q \exp \fint_Q \frac{v}{v_Q} \lg \frac{v}{v_Q} = G_1(v) < \infty \tag{1.7}$$

where $v_Q = \fint_Q v$.

It is evident that the classes $A_1,\ A_\infty,\ G_\infty,\ G_1$ are obtained by performing limits in the defining integral expressions for A_p and G_q, without taking into account the suprema appearing in the corresponding definitions. However in [39] it was proved that actually

$$A_\infty(\omega) = \lim_{p\to\infty} A_p(\omega) \tag{1.8}$$

and in [32] that

$$G_1(v) = \lim_{q\to 1} G_q(v) \tag{1.9}$$

where

$$G_q(v)^{\frac{1}{q'}} = \sup_Q \frac{\left(\fint_Q v^q\right)^{\frac{1}{q}}}{\fint_Q v} \tag{1.10}$$

$q' = \dfrac{q}{q-1}$.

These formulas give a quantitative version of the equalities

$$A_\infty = \bigcup_{p>1} A_p = \bigcup_{q>1} G_q = G_1.$$

In 1974 Coifman and Fefferman extended the theory to general singular integrals in $\mathbb{R}^n$ (see [10], [4], [40]).

Around 1980 there were new results in the theory of A_p-weights and its applications to elliptic PDE's: in [24] P. Jones proved an important result on factorization of A_p-weights which has been recently sharpened for weights with A_p-constant near 1 [28]; in [11] Dahlberg discovered various G_q-properties of harmonic measure on Lipschitz domains, (see [26] for a recent account of the theory of weights and boundary value problems for elliptic equations); in 1982 [13] Fabes, Kenig and Serapioni proved the Hölder - continuity of local solutions of degenerate elliptic equations in the weighted Sobolev space $W^{1,2}(\Omega, \omega)$ when $\omega \in A_2$ (see also [38] and [15]).

Another interesting issue is the relation between A_p-classes and the BMO space of John-Nirenberg. We will exploit this theme in the one dimensional case, where sharp results due to Korenovskii ([30], [31]) are also available (see Section 3).

2. - The exact continuation of reverse Hölder inequalities in one dimension

In this section we confine ourselves to the case $n = 1$.

The Muckenhoupt and Gehring classes A_p and G_q are generated by the reverse Hölder inequalities (1.1) and (1.3) respectively. For $1 < p,\ q < \infty$ we define the A_p-norm of the weight $\omega : [a,b] \to [0,\infty[$

$$A_p(\omega) = \sup_I \fint_I \omega \left(\fint_I \omega^{\frac{-1}{(p-1)}} \right)^{p-1} \tag{2.1}$$

and the G_q-norm of the weight $v : [a,b] \to [0,\infty)$

$$G_q(v) = \sup_I \left[\frac{\left(\fint_I v^q \right)^{\frac{1}{q}}}{\fint_I v} \right]^{q'}, \qquad q' = \frac{q}{q-1} \tag{2.2}$$

where the suprema are taken over all intervals $I \subset [a,b]$.

It is easy to check that if $q = p/(p-1)$, then we have

$$A_q(\omega) = [A_p(\omega^{-\frac{1}{q-1}})]^{q-1} \tag{2.3}$$

while it is obvious that for $1 \leq r \leq p$ we have

$$1 \leq A_p(\omega) \leq A_r(\omega) \leq A_1(\omega) \tag{2.4}$$

Various relations occurring among A_p and A_2 constants of weights and their powers are collected in the following

LEMMA 2.1: *For $p > 1$ we have*

$$[A_2(\omega^{\frac{1}{p-1}})]^{p-1} \leq A_p(\omega)\, A_p(\omega^{-1}) \tag{2.5}$$

