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The General Relative Entropy Principle – Applications in Perron-Frobenius and Floquet Theories and a Parabolic System for Biomotors –

ABSTRACT. — We survey several types of results that follow from the notion of General Relative Entropy Inequality. This concept was introduced in [38, 39, 40] and extends to equations that are not conservation laws, the notion of relative entropy for conservative parabolic, hyperbolic or integral equations.

We first show how this notion arises naturally for positive matrices in the context of Perron-Frobenius theory and Floquet theory. It explains why the solutions to the associated differential systems converge to the first eigenvector (Perron-Frobenius theory) or to the periodic solution (Floquet theory) because they minimize the general relative entropy.

For Partial Differential Equations, we give another example of recent interest where a General Relative Entropy Inequality exists: a parabolic system describing molecular biomotors.

1. - INTRODUCTION

This contribution gives a survey of recent findings on entropy structures of equations with nonconservative forms. The motivation has been to study models arising in biology where birth and death processes are crucial and where it is expected to see (at least in the first phase of the process) a logistic growth along with the Malthus law. We discovered the form of this new entropy structure for the renewal equation in [40]. It turns out that the concept of relative entropy, once correctly extended, covers a larger and well established class of (linear) mathematical models that has been derived and validated, the so-called *structured population dynamic* equations arising in biology. In order to have a concrete example at hand, we can for instance have in mind the 'size structured dynamic'

$$(1) \qquad \begin{cases} \dfrac{\partial}{\partial t} n(t,x) + \dfrac{\partial}{\partial x} n(t,x) + B(x)\, n(t,x) = 4B(2x)\, n(t,2x), \qquad t \geq 0,\ x \geq 0, \\ n(t, x=0) = 0, \end{cases}$$

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which describes the number of individuals $n(t,x)$ with size $x > 0$, growing continuously $\Big($this is the term $\frac{\partial}{\partial x} n(t,x)\Big)$ and that divide with rate $B(x) > 0$ in two equal individuals of size $x/2$. This represents mitosis for instance, and is modeled by the term $B(x)n(t,x) - 4B(2x)n(t,2x)$. A complete book on structured population dynamic is [35], goes back to 1986, and already in 1966 a full study of the simple renewal model was surveyed in [29]. But the subject is still active in terms of modeling and mathematical analysis [1, 7, 13, 37, 43]. Also, as usual in mathematical settings, and considering equation (1), similar models cover broader applications ([4, 9, 27]).

Because they describe growth processes in biology, these linear equations always have variable coefficients, including zeroth order terms or boundary conditions and thus it is not obvious to derive a priori estimates. Especially, one usually expects that the first positive eigenvector of such problems (that can be obtained through Krein-Rutman theorem, see [20] for instance), describes the long time limit of solutions once they have been renormalized in time along with the first eigenvalue, denoted by λ below, in other words we rather consider $n(t,x)e^{-\lambda t}$.

The concept of *entropy* is mostly used in nonlinear conservative problems, and is known to be a fundamental tool there, such as Boltzmann equation and the famous H-theorem ([11, 50, 25]), systems of conservation laws ([46, 24, 42]) and even more general Partial Differential Equations, see [28] and the references therein : there the entropy is an appropriate function $H(n)$ with n the solution of the equation or system under consideration. The notion of *relative entropy* is also usual for conservative models (then $\lambda = 0$ again) and the main difference is that some weights now come in the definition of the entropy as $NH\left(\frac{n}{N}\right)$ and the usual choice of the function H leads to consider more specificall the relative entropy $n \ln\left(\frac{n}{N}\right)$, ([49] for the probabilistic background and [51] for further references and related topics). In the *General Relative Entropy* method, we consider more elaborate weights which now include the steady state N as before but also the adjoint operator solution ϕ, in order to build the quantity $\phi\, NH\left(\frac{ne^{-\lambda t}}{N}\right)$ as stated in [38, 39] for various Partial Differential Equations. To be concrete, in the above example (1), we need to take for (λ, N, ϕ) the solutions to

$$(2)\qquad \begin{cases} \frac{\partial}{\partial x} N(x) + (B(x) + \lambda)\, N(x) = 4B(2x)\, N(2x), & x \geq 0, \\ N(x=0) = 0, \qquad N \geq 0, \qquad \int_0^\infty N = 1, \end{cases}$$

$$(3)\qquad \begin{cases} -\frac{\partial}{\partial x} \phi(x) + (B(x) + \lambda)\, \phi = 2B(x)\, \phi\left(\frac{x}{2}\right), & x \geq 0, \\ \phi(x) \geq 0, \qquad \int_0^\infty N(x)\phi(x) = 1, \end{cases}$$

and then, we have

$$\frac{d}{dt}\int_0^\infty \phi N H\left(\frac{ne^{-\lambda t}}{N}\right)dx \leq 0.$$

The solution N to this equation is depicted in Figure 8 for $B \equiv 1$ and mathematical studies of these equations can be found in [43, 37].

The fact that the general structure encompasses all positive matrices comes from unpublished discussions, [16], and is related to a characterisation of M-matrices in [45] for M-matrices several years ago.

As we mentioned, the main reason why these entropy methods are fundamental tools is that they provide several estimates for the integral or Partial Differential Equation under consideration, as a priori estimates. But more important is that these entropy methods provide a general understanding of long time convergence to steady states. Especially they explain various levels of difficulties depending wether the entropy dissipation controls or not the entropy itself. In the former case exponential rates of convergence can be proved (with relation to concepts of log-Sobolev inequalities, hypercontractivity); in the latter case one can usually at least prove simple convergence to the steady state (usually by arguments that combine informations on the entropy dissipation and the equation itself) [3, 6, 10, 22, 23, 30, 40, 38, 39, 43, 47, 51].

