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Uncertainty Inequalities on Spaces with Polynomial Volume Growth

0. - Introduction

The common feature of uncertainty inequalities is the statement that a non-zero function and its Fourier transform cannot be simultaneously concentrated on small sets or too close to given points. In the classical Heisenberg-Pauli-Weyl inequality (HPW in short), the "concentration" of a function f on $\mathbb{R}^n$ near a point x_0 is measured by the variance of $|f|^2$,

$$V(|f|^2, x_0) = \int_{\mathbb{R}^n} |x - x_0|^2 \, |f(x)|^2 \, dx,$$

and the inequality says that, for every pair of points $x_0, \xi_0 \in \mathbb{R}^n$,

$$V(|f|^2, x_0)\, V(|\hat{f}|^2, \xi_0) \geq C \|f\|_2^4 \tag{0.1}$$

(where the constant C depends only on the precise definition of $\hat{f}$).

There is large varieties of "uncertainty inequalities" and of qualitative "uncertainty principles", as well as of extensions and generalizations to other contexts. We refer the reader to [FS] for a survey on this subject. Our interest here is in an equivalent formulation of HPW, obtained by applying the Plancherel formula to (0.1). Taking $x_0 = \xi_0 = 0$ for simplicity, (0.1) can be rewritten as

$$\Big(\int_{\mathbb{R}^n} |x|^2 \, |f(x)|^2 \, dx\Big)\Big(\int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx\Big) \geq C \|f\|_2^4.$$

More generally, the following inequality holds,

$$\|f\|_2^2 \leq C_{\alpha,\beta}\Big(\int_{\mathbb{R}^n} |x|^{2\alpha} \, |f(x)|^2 \, dx\Big)^{\frac{\beta}{\alpha+\beta}}\Big(\int_{\mathbb{R}^n} \big|(-\Delta)^{\frac{\beta}{2}} f(x)\big|^2 \, dx\Big)^{\frac{\alpha}{\alpha+\beta}}, \tag{0.2}$$

for every $\alpha, \beta > 0$.

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This inequality does not involve the Fourier transform, and the leading rôle is played instead by the gradient operator or by the Laplacian. Once HPW is put into this "differential" form, its validity can be discussed in more general contexts than $\mathbb{R}^n$, as soon as reasonably related analogues of the norm function and of the Laplacian are introduced.

When HPW is taken in the form (0.2), the comparison is made between the "spatial concentration" of f near the origin, measured by the first factor in the l.h.s. of (0.2), and the "spectral concentration" of f near the point 0 in the spectrum of Δ, measured by the second factor. For instance, assume that $\|f\|_2 = 1$ and that $f = E_{[0,\lambda]}\, f$, where E denotes the spectral measure of $-\Delta$. Then

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{a}{2}} f(x)|^2\, dx \le \lambda^a,$$

and (0.2) implies that

$$\int_{\mathbb{R}^n} |x|^{2a} |f(x)|^2\, dx \ge C_a \lambda^{-a}.$$

So the smaller is λ (i.e. f has higher spectral concentration near 0), the larger is the variance of $|f|^2$ (i.e. f has smaller spatial concentration).

What we discuss below is based on recent work by P. Ciatti, M. Sundari and the author [CRS], motivated by extensions of HPW to the setting of Lie groups with polynomial growth, with a left-invariant Carnot-Carathéodory distance replacing the euclidean norm and the corresponding sub-Laplacian replacing Δ. The crucial assumption that is used is the existence of polynomial bounds for the heat kernel associated to the sub-Laplacian.

In this note we develop some arguments that appear in [CRS] in a wider context than that of Lie groups, assuming the existence of certain polynomial bounds on a given hypercontractive semigroup.

Our interest, is focused not so much on HPW, as in [CRS], but rather on the extension of a few of related inequalities, some of which play a rôle in the proof of HPW in [CRS].

To begin with, we present these inequalities in the classical context.

The first inequality is not explicitely an uncertainty inequality, but a weighted L^2-estimate for the heat semigroup $e^{t\Delta}$. It says that, if $0 < a < \frac{n}{2}$, $x_0 \in \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$, then

$$\|e^{t\Delta} f\|_2 \le C_a t^{-a} \big\| |x - x_0|^a f \big\|_2;$$

Inequalities of this type will give us the link between the estimates that we assume on the semigroup and the desired uncertainty inequalities.