For $1 < p \leq 2$ we have

$$A_p(\omega) \leq [A_2(\omega^{\frac{1}{p-1}})]^{p-1} \tag{2.6}$$

For $q > 1$ we have

$$A_2(\omega) \leq A_q(\omega)\, A_q(\omega^{-1}) \tag{2.7}$$

PROOF: For any cube Q, Hölder inequality implies

$$1 \leq \fint_Q \omega \, \fint_Q \omega^{-1}$$

hence

$$\left[\fint_Q \omega^{\frac{1}{p-1}} \fint_Q \omega^{-\frac{1}{p-1}} \right]^{p-1} \leq$$

$$\leq \fint_Q \omega \left(\fint_Q \omega^{-\frac{1}{p-1}} \right)^{p-1} \cdot \fint_Q \omega^{-1} \left(\fint_Q \omega^{\frac{1}{p-1}} \right)^{p-1} \leq A_p(\omega) A_p(\omega^{-1})$$

taking supremum with respect to all cubes Q we obtain (2.5).

Fix a cube Q and take p such that $1 < p \leq 2$; then we have $1 \leq \frac{1}{p-1}$ and Jensen inequality implies

$$\fint_Q \omega \leq \left(\fint_Q \omega^{\frac{1}{p-1}} \right)^{p-1}$$

hence

$$\fint_Q \omega \left(\fint_Q \omega^{-\frac{1}{p-1}} \right)^{p-1} \leq \left[\fint_Q \omega^{-\frac{1}{p-1}} \cdot \fint_Q \omega^{\frac{1}{p-1}} \right]^{p-1} \leq \left[A_2(\omega^{\frac{1}{p-1}}) \right]^{p-1}.$$

Taking supremum with respect to all cubes Q we obtain (2.6).

If $q > 1$ assume

$$A_q(\omega) A_q(\omega^{-1}) = A < \infty.$$

Using (2.3) we have $p = q/(q-1)$

$$A_p(\omega^{\frac{1}{q-1}}) A_p(\omega^{-\frac{1}{q-1}}) = A^{\frac{1}{q-1}}.$$

Replacing ω with $\omega^{\frac{1}{q-1}}$ in (2.5) we get

$$\left[A_2((\omega^{\frac{1}{q-1}})^{\frac{1}{p-1}}) \right]^{p-1} \leq A_p(\omega^{\frac{1}{q-1}})\, A_p(\omega^{-\frac{1}{q-1}})$$

But $(q-1)(p-1) = 1$, hence

$$A_2(\omega) \leq A$$

that is (2.7). □

Our aim is to describe the so-called "sharp self-improvement of exponents" property of the A_p and G_q classes.

The first result in this direction is due to [36], [12] and concerns non-increasing weights of the class G_q. Then in [29] it was shown that it is enough to consider only monotone functions to find the extremal values of the involved exponents and also the A_p-case was exploited.

We have the following theorems from [12] and [29].

THEOREM 2.1: *Let $q > 1$ and assume $v : [a, b] \to [0, \infty)$ satisfy the condition*

$$G_q(v) = G \ < \ \infty$$

Let $q_1 > q$ be the unique solution to the equation

$$\varphi(x) = 1 - G^{q-1}\frac{(x-q)}{x}\left(\frac{x}{x-1}\right)^q = 0. \tag{2.8}$$

Then, for $q \leq \sigma < q_1$ we have

$$[G_\sigma(v)]\frac{1}{\sigma'} \leq G^{\frac{1}{q'}}\left[\frac{q}{\sigma\varphi(\sigma)}\right]^{\frac{1}{q}}$$

The result is sharp.

THEOREM 2.2: *Let $p > 1$ and assume $\omega : [a, b] \to [0, \infty)$ satisfy the condition*

$$A_p(\omega) = A \ < \ \infty$$

Let $1 < p_1 < p$ be the unique solution to the equation

$$\psi(x) = 1 - A^{\frac{1}{p-1}}\frac{p-x}{(p-1)}x^{\frac{1}{p-1}} = 0 \tag{2.9}$$

Then, for $p_1 < \rho \leq p$ we have

$$A_\rho(\omega) \leq A\left[\frac{\rho-1}{(p-1)\psi(\rho)}\right]^{p-1}$$

The result is sharp.