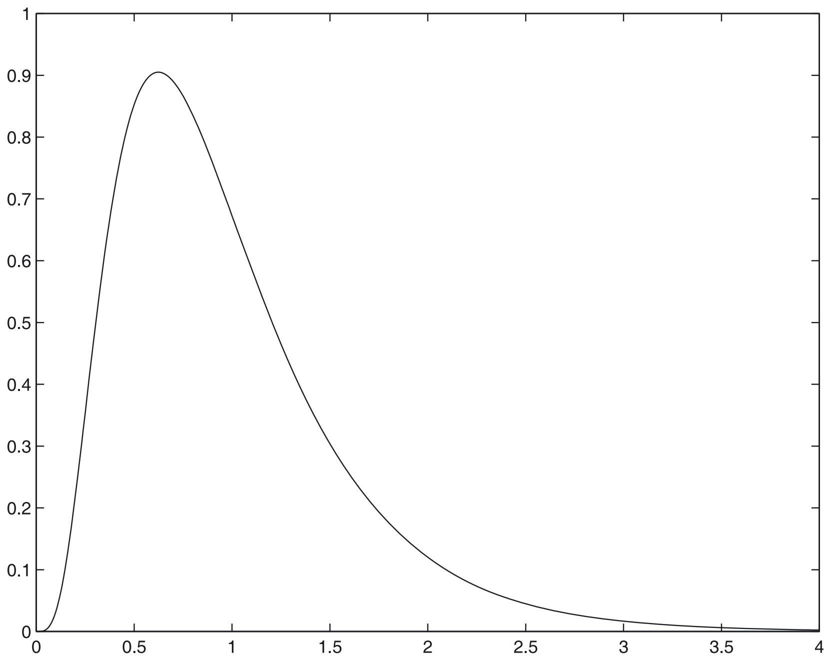

FIGURE 1. – The solution N to equation (2) with $B \equiv 1$. This distribution results of the compromize between continuous growth that pushes N to large x and division in two equal pieces that pushes N to small x.

This paper presents mostly the formalism of *General Relative Entropies* in as elementary as possible examples: we begin with matrices in the frameworks of Perron-Frobenius and Floquet theories (Section 2), and especially we illustrate the long time convergence results that can be derived in this simple and classical case. Then, in Section 3, we turn to general parabolic and scattering equations where we stay at a formal level in order to illustrate the formalism. For more elaborate examples applications, and especially for the case of the cell-division equation (1), we refer to [40, 39, 30]. In order to give another example of system, we consider a recent parabolic system of biophysical interest in Section 4.

2. - Finite dimensional systems

We begin with describing the General Entropy Inequality in the case of matrices and we deal with two theories where it applies to give an entropy based understanding of time relaxation. In the framework of Perron-Frobenius eigenvalue Theorem it explains why the associated dynamic converges to the first (positive) eigenvector (once correctly normalized). In the framework of Floquet's eigenvalue theorem it explains why the associated dynamic converges to the (positive) periodic solution (once correctly normalized).

2.1. *The Perron-Frobenius theory*

Let $a_{ij} > 0$, $1 \leq i, j \leq d$, be the coefficients of a matrix $A \in M_{d\times d}(\mathbb{R})$ (there are interesting issues with the case $a_{ij} \geq 0$ but we try to keep simplicity here). The Perron-Frobenius theorem (see [44] for instance) tells us that there is a first eigenvalue $\lambda > 0$ to A associated with a positive right eigenvector $N \in \mathbb{R}^d$, and a positive left eigenvector $\phi \in \mathbb{R}^d$

$$\begin{cases} A.N = \lambda N, & N_i > 0 \quad \text{for} \quad i = 1, \dots, d, \\ \phi.A = \lambda\phi, & \phi_i > 0 \quad \text{for} \quad i = 1, \dots, d. \end{cases}$$

For later purposes, it is convenient to normalize these vectors, so that they are now uniquely defined. We choose

$$\sum_{i=1}^{d} N_i = 1, \qquad \sum_{i=1}^{d} N_i\, \phi_i = 1.$$

We set $\widetilde{A} = A - \lambda Id$ and consider the problem

$$\frac{d}{dt} n(t) = \widetilde{A}.n(t), \qquad n(0) = n^0. \tag{4}$$

The following result is 'standard'.

Proposition 2.1: *For positive matrices A and solutions to the differential system* (4)*, we have,*

$$\rho := \sum_{i=1}^{d} \phi_i n_i(t) = \sum_{i=1}^{d} \phi_i n_i^0, \tag{5}$$

$$\sum_{i=1}^{d} \phi_i |n_i(t)| \leq \sum_{i=1}^{d} \phi_i |n_i^0|, \tag{6}$$

$$\underline{C} N_i \leq n_i(t) \leq \overline{C} N_i \quad \text{with constants given by} \quad \underline{C} N_i \leq n_i^0 \leq \overline{C} N_i, \tag{7}$$

and there is a constant $\alpha > 0$ *such that, with* ρ *given in* (5)*, we have*

$$\sum_{i=1}^{d} \phi_i N_i \left(\frac{n_i(t) - \rho N_i}{N_i} \right)^2 \leq \sum_{i=1}^{d} \phi_i N_i \left(\frac{n_i^0 - \rho N_i}{N_i} \right)^2 e^{-\alpha t}. \tag{8}$$

Here, we wish to justify it with an entropy inequality.

Proposition 2.2: *Let* $H(\cdot)$ *be a convex function on* $\mathbb{R}$*, then the solution to* (4) *satisfies*

$$\frac{d}{dt} \sum_{i=1}^{d} \phi_i N_i H\left(\frac{n_i(t)}{N_i}\right) =$$

$$= \sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left[H'\left(\frac{n_i(t)}{N_i}\right) \left[\frac{n_j(t)}{N_j} - \frac{n_i(t)}{N_i}\right] - H\left(\frac{n_j(t)}{N_j}\right) + H\left(\frac{n_i(t)}{N_i}\right) \right] \leq 0.$$

Definition 2.3: *We call General Relative Entropy, the quantity* $\sum_{i=1}^{d} \phi_i N_i H\left(\frac{n_i(t)}{N_i}\right)$.