The second inequality is the "local uncertainty inequality" (see [Fa]), stating that if $f \in L^2(\mathbb{R}^n)$, $A \subset \mathbb{R}^n$ is measurable and $0 \le a < \frac{n}{2}$, then

$$\left(\int_A |\hat{f}(\xi)|^2 \, d\xi\right)^{\frac{1}{2}} \le C_a |A|^{\frac{a}{n}} \big\| |x|^a f \big\|_2 . \tag{0.3}$$

Replacing f by $\hat{f}$ and using Plancherel's formula, this inequality becomes

$$\left(\int_A |f(x)|^2 \, dx\right)^{\frac{1}{2}} \le C_a |A|^{\frac{a}{n}} \big\| (-\Delta)^{\frac{a}{2}} f \big\|_2 \tag{0.3'}$$

an equivalent formulation of the Sobolev embedding of $H^a(\mathbb{R}^n)$ into the weak-L^p space $L^p_w(\mathbb{R}^n)$ for $\frac{1}{p} = \frac{1}{2} - \frac{a}{n}$ [KJF].

The last inequality, whose extensions will be briefly discussed at the end, is related to the work of Landau, Slepian and Pollak on approximate concentration of both a function and its Fourier transform (see [S] and the references therein), and it appears in [DS]. It says that, for any pair of measurable subsets A, W of $\mathbb{R}^n$,

$$\left(\int_W |\widehat{f\chi_A}(\xi)|^2 \, d\xi\right)^{\frac{1}{2}} \le \sqrt{|A|\,|W|}\, \|f\|_2 . \tag{0.4}$$

In particular, if f is supported on A and the product $|A||W|$ is small, only a small portion of $\hat{f}$ can be supported on W.

We shall place ourselves in the following context. We considers a locally compact space X endowed with a positive Borel measure m and:

(i) a positive self-adjoint operator T on $L^2(X)$ generating a hypercontractive semigroup

$$e^{-tT} f(x) = \int_X p_t(x, y)\, dm(y),$$

with $p_t(x, \cdot) \in L^1(X)$ for every $x \in X$ and satisfying the estimate (1)

$$\|e^{-tT}\|_{1\to\infty} \lesssim \begin{cases} t^{-q_0} & \text{for } t \to 0 \\ t^{-q_\infty} & \text{for } t \to \infty \end{cases} . \tag{0.5}$$

for some $q_0, q_\infty > 0$;

(ii) a non-negative continuous function $\rho(x,y)$ on $X \times X$ such that

$$\rho(x, y) = 0 \Leftrightarrow x = y , \qquad \rho(x, y) = \rho(y, x), \tag{0.6}$$

(1) Notice that the following quantities are all equal:

$$\|e^{-tT}\|_{1\to\infty} , \ \|e^{-\frac{t}{2}T}\|^2_{1\to 2} , \ \|e^{-\frac{t}{2}T}\|^2_{2\to\infty} \sup_{x\in X} \|p_{\frac{t}{2}}(x, \cdot)\|_2^2 , \ \sup_{x\in X} p_t(x, x) , \ \sup_{x, y\in X} |p_t(x, y)|.$$

and with the property that the "balls" $B(x,r) = \{ y : \rho(x, y) < r\}$ satisfy the volume growth conditions

$$m\big(B(x,r)\big) \lesssim \begin{cases} r^{q_0} & \text{for } r \to 0 \\ r^{q_\infty} & \text{for } r \to \infty \end{cases} \tag{0.7}$$

uniformly in x.

These assumptions are satisfied, with $q_0 = q_\infty = n$, if $X = \mathbb{R}^n$, $T = \sqrt{-\Delta}$ and ρ is the euclidean distance. For $T = -\Delta$, they are satisfied with $q_0 = q_\infty = \frac{n}{2}$ and ρ equal to the square of the euclidean distance. In the first case, $p_t(x,y) = c_n\, t(t^2 + |x-y|^2)^{-\frac{n+1}{2}}$ is the Poisson kernel for the upper half-space in $\mathbb{R}^{n+1}$; in the second case, $p_t(x,y) = c'_n\, t^{-\frac{n}{2}}\, e^{-\frac{|x-y|^2}{4t}}$ is the heat kernel in $\mathbb{R}^n$.