REMARK 2.1: Notice that if we perform in (2.8) the transform $t \longrightarrow \dfrac{t}{t-1}$ on the variable x and on the exponent q, then φ turns into ψ, for $A = G$.

It is worth noting that in the special case $p = q = 2$ we have explicit values of q_1 and p_1 and the above theorems enjoy a simpler presentation.

COROLLARY 2.1: *If $G_2(v) = G < \infty$, then for $2 \leq r < 1 + \sqrt{\dfrac{G}{G-1}}$*

$$[G_r(v)]^2 \leq \frac{2G(r-1)^2}{r[(r-1)^2 - Gr(r-2)]}$$

COROLLARY 2.2: *If $A_2(\omega) = A < \infty$, then for $1 + \sqrt{\frac{A-1}{A}} < s \leq 2$*

$$A_s(\omega) \leq \frac{A(s-1)}{1 - As(2-s)}$$

A more general class of weights, including both the A_p and G_q classes was proposed by B. Bojarski [6] and I. Wik [44]. We confine here ourselves to case of dimension one [35].

Let $r, s \in \mathbb{R} - \{0\}$ with $r < s$ and let $u : [a, b] \longrightarrow [0, \infty)$ be a weight. Define the B_r^s-norm of u as follows

$$B_r^s(u) = \sup_I \frac{\left(\fint_I u^s \right)^{1/s}}{\left(\fint_I u^r \right)^{1/r}}$$

The B_r^s-class consists of all weights u such that $B_r^s(u) < \infty$. It is immediate to check that

$$B_1^q(u) = G_q(u)^{1/q'}$$

$$B^1_{-\frac{1}{p-1}}(u) = A_p(u)$$

hence A_p and G_q classes are included in the system of B_r^s classes. To state an exact continuation theorem in B_r^s-classes, in view of the optimal integrability results established in [35], let us introduce two auxiliary functions on $[0, 1]$. Given $B \geq 1$, if $0 < r < s$ let us define

$$\varphi(y) = 1 - B^s(1 - y)\left(\frac{s}{s - ry} \right)^{s/r} \qquad y \in [0, 1]$$

while, if $r < 0 < s$ we define

$$\chi(y) = 1 - B^{-r}(1 - y)\left(\frac{r}{r - sy} \right)^{r/s} \qquad y \in [0, 1]$$

THEOREM 2.3: *Assume the weight* $u : [a, b] \longrightarrow [0, \infty)$ *satisfy the condition*

$$B_r^s(u) = B < \infty$$

and let x_0 *be the unique solution to the equation*

$$\left(\frac{x}{x - s} \right)^{1/s} = B\left(\frac{x}{x - r} \right)^{1/r}$$

Then we have:

1. *if* $0 < r < s$,

$$B_r^\sigma(u) \leq B\left[\frac{s}{\sigma\varphi(\frac{s}{\sigma})} \right]^{1/s}$$

for $s \leq \sigma < x_0$ $\left(\textit{and } \varphi\left(\frac{s}{x_0}\right) = 0\right)$.

2. *if* $r < 0 < s$,

$$B_\rho^s(u) \leq B\left[\frac{r}{\rho\chi(\frac{r}{\rho})} \right]^{-1/r}$$

for $x_0 < \rho \leq r$ $\left(\textit{and } \chi\left(\frac{r}{x_0}\right) = 0\right)$.

The proof follows through suitable calculations from the proof of Theorem 1.3 in [35]. The main idea in [12], [35], [36] is to deal with some Hardy inequality and we will illustrate it to prove the forthcoming Theorem 2.6.

Up to now we have been dealing with the self-improvement of exponents p and q in A_p and G_q classes respectively or of exponents r and s in B_r^s classes.