Proof of Proposition 2.2: We denote by $\widetilde{a_{ij}}$ the coefficients of the matrix $\widetilde{A}$ and compute

$$\begin{aligned} \frac{d}{dt} \sum_i \phi_i N_i H\left(\frac{n_i(t)}{N_i}\right) &= \sum_{i,j} \phi_i H'\left(\frac{n_i(t)}{N_i}\right) \widetilde{a_{ij}} n_j(t) \\ &= \sum_{i,j} \phi_i \widetilde{a_{ij}} N_j H'\left(\frac{n_i(t)}{N_i}\right) \left[\frac{n_j(t)}{N_j} - \frac{n_i(t)}{N_i}\right], \end{aligned}$$

because the additional $\frac{n_i(t)}{N_i}$ term vanishes since $\widetilde{A}.N = 0$. But we also have, again thanks to the equation on N and ϕ, that

$$\sum_{i,j} \phi_i \widetilde{a_{ij}} N_j \left[H\left(\frac{n_j(t)}{N_j}\right) - H\left(\frac{n_i(t)}{N_i}\right) \right] = 0.$$

Combining these two identities, we arrive to the equality in Proposition 2.2. The inequality follows because only the coefficients out of the diagonal, that satisfy $\widetilde{a_{ij}} = a_{ij} \geq 0$, enters here. □

PROOF OF PROPOSITION 2.1: Notice that, as a special case of H in Proposition 2.2, we can choose $H(u) = u$, which being convex together with $-H$ gives the equality

$$\frac{d}{dt}\sum_{i=1}^{d} \phi_i n_i(t) = 0.$$

And (5) follows. In particular this identifies the value ρ mentioned in (8).

The second statement (6) follows immediately by choosing the (convex) entropy function $H(u) = |u|$.

As for the third statement (7), let us consider for instance the upper bound. It follows choosing the (convex) entropy function $H(u) = (u - \overline{C})_+^2$ because for this nonnegative function we have

$$\sum_{i=1}^{d} \phi_i N_i H\left(\frac{n_i^0}{N_i}\right) = 0.$$

Therefore, because the General Relative Entropy decays, it remains zero for all times,

$$\sum_{i=1}^{d} \phi_i N_i H\left(\frac{n_i(t)}{N_i}\right) = 0,$$

which proves the result.

It remains to prove the exponential decay statement (8). To do that, we work on

$$m(tx) = n(t,x) - \rho N,$$

which verifies $\int \varphi[n(t,x) - \rho N]dx = 0$ and satisfies the same equation as n. Then, we use the quadratic entropy function $H(u) = u^2$ and the General Entropy Inequality gives

$$\frac{d}{dt}\sum_{i=1}^{d} \phi_i N_i \left(\frac{m_i(t)}{N_i}\right)^2 = -\sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left(\frac{m_j(t)}{N_j} - \frac{m_i(t)}{N_i}\right)^2 \leq 0.$$

Then, we need a lemma (Poincaré inequality).

LEMMA 2.4: *Being given* $\phi_i > 0$, $N_i > 0$, $a_{ij} > 0$ *for* $i = 1, \dots, d$, $j = 1, \dots, d$, $i \neq j$, *there is a constant* $\alpha > 0$ *such that for all vector* m *of components* m_i, $1 \leq i \leq d$ *satisfying*

$$\sum_{i=1}^{d} \phi_i m_i = 0,$$

we have

$$\sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left(\frac{m_j}{N_j} - \frac{m_i}{N_i}\right)^2 \geq \alpha \sum_{i=1}^{d} \phi_i N_i \left(\frac{m_i}{N_i}\right)^2.$$

With this lemma, we conclude

$$\frac{d}{dt}\sum_{i=1}^{d} \phi_i N_i \left(\frac{m_i(t)}{N_i}\right)^2 \leq -\alpha \sum_{i=1}^{d} \phi_i N_i \left(\frac{m_i(t)}{N_i}\right),$$

and then, (8) follows by a simple use of Gronwall lemma. □

PROOF OF LEMMA 2.4. After renormalizing the vector m (when it does not vanish, otherwise the result is obvious), we may suppose that

$$\sum_{i=1}^{d} \phi_i m_i = 0, \qquad \sum_{i=1}^{d} \phi_i N_i \left(\frac{m_i}{N_i}\right)^2 = 1.$$

Then we argue by contradiction. If such a α does not exist, this means that we can find a sequence of vectors $(m^k)_{(k\geq 1)}$ such that

$$\sum_{i=1}^{d} \phi_i m_i^k = 0, \qquad \sum_{i=1}^{d} \phi_i N_i \left(\frac{m_i^k}{N_i}\right)^2 = 1, \qquad \sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left(\frac{m_j^k}{N_j} - \frac{m_i^k}{N_i}\right)^2 \leq 1/k.$$

After extraction of a subsequence, we may pass to the limit $m^k \to \bar{m}$ and this vector satisfies

$$\sum_{i=1}^{d} \phi_i \bar{m}_i = 0, \qquad \sum_{i=1}^{d} \phi_i N_i \left(\frac{\bar{m}_i}{N_i}\right)^2 = 1,$$

$$\sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left(\frac{\bar{m}_j}{N_j} - \frac{\bar{m}_i}{N_i}\right)^2 = 0.$$

Therefore, from this last relation, for all i and $j = 1, \ldots, d$, we have

$$\frac{\bar{m}_i}{N_i} = \frac{\bar{m}_j}{N_j} := \upsilon.$$

By the zero sum condition, we have $\upsilon = 0$ because

$$\upsilon \sum_{i=1}^{d} \phi_i = 0.$$

In other words, $\bar{m} = 0$ which contradicts the normalization and thus such a α should exist. □

REMARK 1: This entropy structure is related to a characterization of M-matrices, i.e., those whose terms out of the diagonal are negative, diagonal terms are positive and dominate the corresponding line. Such a matrix has an inverse with positive coefficients. It was noticed in [45] that a characterization of M-matrices uses the existence of positive eigenvectors as N and ϕ above. Let us point out that the General Relative Entropy inequality also holds for M-matrices because the diagonal terms do not appear in the inequality of Lemma 2.4.