Other concrete examples are:

(1) $X = \mathbb{R}^n$, $T = P(i\partial)$ a self-adjoint hypoelliptic differential operator, with P a polynomial such that $P(\xi) \sim |\xi|^{2m} + |\xi|^{2m'}$, $0 < m \leq m'$; then (0.5) holds with $q_0 = \frac{n}{2m'}$ and $q_\infty = \frac{n}{2m}$, and (0.7) holds with $\rho(x,y) = P\big(|x-y|^{-1}\big)^{-2}$;

(2) as in [CRS], X a Lie group with polynomial volume growth, m being the Haar measure, $\rho = d^2$, where d is the Carnot-Carathéodory distance induced by a system $\{X_1, \cdots, X_m\}$ of left-invariant vector fields generating the full Lie algebra, and $T = -\sum_{j=1}^m X_j^2$ being the corresponding hypoelliptic sub-Laplacian;

(3) Laplace-Beltrami operators T on Riemannian manifolds X satisfying the above assumptions; e.g. X a complete noncompact Riemannian manifold of nonnegative Ricci curvature, and ρ equal to the square of the Riemannian distance (see [LY, C, G]);

(4) transition semigroups for Brownian motions on unbounded self-similar fractals (like the infinite Sierpinski gasket) of Hausdorff dimension α, m being the α-Hausdorff measure, ρ the intrinsic distance raised to the "walk dimension" β, and $q_0 = q_\infty = \frac{\alpha}{\beta}$ [BP, B];

(5) by subordination, any of the above, with T replaced by T^δ and ρ by ρ^δ, $0 < \delta < 1$.

Notice that in all these examples, the function ρ is a power of a distance naturally related to the operator T (2).

The inequalities we shall discuss are instances of the principle that, under the above assumptions, there are quantitative obstructions to simultaneous spatial concentration of a function f on X and its spectral concentration, measured by quantities like the L^2-norm of $E(\omega)f$ for $\omega \subset [0,\infty)$, or of $T^a f$ for $a > 0$.

(2) In general, as soon as X has a distance d, it is possible to construct from it a function ρ satisfying (0.6) and (0.7).

From this point of view, the analogue of HPW – in the differential form (0.2) – is as follows.

THEOREM 0 [CRS]: *Suppose that X, m, ρ, T satisfy the assumptions (i)-(iii) above. For every $\alpha, \beta > 0$ there is a constant $C_{\alpha,\beta}$ such that, if $f \in L^2(X)$ and $x_0 \in X$,*

$$\|f\|_2^2 \le C_{\alpha,\beta}\left(\int\limits_X \left|\rho(x,x_0)^\alpha f(x)\right|^2 dm(x)\right)^{\frac{\beta}{\alpha+\beta}}\left(\int\limits_X \left|T^\alpha f(x)\right|^2 dm(x)\right)^{\frac{\alpha}{\alpha+\beta}}.$$

Notice that, for $X = \mathbb{R}^n, T = \sqrt{-\Delta}$ and ρ the Euclidean distance, the Fourier transform interchanges T with $\rho(x, 0) = |x|$, a fact that is used in deriving (0.3') from (0.3). In the general case, the absence of an involutive Fourier transform leads us to discuss the existence of pairs of "companion" inequalities, one formally obtainable from the other by interchanging the rôles of T and ρ (from this point of view, HPW is self-companion, being formally symmetric in T and ρ). Clearly, the two inequalities in a pair will require in general independent proofs. We find the existence of such pairs of inequalities interesting because it recaptures to some extent the symmetry that is intrinsic in the Fourier transform in $\mathbb{R}^n$.

1. - L^2-ESTIMATES WITH WEIGHTS

The proof of HPW in [CRS] is based on the following estimate, and the proof given there can be easily adapted to our general setting. We give it here for completeness and as an introcution to the proof of the following Theorem 1'. We set

$$q = \min\{q_0, q_\infty\}.$$

THEOREM 1: *Suppose that X, m, ρ, T satisfy the assumptions (i)-(iii) in the Introduction and let $\alpha > 0$. The inequality*

$$\|e^{-tT} f\|_2 \le C_\alpha t^{-\alpha} \|\rho(\cdot, x_0)^\alpha f\|_2, \qquad (t > 0) \tag{1.1}$$

holds for every $f \in L^2(X)$ and $x_0 \in X$, with a constant C_α depending only on α,