Now we consider the problem of the exact G_q-class pertaining to all A_p-weights. This was solved for $p = 1$ in [7] and for $p > 1$ has been recently settled by Vasyunin [41], who found the exact range of exponents q so that a weight in the A_p-class belongs to the G_q-class. Let us first state the main result in [7].

THEOREM 2.4: *Let w belong to the A_1-class with $A_1(w) = A$. Then for every* $1 \leq q < \dfrac{A}{A-1}$

$$[G_q(w)]^{q-1} \leq \frac{1}{A^{q-1}(A + q - qA)} \tag{2.10}$$

The constant on the right hand side as well as the upper bound of q cannot be improved. In fact, the weight $w(t) = \dfrac{t^{\frac{1}{(A-1)}}}{A}$ is an extremal, which gives equality in (2.10) and lies in L^q if and only if $q < \dfrac{A}{A-1}$.

In order to state the result from [41], we fix $p > 1$ and $\delta > 1$ and denote by $x = x(p, \delta)$ the positive solution to the equation

$$(1-x)(1-x/p)^{-p} = \frac{1}{\delta}.$$

Then $0 < x \leq 1$ and we put

$$p^* = p^*(p, \delta) = \frac{p-x}{x(p-1)}$$

we have the following.

THEOREM 2.5: *Suppose that a weight ω belongs to A_p and let $A = A_p(\omega)$. Then ω belongs to G_q for each $1 \leq q < p^*(p, A)$. The bound for q is optimal.*

Our aim here is to give a simple proof of previous theorem in a special case.

THEOREM 2.6: *Suppose that a non-decreasing weight $\omega : [a, b] \to [0, \infty)$ belongs to A_2 and $A = A_2(\omega)$. Then for $1 \leq q < \sqrt{\dfrac{A}{A-1}}$, ω^{-1} belongs to G_q and for any $[c, d] \subset [a, b]$*

$$\left(\fint_c^d \omega^{-q}\, dx \right)^{1/q} \leq \frac{q}{A - q^2(A-1)} \fint_c^d \omega^{-1}\, dx \tag{2.11}$$

The result is sharp.

Before proving Theorem 2.6 we state an useful Lemma ([29],[36]).

LEMMA 2.2: *Let ω be a non-decreasing function in $[a,b]$ and $0<\alpha<1$. Then*

$$\left(\int_a^b \omega^{-1/\alpha}\,dx\right)^{\alpha} \leq \alpha \int_a^b (x-a)^{\alpha-1}\omega^{-1}\,dx.$$

PROOF: *(of Theorem 2.6)* We prove our theorem by the same method first adopted in [36] for reverse Hölder inequalities of G_2 type.

Let us define for $0<\alpha<1$

$$\gamma(\alpha)=1-A(1-\alpha^2)$$

and note that $\gamma\left(\sqrt{\frac{A-1}{A}}\right)=0$, $\gamma(\alpha)>0$ for $\alpha>\sqrt{\frac{A-1}{A}}$.

Let us prove that, for any $c<d$

$$\int_c^d (x-c)^{\alpha-1}\omega^{-1} \leq \frac{(d-c)^{\alpha-1}}{\gamma(\alpha)}\int_c^d \omega^{-1} \tag{2.12}$$

for $\alpha>\sqrt{\frac{A-1}{A}}$.

By Fubini's theorem and our assumption on ω, we have

$$\frac{1}{\alpha-1}\left[(d-c)^{\alpha-1}\int_c^d \omega^{-1} - \int_c^d (x-c)^{\alpha-1}\omega^{-1}\right] = \tag{2.13}$$

$$= \int_c^d (x-c)^{\alpha-1} \fint_c^x \omega^{-1} \leq A\int_c^d (x-c)^{\alpha-1}\left(\fint_c^x \omega\right)^{-1}$$

We invoke now the weighted Hardy's inequality [29]

$$\int_c^d (x-c)^{\alpha-1}\left(\fint_c^x \omega\right)^{-1} \leq (1+\alpha)\int_c^d (x-c)^{\alpha-1}\omega^{-1}(x)$$

which enables us to deduce by (2.13) that

$$\frac{1}{\alpha-1}(d-c)^{\alpha-1}\int_c^d \omega^{-1} \leq \left[\frac{1}{\alpha-1}+A(1+\alpha)\right]\int_c^d (x-c)^{\alpha-1}\omega^{-1}(x).$$

Hence

$$[1-A(1-\alpha^2)]\int_c^d (x-c)^{\alpha-1}\omega^{-1}(x) \leq (d-c)^{\alpha-1}\int_c^d \omega^{-1}$$

which is (2.12).