REMARK 2: The matrix with (positive) coefficients $b_{ij} = \phi_i \, a_{ij} \, N_j$ is doubly stochastic, i.e., the sum of the lines and columns is 1 (see for instance [44]).

REMARK 3: Notice that $a_{ii} - \lambda < 0$ because $\sum_j \widetilde{a_{ij}} N_j = 0$. Therefore the matrix C with coefficients $c_{ij} = \frac{1}{N_i} \widetilde{a_{ij}}\, N_j$ is that of a Markov process. In other words, we set $y_i = x_i/N_i$, then it satisfies

$$\frac{d}{dt} y_i(t) = c_{ij} y_j(t),$$

and the vector $(1, 1, \ldots, 1)$ is the (positive) eigenvector associated to the eigenvalue 0 of the matrix C, i.e., $c_{ii} = \sum_{j \neq i} c_{ij}$ and $c_{ij} \geq 0$. Then, $(N_i \phi_i)_{(i=1,\ldots,d)}$ is the invariant measure of the Markov process. In particular this explains the entropy property which is classical for Markov processes ([48]).

2.2. *The Floquet theory*

We now consider T-periodic coefficients $a_{ij}(t) > 0$, $1 \leq i, j \leq d$, i.e., $a_{ij}(t + T) = a_{ij}(t)$. And we denote by $A(t) \in M_{d \times d}$ the corresponding matrix. Again our motivation comes from several questions in biology where such structures arise as seasonal rhythm, circadian rhythm, see [31, 17, 36, 5] for instance.

The Floquet theorem tells us that there is a first 'Floquet eigenvalue' $\lambda_{per} > 0$ and two positive T-periodic functions $N(t) \in \mathbb{R}^d$, $\phi(t) \in \mathbb{R}^d$ that are periodic solutions (uniquely defined up to multiplication by a constant) to the differential systems

$$\frac{d}{dt} N(t) = [A(t) - \lambda_{per} Id].N(t), \tag{9}$$

$$\frac{d}{dt} \phi(t) = \phi(t).[A(t) - \lambda_{per} Id]. \tag{10}$$

Up to a normalization, these elements $(\lambda_{per} > 0, N(t) > 0, \phi(t))$ are unique and we normalize again as

$$\int_0^T \sum_{i=1}^d N_i(t) dt = 1, \qquad \int_0^T \sum_{i=1}^d \phi_i(t) N_i(t) dt = 1.$$

We recall that this case of Floquet theory (which applies to more general situations than positive matrices) is an application of Perron-Frobenius theory to the resolvent matrix

$$S(t) = e^{\int_0^t A(s) ds},$$

which has positive coefficients also. A classical introduction to the subject can be found in Coddington and Levinson [18].

Again, we set $\widetilde{A(t)} = A(t) - \lambda_{per} Id$ and consider the differential system

$$\frac{d}{dt} n(t) = \widetilde{A}.n(t), \qquad n(0) = n^0.$$

In the present context we obtain the following version of Proposition 2.1, PROPOSITION 2.5: *For positive matrices A we have,*

$$\rho := \sum_{i=1}^{d} \phi_i(t) n_i(t) = \sum_{i=1}^{d} \phi_i(t=0) n_i^0, \tag{11}$$

$$\sum_{i=1}^{d} \phi_i(t) |n_i(t)| \leq \sum_{i=1}^{d} \phi_i(t=0) |n_i^0|, \tag{12}$$

$$\underline{C} N_i(t) \leq n_i(t) \leq \overline{C} N_i(t) \quad \text{with constants given by} \quad \underline{C} N_i(t=0) \leq n_i^0 \leq \overline{C} N_i(t=0), \tag{13}$$

and there is a constant $\alpha > 0$ such that

$$\sum_{i=1}^{d} \phi_i N_i \left(\frac{n_i(t) - \rho N_i(t)}{N_i(t)} \right)^2 \leq \sum_{i=1}^{d} \phi_i N_i \left(\frac{n_i^0 - N_i^0}{N_i^0} \right)^2 e^{-\alpha t}. \tag{14}$$

Again, this can be justified thanks to entropy inequalities.

PROPOSITION 2.6: *Let $H(\cdot)$ be a convex function on $\mathbb{R}$, then we have*

$$\frac{d}{dt} \sum_{i=1}^{d} \phi_i(t) N_i(t) H\left(\frac{n_i(t)}{N_i(t)} \right) =$$

$$= \sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left[H'\left(\frac{n_i}{N_i} \right) \left[\frac{n_j}{N_j} - \frac{n_i}{N_i} \right] - H\left(\frac{n_j}{N_j} \right) + H\left(\frac{n_i}{N_i} \right) \right] \leq 0.$$

These two propositions are variants of the corresponding ones in Perron-Frobenius theory and we leave the proofs to the reader. Of course, adapting the Lemma 2.4 requires an additional compactness argument based on the Ascoli Theorem.

3. - PARABOLIC AND INTEGRAL PDE's

We now explain the notion of General Relative Entropy on continuous models. We begin with the most classical equation, namely the parabolic equation for the unknown $n(t, x)$,

$$\frac{\partial n}{\partial t} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial n}{\partial x_j} \right) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (b_i n) + dn = 0, \qquad x \in \mathbb{R}^d, \tag{15}$$

where the coefficients depend on t and x, $d \equiv d(t, x)$ (no sign assumed), $b_i \equiv b_i(t, x)$, and the symmetric matrix $A(t, x) = (a_{ij}(t, x))_{1 \leq i, j \leq d}$ satisfies $A(t, x) \geq 0$. We could also set the equation on a domain and assume Dirichlet, zero-flux, mixed or periodic boundary conditions and then include them in the above calculation.