(i) *for t small and $\alpha < \frac{q_0}{2}$;*

(ii) *for t large and $\alpha < \frac{q}{2}$.*

PROOF: Decompose f as $f_t + f^t$, where $f_t = f\chi_{B(x_0,\,t)}$. Then

$$\|e^{-tT} f_t\|_2 \le \|e^{-tT}\|_{1\to 2}\|f_t\|_1 = \|e^{-tT}\|_{1\to\infty}^{\frac{1}{2}}\|f_t\|_1.$$

By Hölder's inequality,

$$\begin{aligned}\|f_t\|_1 &= \int\limits_{B(x_0,\,t)} |f(x)|\,dx \\ &\le \Bigg(\int\limits_{B(x_0,\,t)} \rho(x,x_0)^{-2\alpha}\,dx\Bigg)^{\frac{1}{2}}\Bigg(\int\limits_{B(x_0,\,t)} \big|\rho(x,x_0)^{\alpha} f(x)\big|^2\,dx\Bigg)^{\frac{1}{2}} \\ &\le \Bigg(\int\limits_{B(x_0,\,t)} \rho(x,x_0)^{-2\alpha}\,dx\Bigg)^{\frac{1}{2}} \big\|\rho(\cdot,x_0)^{\alpha} f\big\|_2.\end{aligned}$$

If $t < 1$, and if $2\alpha < q_0$,

$$\int\limits_{B(x_0,\,t)} \rho(x,x_0)^{-2\alpha}\,dx \le \sum_{j=0}^{\infty} \int\limits_{2^{-j-1}t \le \rho(x,x_0) \le 2^{-j}t} \rho(x,x_0)^{-2\alpha}\,dx \lesssim t^{q_0-2\alpha},$$

with a constant that only depends on α. Hence

$$\|e^{-tT} f_t\|_2 \lesssim t^{\frac{q_0}{2}-\alpha}\,\|e^{-tT}\|_{2\to\infty}\,\|\rho(\cdot,x_0)^{\alpha} f\|_2 \lesssim t^{-\alpha}\|\rho(\cdot,x_0)^{\alpha} f\|_2, \tag{1.2}$$

by comparing (0.5) and (0.7). For $t \ge 1$,

$$\int\limits_{B(x_0,\,t)} \rho(x,x_0)^{-2\alpha}\,dx \le \int\limits_{B(x_0,\,1)} \rho(x,x_0)^{-2\alpha}\,dx + \sum_{j\,:\,2^j<t}\ \int\limits_{2^j \le \rho(x,x_0) \le 2^{j+1}} \rho(x,x_0)^{-2\alpha}\,dx.$$

By the previous argument, the first integral is convergent for $2\alpha < q_0$. On the other hand, if $2\alpha < q_\infty$,

$$\begin{aligned}\sum_{j\,:\,2^j<t}\ \int\limits_{2^j \le \rho(x,x_0) \le 2^{j+1}} \rho(x,x_0)^{-2\alpha}\,dx &\lesssim \sum_{j\,:\,2^j<t} 2^{(q_\infty-2\alpha)j} \\ &\lesssim t^{(q_\infty-2\alpha)}.\end{aligned}$$

This gives (1.2) with q_0 replaced by q_∞.

For f^t, we just use the trivial estimate, holding for every $\alpha > 0$,

$$\|e^{-tT} f^t\|_2 \le \|f^t\|_2 \le t^{-\alpha}\big\|\rho(\cdot,x_0)^{\alpha} f^t\big\|_2. \qquad \square$$

We shall prove now the companion estimate, in the sense presented in the introduction. It requires a preliminary remark (see also [A]).

LEMMA 2: *Let*

$$T = \int_0^{\infty} \lambda\,dE(\lambda)$$

be the spectral decomposition of T. *Then* $E\big(\{0\}\big) = 0$.

Proof: This is equivalent to proving that, for $f \in L^2(X)$,

$$\lim_{t\to\infty} \langle e^{-tT} f, f\rangle = 0 \ .$$

By density and by contractivity of the semigroup, it is sufficient to assume that $f \in C_c(X)$. Then

$$\big|\langle e^{-tT} f, f\rangle\big| \le \|e^{-tT}\|_{1\to 2}\, \|f\|_2\, \|f\|_1,$$

which tends to 0 by (0.5). □

Theorem 1′: *The inequality*

$$\|e^{-t\rho(\cdot,\, x_0)} f\|_2 \le C_\alpha t^{-\alpha} \|T^\alpha f\|_2. \tag{1.3}$$

holds for every $f \in L^2(X)$ and $x_0 \in X$, with a constant C_α depending only on α,