Next, we combine (2.12) with Lemma 2.2 obtaining

$$\frac{1}{a}\left(\int_c^d \omega^{-1/a}\, dx\right)^a \leq \frac{(d-c)^{(a-1)}}{\gamma(a)}\int_c^d \omega^{-1} dx$$

hence

$$\left(⨍_c^d \omega^{-1/a}\, dx\right)^a \leq \frac{a}{1-A+Aa^2}⨍_c^d \omega^{-1} dx$$

and (2.11) follows, for $\sqrt{\frac{A-1}{A}} < a < 1$. □

Another point of view concerns the improvement of power exponents pertaining to A_2 weights. Namely, assume that the weights ω belongs to A_2 and set $A = A_2(\omega)$.

Then, it is easy to check that

$$A_2(\omega^\theta) \leq A^\theta \qquad \text{for } 0 \leq \theta \leq 1$$

Passing to exponents $\tau > 1$ is possible, as a consequence of Muckenhoupt's work, as we have already seen in this section.

Precisely, we have the following theorem.

THEOREM 2.7: *Assume* $A_2(\omega) = A < \infty$, *then for* $1 \leq \tau < \sqrt{\frac{A}{A-1}}$ *we have* $\omega^\tau \in A_2$ *and*

$$A_2(\omega^\tau)^{\frac{1}{2\tau}} \leq \frac{\tau A}{A-\tau^2(A-1)}$$

The upper bound on τ *cannot be improved.*

Proof [2].

3. - The space BMO and its interplay with the A_2-class

The space BMO -functions of bounded mean oscillation- was introduced in 1961 by John and Nirenberg [23]. A real valued locally integrable function f on $\mathbb{R}^n$ has bounded mean oscillation, $f \in BMO(\mathbb{R}^n)$ if

$$\sup_Q ⨍_Q |f - f_Q| dx = \|f\|_* < \infty$$

where

$$f_Q = \frac{1}{|Q|}\int_Q f\, dx = ⨍_Q f\, dx$$

is the mean of f over Q. $L^\infty(\mathbb{R}^n)$ is contained in BMO($\mathbb{R}^n$) and constant functions have zero BMO-norm. Notice that the function $\log|x|$ on $\mathbb{R}$ is in BMO but it is not bounded.

A substantial result in the theory of BMO is the John and Nirenberg inequality that covers the following remarkable fact: the L^1-inequalities that define BMO functions imply similar L^p-inequalities for every $1 < p < \infty$, and in fact imply also exponential bounds.

THEOREM 3.1: ([40], [4]) *Suppose that f is in BMO. Then*

(a) *For any* $1 < p < \infty$, $f \in L^p_{loc}$ *and*

$$\sup_Q \fint_Q |f - f_Q|^p dx \le c_p \|f\|_*^p$$

(b) *There exist positive constants c_1 and c_2 such that, for every $\lambda > 0$ and any cube Q*

$$\frac{1}{|Q|} |\{x \in Q : |f(x) - f_Q| > \lambda\}| \le c_1 e^{-\lambda \frac{c_2}{\|f\|_*}}$$

Notice that (b) corresponds to the celebrated John and Nirenberg theorem [23].