Here, it is not obvious to derive a priori bounds on the solution $n(t,x)$, by opposition to the case $A \geq \upsilon Id > 0$, $b_i \equiv 0$, $d(x) \geq 0$ where we have, multiplying the equation by $n|n|^{p-2}$ with $p > 1$,

$$\frac{d}{dt}\int \frac{|n(t,x)|^p}{p}dx + \frac{4\upsilon(p-1)}{p^2}\int |\nabla n^{p/2}|^2 dx \leq 0.$$

Indeed the only remarkable property of (15) is the L^1 contraction property

$$\frac{d}{dt}\int n(t,x)dx + \int d(t,x)n(t,x)dx = 0, \quad \frac{d}{dt}\int (n(t,x))_+ dx + \int d(t,x)(n(t,x))_+ dx \leq 0.$$

On the other hand the conservative Fokker-Planck equation is very standard when $b = -\nabla V$ for some convex potential with enough growth at infinity

$$\frac{\partial n}{\partial t} - \Delta n - \mathrm{div}(\nabla V\, n) = 0.$$

Then, one has $N = e^{-V}$ and the relative entropy $\int n \ln\left(\frac{n}{N}\right)dx$ is a standard object related to log-sobolev inequalities, [51]. Of course, here we still have the family $\int NH\left(\frac{n}{N}\right)dx$ of relative entropies for all convex functions $H(\,\cdot\,)$ and not only $H(u) = u \ln(u)$.

3.1. *Coefficients independent of time*

In the case of coefficients independent of time, and depending on the values of $a_{ij}(x)$, div $b(x)$ and $d(x)$, the solution can grow or decay exponentially. Therefore, we will assume that 0 is the first eigenvalue and, following Krein-Rutman theorem (see [20]), we also assume that we can find two functions $N(x) > 0$, $\phi(x) > 0$, such that

$$\text{(16)} \qquad \begin{cases} -\displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial N}{\partial x_j}\right) + \sum_{i=1}^{d}\frac{\partial}{\partial x_i}(b_i(x)N) + d(x)N = 0, \\ \qquad\qquad N(x) > 0, \qquad \int N(x)dx = 1, \end{cases}$$

$$\text{(17)} \qquad \begin{cases} -\displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial \phi}{\partial x_j}\right) - \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}\phi + d(x)\phi = 0, \\ \qquad\qquad \phi(x) > 0, \qquad \int N(x)\phi(x)dx = 1. \end{cases}$$

These are the first eigenvectors; N for the direct problem and ϕ for the dual operator. Notice that such eigenelements do not always exist but there are standard examples, namely when $d \equiv 0$, $A = Id$ and there is a potential V such that $b = -\nabla V$. Then, one can readily check that solutions to (16)-(17) are

$$N = e^{-V} \qquad \phi \equiv 1,$$

when $V(x) \to \infty$ as $|x| \to \infty$ fast enough in order to fulfill the integrability conditions.

The general relative entropy property of the parabolic equation (15) can be expressed as

LEMMA 3.1: *For coefficients independent of t, assume that there exist eigenelements N, ϕ satisfying* (16)-(17). *Then for all convex function $H : \mathbb{R} \to \mathbb{R}$, and all solutions n to* (15) *with sufficient decay to zero at infinity, we have*

$$\frac{d}{dt}\int \phi(x)\, N(x)\, H\Big(\frac{n(t,x)}{N(x)}\Big) dx = -\int \phi\, N\, H''\Big(\frac{n(t,x)}{N(x)}\Big) \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i}\Big(\frac{n}{N}\Big)\, \frac{\partial}{\partial x_j}\Big(\frac{n}{N}\Big)\, dx \le 0.$$

For conservative equations, i.e., $d \equiv 0$, it is usual to take $\phi \equiv 1$, and then the corresponding principle is classical (especially related to stochastic differential equations and Markov processes, [48]).

PROOF OF LEMMA 3.1. We just calculate (leaving the details to the reader)

$$\frac{\partial}{\partial t}\Big(\frac{n}{N}\Big) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big[a_{ij}\frac{\partial}{\partial x_j}\Big(\frac{n}{N}\Big)\Big] + 2N\sum_{i,j=1}^{d} a_{ij}\frac{\partial}{\partial x_i}\Big(\frac{n}{N}\Big)\, \frac{\partial}{\partial x_j}\Big(\frac{1}{N}\Big) + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}\Big(\frac{n}{N}\Big) = 0.$$

Therefore, for any smooth function H, we arrive at

$$\frac{\partial}{\partial t}NH\Big(\frac{n}{N}\Big) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big[a_{ij}\frac{\partial}{\partial x_j}NH\Big(\frac{n}{N}\Big)\Big] + NH''\Big(\frac{n}{N}\Big)\sum_{i,j=1}^{d} a_{ij}\frac{\partial}{\partial x_i}\Big(\frac{n}{N}\Big)\, \frac{\partial}{\partial x_j}\Big(\frac{n}{N}\Big) +$$
$$+\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Big[b_i NH\Big(\frac{n}{N}\Big)\Big] + dNH\Big(\frac{n}{N}\Big) = 0.$$

Finally, combining it with the equation on ϕ, we deduce that

$$\frac{\partial}{\partial t}\phi NH\Big(\frac{n}{N}\Big) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big[\phi a_{ij}\frac{\partial}{\partial x_j}NH\Big(\frac{n}{N}\Big)\Big] + \sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\Big[a_{ij}NH\Big(\frac{n}{N}\Big)\frac{\partial}{\partial x_j}\phi\Big] +$$
$$+\phi NH''\Big(\frac{n}{N}\Big)\sum_{i,j=1}^{d} a_{ij}\frac{\partial}{\partial x_i}\Big(\frac{n}{N}\Big)\, \frac{\partial}{\partial x_j}\Big(\frac{n}{N}\Big) = 0.$$

After integration in x (because we have assumed sufficient decay to zero at infinity), we arrive at the result stated in Lemma 3.1. □

This Lemma can be used in the directions indicated in Section 2 (a priori estimates, long time convergence to a steady state) and we refer to [40, 38, 39] for specific examples.