(i) *for t small and $\alpha < \frac{q_\infty}{2}$;*

(ii) *for t large and $\alpha < \frac{q}{2}$.*

Proof: We set $E_\lambda = E\big([0,\lambda]\big)$ and decompose f spectrally as $f_t + f^t$, where now $f_t = E_t f$. Then

$$\|e^{-t\rho(\cdot,\, x_0)} f_t\|_2 \le \|e^{-t\rho(\cdot,x_0)}\|_2\, \|f_t\|_\infty \le \|e^{-t\rho(\cdot,\, x_0)}\|_2\, \|E_t T^{-\alpha}\|_{2\to\infty} \|T^\alpha f\|_2. \tag{1.4}$$

We have

$$\begin{aligned}\|e^{-t\rho(\cdot,\, x_0)}\|_2^2 &= \sum_{j=0}^{\infty} \int_{jt^{-1}\le \rho(x,\, x_0) < (j+1)\, t^{-1}} e^{-2t\rho(x,\, x_0)}\, dm(x)\\ &\le \sum_{j=0}^{\infty} e^{-2j} m\big(B(x_0, (j+1)t^{-1})\big) \ .\end{aligned}$$

If $t \le 1$, $m\big(B(x_0,(j+1)t^{-1}\big) \lesssim (j+1)^{q_\infty} t^{q_\infty}$ for every j, so that

$$\|e^{-t\rho(\cdot,x_0)}\|_2^2 \lesssim t^{-q_\infty} \sum_{j=0}^{\infty} e^{-2j}(j+1)^{q_\infty} \lesssim t^{-q_\infty} \ . \tag{1.5}$$

If $t > 1$,

$$\begin{aligned}\|e^{-t\rho(\cdot,x_0)}\|_2^2 &\lesssim t^{-q_0} \sum_{1\le j+1\le t} e^{-2j}(j+1)^{q_0} + t^{-q_\infty} \sum_{j+1>t} e^{-2j}(j+1)^{q_\infty}\\ &\lesssim t^{-q_0} + t^{-q_\infty} \sum_{j+1>t} e^{-j}\\ &\lesssim t^{-q_0} + t^{-q_\infty} e^{-t}\\ &\lesssim t^{-q_0} \ .\end{aligned} \tag{1.6}$$

We estimate now $\|E_t T^{-a}\|_{2\to\infty}$ by first looking at the quantities

$$\|(E_{2^{-j}t} - E_{2^{-j-1}t})T^{-a}\|_{2\to\infty} = \|E([2^{-j-1}t, 2^{-j}t])T^{-a}\|_{2\to\infty}$$

for $j \geq 0$. By the spectral theorem,

$$\begin{aligned}\|E([2^{-j-1}t, 2^{-j}t])T^{-a}\|_{2\to\infty} &\leq \|E([2^{-j-1}t, 2^{-j}t])T^{-a}e^{2^j t^{-1}T}\|_{2\to 2} \\ &\quad \times \|e^{-2^j t^{-1}T}\|_{2\to\infty} \\ &\lesssim (2^{-j}t)^{-a}\|e^{-2^j t^{-1}T}\|_{2\to\infty} ,\end{aligned}$$

since the spectral multiplier of the first factor, $m(\lambda) = \chi_{[2^{-j-1}t, 2^{-j}t]}(\lambda)\lambda^{-a}e^{2^j t^{-1}\lambda}$ is bounded by a constant times $(2^{-j}t)^{-a}$. Hence, by (0.5),

$$\|(E_{2^{-j}t} - E_{2^{-j-1}t})T^{-a}\|_{2\to\infty} \lesssim \begin{cases} (2^{-j}t)^{\frac{q_0}{2}-a} & \text{for } 2^j < t \\ (2^{-j}t)^{\frac{q_\infty}{2}-a} & \text{for } 2^j \geq t\ . \end{cases} \tag{1.7}$$