COROLLARY 3.1: *Let $f \in L^1_{loc}(\mathbb{R}^n)$ verify condition (b) of* Theorem 3.1. *Then, for* $\lambda > \frac{\|f\|_*}{c_2}$ *and for any cube Q*

$$\fint_Q e^{\frac{|f(x) - f_Q|}{\lambda}} dx \le \frac{c_1}{(c_2\, \lambda / \|f\|_*)\; -\; 1} \tag{3.1}$$

The preceding results can be precisely reformulated in case of dimension $n = 1$ [30].

Actually A. Korenovskii proved that optimal constants c_1 and c_2 in (b) of Theorem 3.1 and in Corollary 3.1 are as follows

$$c_1 = e^{1+\frac{2}{e}} \qquad c_2 = \frac{2}{e}. \tag{3.2}$$

In [17] Garnett and Jones gave upper and lower bounds for the distance

$$dist_{BMO}(f, L^\infty) = \inf_{g \in L^\infty} \|f - g\|_*$$

by mean of the quantity

$$\varepsilon(f) = \inf\{\lambda > 0 : \sup_Q \fint_Q e^{\frac{|f - f_Q|}{\lambda}} dx < \infty\}. \tag{3.3}$$

THEOREM 3.2: ([17]) *If $f \in L^1_{loc}(\mathbb{R}^n)$ then*

$$k_1\, \varepsilon(f) \le dist_{BMO}(f, L^\infty) \le k_2\, \varepsilon(f) \tag{3.4}$$

where k_i are constants depending only on the dimension.

We now digress for a second to mention a parallel result to Theorem 3.2 in which BMO is replaced by EXP, the space of exponentially integrable functions.

Let Ω be a measurable set with finite measure $|\Omega|$. We denote by EXP=EXP(Ω) the set of functions $g : \Omega \longrightarrow \mathbb{R}$ such that there exists $\lambda > 0$ for which

$$\fint_{\Omega} e^{|g|/\lambda}\, dx < \infty$$

equipped with the norm

$$\|g\|_{EXP(\Omega)} = \inf\left\{ \lambda > 0 : \fint_{\Omega} e^{|g|/\lambda}\, dx \leq 2 \right\}. \tag{3.5}$$

THEOREM 3.3: ([16], [9]) *For every $g \in EXP(\Omega)$ we have*

$$dist_{EXP}(g, L^{\infty}) = \inf\left\{ \lambda > 0 : \fint_{\Omega} e^{|g|/\lambda}\, dx < \infty \right\}$$

where the distance is evalued with respect to norm (3.5).

In the following, restricting ourselves to the one dimensional case, we will take advantage of the results of the previous section to estimate the constant k_1 in Theorem 3.2.

We will essentially describe some results from [2]. The preceding results imply that a locally integrable function f belongs to BMO if and only if there exists $\lambda > 0$ such that

$$s(f, \lambda) = \sup_{Q} \fint_{Q} e^{|f - f_Q|/\lambda}\, dx < \infty. \tag{3.6}$$

The following Proposition, whose proof is contained in [40] (Prop. 6.1 of ch IX; see also [17]), illustrates the connection among A_2 and BMO.

PROPOSITION 3.1: *A locally integrable function f belongs to BMO if and only if there exists $\lambda > 0$ such that $A_2(e^{f/\lambda}) < \infty$. Actually, for any $\lambda > 0$ the following inequalities hold:*

$$\frac{1}{2}\, s(f, \lambda) \leq A_2(e^{f/\lambda}) \leq s(f, \lambda)^2 \tag{3.7}$$

where $s(f, \lambda)$ is defined by (3.6).

Let us now restrict ourselves to the one-dimensional case.

For $f \in L^1_{loc}$, consider the set

$$I_f = \{\lambda > 0 : A_2(e^{f/\lambda}) < \infty\} \tag{3.8}$$

and let us describe its properties.