As far as long time convergence is concerned, we notice that, as in Lemma 2.4, a control of entropy by entropy dissipation is useful, i.e., for the quadratic entropy, from the Poincaré inequality

$$\nu\int \phi N\Big(\frac{m}{N}\Big)^2 \le 2\int \phi N\sum_{i,j=1}^{d} a_{ij}\frac{\partial}{\partial x_i}\Big(\frac{m}{N}\Big)\, \frac{\partial}{\partial x_j}\Big(\frac{m}{N}\Big), \quad \text{when} \int \phi m = 0.$$

Indeed, this inequality implies exponential decay to the steady state as in 8. Such inequalities, as well as log-Sobolev inequalities, are classical when $b = \nabla V$, $d = 0$. We are not aware of any result when $\phi \neq 1$.

3.2. *Time dependent coefficients*

In fact the above manipulations are also valid for time dependent coefficients. A situation similar to the Floquet Theory and which is therefore useful for periodic coefficients for instance. We now consider solutions to

$$(18)\qquad \begin{cases} \dfrac{\partial}{\partial t}N(t,x) - \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial N}{\partial x_j}\right) + \sum_{i=1}^{d}\frac{\partial}{\partial x_i}(b_i(x)N) + d(x)N = 0, \\ \hfill N(x) > 0, \end{cases}$$

$$(19)\qquad \begin{cases} \dfrac{\partial}{\partial t}\phi(t,x) - \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial \phi}{\partial x_j}\right) - \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}\phi + d(x)\phi = 0, \\ \hfill \phi(x) > 0. \end{cases}$$

Then we have

LEMMA 3.2: *For all convex function $H : \mathbb{R} \to \mathbb{R}$, and all solutions n to* (15) *with sufficient decay to zero at infinity, we have*

$$\frac{d}{dt}\int \phi(t,x)\, N(t,x)\, H\left(\frac{n(t,x)}{N(t,x)}\right) dx =$$
$$= -\int \phi\, N\, H''\left(\frac{n(t,x)}{N(t,x)}\right) \sum_{i,j=1}^{d} a_{ij}\frac{\partial}{\partial x_i}\left(\frac{n}{N}\right)\,\frac{\partial}{\partial x_j}\left(\frac{n}{N}\right)\, dx \le 0.$$

Again we leave the proof of this variant to the reader.

3.3. *Scattering equations*

To exhibit another class of equation where the General Relative Entropy inequality holds true, let us mention the scattering (also called linear Boltzman) equation

$$(20)\qquad \frac{\partial}{\partial t}n(t,x) + k_T(x)\,n(t,x) = \int_{\mathbb{R}^d} k(y,x)\,n(t,y)\,dy.$$

Here we restrict ourselves to coefficients independent of time for simplicity, but the same inequality holds in the time dependent case as before. We assume that

$$0 \le k_T(\,\cdot\,) \in L^\infty(\mathbb{R}^d), \qquad 0 \le k(x,y) \in L^1 \cap L^\infty\left(\mathbb{R}^{2d}\right).$$

And we do not make special assumption on the symmetry of the cross-section $k(x,y)$ as motivated by turning kernels that appear in various applications as bacterial movement [2, 21, 34, 12, 41].

Again, we assume that there are solutions $N(x)$ and $\phi(x)$ to the stationary equation and its adjoint, namely

$$k_T(x)\,N(x) = \int_{\mathbb{R}^d} k(y,x)\;N(y)\;dy, \qquad N(x) > 0. \tag{21}$$

$$k_T(x)\,\phi(x) \;=\; \int_{\mathbb{R}^d} k(x,y)\;\;\phi(y)\;dy, \qquad \phi(x) > 0. \tag{22}$$

Then we have the

LEMMA 3.3:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\phi(x)\;N(x)\;H\left(\frac{n(x)}{N(x)}\right)\right] + \\
&\quad + \int_{\mathbb{R}^d} k(x,y)\left[\phi(y)N(x)H\left(\frac{n(t,x)}{N(x)}\right) - k(y,x)\phi(x)N(y)H\left(\frac{n(t,y)}{N(y)}\right)\right]dy \;= \\
&\quad = \int k(y,x)\phi(x)N(y)\left[H\left(\frac{n(t,x)}{N(x)}\right) - H\left(\frac{n(t,y)}{N(y)}\right) + H'\left(\frac{n(t,x)}{N(x)}\right)\left[\frac{n(t,y)}{N(y)} - \frac{n(t,x)}{N(x)}\right]\right]dy,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbb{R}^d}\left[\phi(x)\;N(x)\;H\left(\frac{n(x)}{N(x)}\right)\right] = \\
&= \int k(y,x)\phi(x)N(y)\left[H\left(\frac{n(t,x)}{N(x)}\right) - H\left(\frac{n(t,y)}{N(y)}\right) + H'\left(\frac{n(t,x)}{N(x)}\right)\left[\frac{n(t,y)}{N(y)} - \frac{n(t,x)}{N(x)}\right]\right]dy \le 0.
\end{aligned}$$

Again we leave to the reader the easy computation that leads to this result and just indicate a class of classical examples where N and ϕ can be computed explicitly.