It follows that the series

$$\sum_{j=0}^{\infty} (E_{2^{-j}t} - E_{2^{-j-1}t})T^{-a}$$

converges in the $L^2 \to L^\infty$ operator norm, for $a < \frac{q_\infty}{2}$. Moreover,

$$\begin{aligned}\sum_{j=0}^{\infty} \|(E_{2^{-j}t} - E_{2^{-j-1}t})T^{-a}\|_{2\to\infty} &\lesssim t^{\frac{q_0}{2}-a} \sum_{1\leq 2^j<t} 2^{-j(\frac{q_0}{2}-a)} \\ &\quad + t^{\frac{q_\infty}{2}-a}\sum_{2^j\geq t} 2^{-j(\frac{q_\infty}{2}-a)} \\ &\lesssim \begin{cases} t^{\frac{q_\infty}{2}-a} & \text{if } t \leq 1 \text{ and } a < \dfrac{q_\infty}{2} \\ t^{\frac{q_0}{2}-a} & \text{if } t > 1 \text{ and } a < \dfrac{q}{2}\ . \end{cases}\end{aligned} \tag{1.8}$$

By Lemma 2, the space

$$H_0 = \bigcup_{\delta>0} (I - E_\delta)L^2(X)$$

is dense in $L^2(X)$. For $f \in H_0$, the series

$$\sum_{j=0}^{\infty} (E_{2^{-j}t} - E_{2^{-j-1}t})T^{-a}f$$

is finite and its sum is equal to $E_t T^{-a}f$. By (1.8),

$$\|E_t T^{-a}f\|_\infty \lesssim \|f\|_2 \begin{cases} t^{\frac{q_\infty}{2}-a} & \text{if } t \leq 1 \text{ and } a < \dfrac{q_\infty}{2} \\ t^{\frac{q_0}{2}-a} & \text{if } t > 1 \text{ and } a < \dfrac{q}{2}\ . \end{cases}$$

By density, this gives bounds for the operator norms $\|E_t T^{-a}\|_{2\to\infty}$. Together with (1.5) and (1.6), we obtain that

$$\|e^{-t\rho(\cdot,x_0)}\|_2 \|E_t T^{-a}\|_{2\to\infty} \lesssim t^{-a} \tag{1.9}$$

for every t.

Finally, for f^t we have the rather trivial estimate

$$\|e^{-t\rho(\cdot,x_0)} f^t\|_2 \le \|f^t\|_2 \le \|(1-E_t)T^{-a}\|_{2\to 2}\|T^a f\|_2 \lesssim t^{-a}\|T^a f\|_2 \ .$$

Combining this with (1.4) and (1.9), we obtain the conclusion. □

2. - Local uncertainty inequalities

By the spectral theorem, Theorem 1 is equivalent to the statement that

$$\|E_s f\|_2 \le C_a s^a \left\|\rho(\cdot,x_0)^a f\right\|_2 \ , \tag{2.1}$$

with C_a independent of f and s, for s large if $a < \frac{q_0}{2}$, and for s small if $a < \frac{q}{2}$. To see this, just observe that, for $\lambda \ge 0$,

$$e^{-1}\chi_{[0,s]}(\lambda) \le e^{-\frac{\lambda}{s}} \le (e-1)\sum_{j=1}^{\infty} e^{-j}\chi_{[0,js]} \ .$$

We are interested in analogues of (2.1), with E_s replaced by the spectral measure of a general Borel subset $\omega \subset [0,\infty)$. In order to do so, consider the quantity $\|E(\omega)\|_{1\to\infty} = \|E(\omega)\|^2_{2\to\infty}$. This is surely finite if ω is bounded, because, if $\omega \subset [0,s]$

$$\|E(\omega)\|_{2\to\infty} \le \|E(\omega)e^{\frac{1}{s}T}\|_{2\to 2}\|e^{-\frac{1}{s}T}\|_{2\to\infty} < \infty \ . \tag{2.2}$$

We set

$$\upsilon(\omega) = \begin{cases} \|E(\omega)\|_{1\to\infty}^{\frac{1}{q_\infty}} & \text{if } \|E(\omega)\|_{1\to\infty} \le 1 \\ \|E(\omega)\|_{1\to\infty}^{\frac{1}{q_0}} & \text{if } \|E(\omega)\|_{1\to\infty} > 1 \ . \end{cases}$$

Notice that, by (2.2) and (0.5), $\upsilon\big([0,s]\big) \lesssim s$.