PROPOSITION 3.2: *The function f belongs to BMO($\mathbb{R}$) if and only if I_f is not empty. If we set*

$$\epsilon(f) = \inf I_f$$

then

$$I_f =]\epsilon(f), \infty[\tag{3.9}$$

and

$$\epsilon(f) \leq \frac{e}{2}\|f\|_*. \tag{3.10}$$

PROOF: From Proposition 3.1 we known that the condition $A_2(e^{f/\lambda}) < \infty$ is equivalent to

$$\sup_I \fint_I e^{\frac{|f-f_I|}{\lambda}}\, dx \ < \infty.$$

Therefore, it is obvious that

$$\lambda_0 \in I_f, \quad \lambda_1 > \lambda_0 \implies \lambda_1 \in I_f.$$

Moreover, due to Theorem 2.7 the set I_f does not contain its infimum $\epsilon(f)$. This means that (3.9) holds true. To establish (3.10) we invoke Corollary 3.1 with constants c_i given by (3.2), obtaining $\lambda \in I_f$ for any $\lambda > \frac{e}{2}\|f\|_*$. □

COROLLARY 3.2: *For $n = 1$, the constant k_1 in Theorem 3.2 can be chosen equal to $2/e$.*

PROOF: It suffices to notice that for any $g \in L^\infty$, by (3.10)

$$\epsilon(f) = \epsilon(f - g) \leq \frac{e}{2}\|f - g\|_*$$ □

In the one dimensional case it is possible to get another expression of the functional $\epsilon(f)$ in which A_2 constants appear. In [2] the following theorem is proved.

THEOREM 3.4: *For any $f \in BMO$*

$$\epsilon(f) = \inf\left\{\lambda\sqrt{\frac{A_2(e^{f/\lambda}) - 1}{A_2(e^{f/\lambda})}} : \lambda \in I_f\right\}$$

PROOF: (see [2]). □

4. - SHARP WEAK-TYPE INEQUALITIES FOR THE MAXIMAL OPERATOR ON $L^p(\mathbb{R}, wdx)$

Let Mf denote the (uncentered) maximal function of $f \in L^1_{loc}(\mathbb{R}^n, \omega dx)$ defined by

$$Mf(x) = \sup_{I \ni x} \fint_I |f(y)|dy. \tag{4.1}$$

It is well known that M is a weak-type operator on $L^p(\mathbb{R}^n, \omega dx)$, $p \geq 1$, if ω is an A_p weight [34]. Moreover in [8] the following precise inequality was established

$$\int_{\{Mf>\lambda\}} \omega \leq c(p,n)\frac{A_p(\omega)}{\lambda^p}|f|^p_{L^p(\mathbb{R}^n,\,\omega)} \tag{4.2}$$

where $c(p,n)$ is a positive constant, depending only on $p \geq 1$ and $n \in \mathbb{N}$. Here we set

$$|g|_{L^p(\mathbb{R}^n,\omega)} = \left(\int_{\mathbb{R}^n} |g|^p \omega dx\right)^{\frac{1}{p}}$$

and the dependance on $A_p(\omega)$ is the best possible. In [1] the following sharp result in one dimension was recently established.

THEOREM 4.1: *The best constant $c(p,1)$ in the inequality* (1.4) *is* $c(p,1) = 2$.

In the special case $\omega(x) = 1$ we reobtain a sharp result due to [5], [21] and [20]. In the same spirit of [20] we actually have the following weighted double weak type inequality (see [1]).

THEOREM 4.2: *Let ω be a A_p weight on $\mathbb{R}$, $1 \leq p < \infty$, and assume that $f : \mathbb{R} \to [0,\infty)$ belongs to $L^p_{loc}(\mathbb{R},\omega)$. Then we have, $\forall\lambda > 0$:*

$$\int_{\{Mf>\lambda\}} \omega(x)dx + \int_{\{f>\lambda\}} \omega(x)dx \leq \tag{4.3}$$

$$\leq \frac{2A_p(\omega)-1}{\lambda^p}\int_{\{Mf>\lambda\}} f^p\omega(x)dx + \frac{1}{\lambda^p}\int_{\{f>\lambda\}} f^p\omega(x)dx.$$

The inequality is sharp.

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