EXAMPLE 1: We consider the case where the scattering cross-section is given by

$$k(x,y) = k_1(x)k_2(y).$$

Then we arrive at (up to a multiplicative constant)

$$N(x) = \frac{k_2(x)}{k_T(x)}, \qquad \phi(x) = \frac{k_1(x)}{k_T(x)},$$

and we need the compatibility condition

$$\int_{\mathbb{R}^d} \frac{k_2(x)k_1(x)}{k_T(x)} = 1.$$

As in the Perron-Frobenius case in Section 2.1, this means that 0 is the first eigenvalue, a condition that can always be met changing if necessary k_T in $\lambda + k_T$.

EXAMPLE 2: We consider the more general case where there exists a function $N(x) > 0$ such that the scattering cross-section satisfies the symmetry condition

$$k(x, y)N(x) = k(y, x)N(y).$$

Then the choice $k_T(x) = \int_{\mathbb{R}^d} k(x, y)dy$ gives the solutions $N(x)$ to (21), and $\phi(x) = 1$ to equation (22).

Again we conclude on long time convergence and the possibility to prove exponential decay to the steady state. As in Lemma 2.4, this follows from a control of entropy by entropy dissipation, i.e., still for the quadratic entropy, from the inequality

$$v \int \phi(x)N(x)\left(\frac{m}{N}\right)^2 dx \leq 2 \int_{\mathbb{R}^d} k(y, x)\phi(x)N(y)\left[\frac{m(x)}{N(x)} - \frac{m(y)}{N(y)}\right]^2 dy\, dx,$$

whenever

$$\int_{\mathbb{R}^d} \phi(x)m(x)dx = 0.$$

This is not always true but holds whenever (again we leave this as an exercise) we have

$$v\phi(x)N(y) \leq k(x, y),$$

a condition that is fulfilled for instance if we work on a bounded domain in velocity and k positive (the difficulties in practical examples as cell division is that ϕ needs not be bounded in unbounded domains and N can vanish at infinity).

4. - APPLICATION: A PARABOLIC SYSTEM FOR MOLECULAR BIOMOTORS

In several papers (see the review [32]), simple models for molecular motors have been derived where chemical energy is transformed in mechanical energy. The principle we consider here is that some molecule can reach two conformations (the density of each being denoted by n^1 and n^2 below). A bath a such molecules is moving in a filament and subject to two physical events. First, the filament induces a smooth, periodic and asymmetric potential seen differently by the two conformations (and denoted by $\psi^i(x)$, $i = 1, 2$ below). Second, fuel consumption triggers a conformational change between states 1 and 2 with rates denoted by $v^i > 0$ below. Being given that, at molecular scale, viscosity is important, this leads to the system of parabolic equations for the evolution of the densities $n^i(t, x)$

$$(23) \qquad \begin{cases} \dfrac{\partial}{\partial t} n^1 - \dfrac{\partial^2}{\partial x^2} n^1 - \dfrac{\partial}{\partial x}(\nabla \psi^1\, n^1) + v^1 n^1 = v^2 n^2, \qquad 0 \leq x \leq 1,\ t > 0 \\ \dfrac{\partial}{\partial t} n^2 - \dfrac{\partial^2}{\partial x^2} n^2 - \dfrac{\partial}{\partial x}(\nabla \psi^2 n^2) + v^2 n^2 = v^1 n^1, \\ \dfrac{\partial}{\partial x} n^i(x) - \nabla \psi^i\, n^i(x) = 0 \text{ at } x = 0, 1, \qquad i = 1, 2. \end{cases}$$

Notice that the zero flux condition makes this system conservative

$$\frac{d}{dt}\int_0^1 [n^1(t,x) + n^2(t,x)]dx = 0.$$

This can be interpreted in terms of our previous theory by noticing that in conservative cases the adjoint problem admits trivial solutions, here $\phi^1 = \phi^2 = Cst$, see (25) below.

This model, as well as several other biomotors, was analyzed in [14, 15, 26, 33]. Especially, in [14] it is proved that there is a positive steady state solution (N^1, N^2), that one can normalize by $\int_0^1 [N^1(x) + N^2(x)]dx = 1$.

$$(24)\qquad \begin{cases} -\dfrac{\partial^2}{\partial x^2}N^1 - \dfrac{\partial}{\partial x}(\nabla\psi^1\, N^1) + \upsilon^1 N^1 = \upsilon^2 N^2, & 0 \le x \le 1, \\ -\dfrac{\partial^2}{\partial x^2}N^2 - \dfrac{\partial}{\partial x}(\nabla\psi^2 N^2) + \upsilon^2 N^2 = \upsilon^1 N^1, & \\ \dfrac{\partial}{\partial x}N^i(x) - \nabla\psi^i\, N^i(x) = 0 \text{ at } x = 0, 1, & i = 1, 2. \end{cases}$$

A simple way to see this goes again through the adjoint system, which is given by

$$(25)\qquad \begin{cases} -\dfrac{\partial^2}{\partial x^2}\phi^1 + \nabla\psi^1\, \dfrac{\partial}{\partial x}\phi^1 + \upsilon^1\phi^1 = \upsilon^1\phi^2, & 0 \le x \le 1, \\ -\dfrac{\partial^2}{\partial x^2}\phi^2 + \nabla\psi^2\, \dfrac{\partial}{\partial x}\phi^2 + \upsilon^2\phi^2 = \upsilon^2\phi^1, & \\ \dfrac{\partial}{\partial x}\phi^i(x) = 0 \text{ at } x = 0, 1, & i = 1, 2. \end{cases}$$

As already mentioned, it admits the trivial solution $\phi^1 = \phi^2 = Cst$, which proves that 0 is the first eigenvalue.

The very deep result in [14], is that the system exhibits a motor effect (the densities are higher near $x = 1$ than near $x = 0$ as shown in Figure 2) under some precise asymmetry conditions on the potentials ψ^i and size conditions on the transition rates υ^i.