Theorem 3: *Let $a > 0$ and ω a Borel subset of* $[0,\infty)$. *The inequality*

$$\|E(\omega)f\|_2 \le C_a \upsilon(\omega)^a \left\|\rho(\cdot,x_0)^a f\right\|_2 \ . \tag{2.3}$$

holds for every $f \in L^2(X)$ and $x_0 \in X$, with a constant C_a depending only on a,

(i) *for $\upsilon(\omega)$ small and $a < \frac{q}{2}$*;

(ii) *for $\upsilon(\omega)$ large and $a < \frac{q_0}{2}$*.

The proof follows the same lines as that of Theorem 1 (see also [CRS]), after decomposing $f \in L^2(X)$ as $f_r + f^r$, with $r = \upsilon(\omega)^{-1}$, $f_r = f\chi_{B(x_0,r)}$, and using the estimate

$$\|E(\omega)f_r\|_2 \le \|E(\omega)\|_{2\to\infty}\|f_r\|_1 \ .$$

Notice that this inequality contains (0.3) when $X = \mathbb{R}^n$, ρ is the Euclidean distance, and $T = \sqrt{-\Delta}$. The set $A \subset \mathbb{R}^n$ in (0.3) is then $A = \{\xi : |\xi| \in \omega\}$.

Theorem 3 also has a companion statement, with the rôle of ρ replaced by T, and $\omega \subset [0,\infty)$ replaced by a subset A of X. We set

$$\mu(A) = \begin{cases} m(A)^{\frac{1}{q_0}} & \text{if } m(A) \le 1 \\ m(A)^{\frac{1}{q_\infty}} & \text{if } m(A) > 1 \ . \end{cases}$$

THEOREM 3′: *Let $\alpha > 0$ and A a measurable subset of X. The inequality*

$$\left(\int_A |f(x)|^2 \, dm(x)\right)^{\frac{1}{2}} \le C_\alpha \mu(A)^\alpha \|T^\alpha f\|_2 \ . \tag{2.4}$$

holds for every $f \in L^2(X)$, with a constant C_α depending only on α,

(i) *for $\mu(A)$ small and $\alpha < \dfrac{q}{2}$;*

(ii) *for $\mu(A)$ large and $\alpha < \dfrac{q_\infty}{2}$.*

In particular, If $q_0 \le q_\infty$, $f, T^\alpha f \in L^2(X)$ implies that $f \in L^p_w(X)$ for $\dfrac{1}{2} - \dfrac{\alpha}{q_0} \le \dfrac{1}{p} \le$ $\le \dfrac{1}{2} - \dfrac{\alpha}{q_\infty}$.

Again, the proof follows the same lines as that of Theorem 1′, with $t = \mu(A)^{-1}$. It is interesting to notice that the choice between q_0 and q_∞ is determined only by the measure of A, and not by its concentration near a given point.

We see therefore that the quantity $\|E(\omega)\|_{1\to\infty}$ plays, on the spectral side, the same rôle played by $m(A)$ on the space side. Keeping this point of view, it is immediate to find the analogues of (0.4).

PROPOSITION 4: *Let A be a measurable subset of X and ω a Borel subset of* $[0,\infty)$. *For $f \in L^2(X)$, then*

$$\|E(\omega)(\chi_A f)\|_2 \le \sqrt{m(A)\|E(\omega)\|_{1\to\infty}}\|f\|_2; \tag{2.6}$$

$$\|\chi_A E(\omega) f\|_2 \le \sqrt{m(A)\|E(\omega)\|_{1\to\infty}}\|f\|_2; \tag{2.7}$$

In particular, for $\omega = [0,s]$,

$$\|E_s(\chi_A f)\|_2 \le \begin{cases} \sqrt{m(A)s^{q_\infty}}\|f\|_2 & \text{if } s \le 1 \ , \\ \sqrt{m(A)s^{q_0}}\|f\|_2 & \text{if } s > 1 \ , \end{cases}$$

and similarly for $\|\chi_A E_s f\|_2$.

PROOF: The trivial estimate

$$\|E(\omega)(\chi_A f)\|_2 \leq \|E(\omega)\|_{1\to 2}\|\chi_A f\|_1 \leq \|E(\omega)\|_{2\to\infty} m(A)^{\frac{1}{2}}\|f\|_2 ,$$

gives (2.6), and

$$\|\chi_A E(\omega) f\|_2 \leq \|\chi_A\|_2 \|E(\omega) f\|_\infty \leq m(A)^{\frac{1}{2}}\|E(\omega)\|_{2\to\infty}\|f\|_2 ,$$

gives (2.7). □

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