Our purpose here less ambitious and is to give, without structure conditions on ψ^i and υ^i, an extension to this system of the General Relative Entropy property (the structure behind is of course more general and relies on the coupling through zeroth order terms). It is not surprising that such systems also admit an entropy principle because they are positivity preserving. Another example is given in [19] for a system in population dynamic.

As a consequence we can study the solution of the parabolic system (23) and especially prove the following properties of the solution.

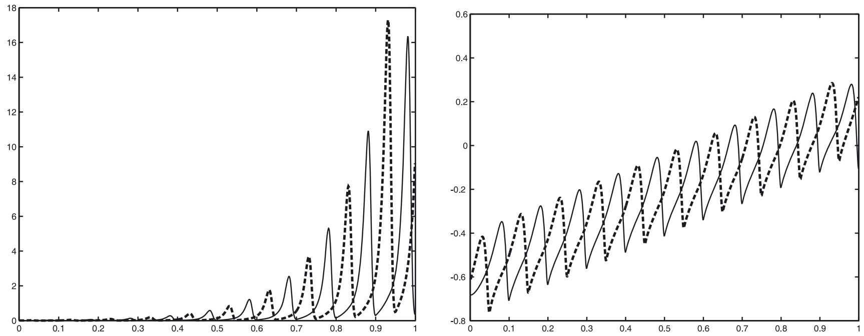

FIGURE 2. –Motor effect for assymetric potentials exhibited by the parabolic system (23). The figure represent the steady state given by (24) (left the densities themselves, right their logarithm). One can observe that the densities concentrate preferentially on one hand as proved in [14].

THEOREM 4.1: *Assume the potential ψ^i are smooth and $n^i(t=0) \geq 0$ are integrable and bounded. Then,*

$$0 \leq n^i(t,x) \leq \max\left(\left\| \frac{n^1(t=0,\cdot)}{N^1(\cdot)} \right\|_{L^\infty}, \left\| \frac{n^2(t=0,\cdot)}{N^2(\cdot)} \right\|_{L^\infty} \right) N^i(x), \qquad \forall x \in (0,1), \quad i = 1,2.$$

We define ρ by

$$\int_0^1 [n^1(t=0,x) + n^2(t=0,x)]dx = \rho \int_0^1 [N^1(x) + N^2(x)]dx,$$

then as $t \to \infty$, $\int_0^1 [|n^1(t,x) - \rho N^1(x)| + |n^2(t,x) - \rho N^2(x)|]dx$ decays to zero and

$$n^1(t,\cdot) \to \rho N^1(\cdot), \quad n^2(t,\cdot) \to \rho N^2(\cdot), \qquad \text{in } L^p(0,1),\ \forall p \in [1,\infty[.$$

Again, the proof relies on a General Relative Entropy property of the parabolic equation (23) that can be expressed as

LEMMA 4.2: *For all convex function $H : \mathbb{R} \to \mathbb{R}$, and all solutions (n^1, n^2) to (23), we have the General Relative Entropy Inequality*

$$\frac{d}{dt}\int_0^1 \left[N^1(x)\, H\left(\frac{n^1(t,x)}{N^1(x)}\right) + N^2(x)\, H\left(\frac{n^2(t,x)}{N^2(x)}\right) \right] dx =$$

$$= -\int \left[N^1\, H''\left(\frac{n^1(t,x)}{N^1(x)}\right) \left[\frac{\partial}{\partial x}\left(\frac{n^1}{N^1}\right)\right]^2 + N^2\, H''\left(\frac{n^2(t,x)}{N^2(x)}\right) \left[\frac{\partial}{\partial x}\left(\frac{n^2}{N^2}\right)\right]^2 \right] dx =$$

$$-\int \upsilon^2 N^2 \left[H'\left(\frac{n^1(t,x)}{N^1(x)}\right) \left[\frac{n^2(t,x)}{N^2(x)} - \frac{n^1(t,x)}{N^1(x)}\right] + H\left(\frac{n^1(t,x)}{N^1(x)}\right) - H\left(\frac{n^2(t,x)}{N^2(x)}\right) \right] dx -$$

$$-\int \upsilon^1 N^1 \left[H'\left(\frac{n^2(t,x)}{N^2(x)}\right) \left[\frac{n^1(t,x)}{N^1(x)} - \frac{n^2(t,x)}{N^2(x)}\right] + H\left(\frac{n^2(t,x)}{N^2(x)}\right) - H\left(\frac{n^1(t,x)}{N^1(x)}\right) \right] dx \leq$$

$$\leq 0.$$

PROOF OF LEMMA 4.2. Since this computation follows exactly that of the similar principle for a parabolic equation (23), we just indicate again the main intermediary steps without details. We have

$$\frac{\partial}{\partial t}\left(\frac{n^1}{N^1}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{n^1}{N^1}\right) + 2N^1\frac{\partial}{\partial x}\left(\frac{n^1}{N^1}\right)\frac{\partial}{\partial x}\left(\frac{1}{N^1}\right) - \frac{\partial \psi^1}{\partial x}\frac{\partial}{\partial x}\left(\frac{n^1}{N^1}\right) = v^2\frac{N^2}{N^1}\left[\frac{n^2}{N^2} - \frac{n^1}{N^1}\right].$$

Therefore, for any smooth function H, we arrive at

$$\frac{\partial}{\partial t}N^1H\left(\frac{n^1}{N^1}\right) - \frac{\partial^2}{\partial x^2}N^1H\left(\frac{n^1}{N^1}\right) + N^1H''\left(\frac{n^1}{N^1}\right)\left(\frac{\partial}{\partial x}\frac{n^1}{N^1}\right)^2 - \frac{\partial}{\partial x}\left[\frac{\partial \psi^1}{\partial x}N^1H\left(\frac{n^1}{N^1}\right)\right] =$$

$$= v^2N^2H'\left(\frac{n^1}{N^1}\right)\left[\frac{n^2}{N^2} - \frac{n^1}{N^1}\right] + (v^2N^2 - v^1N^1)H\left(\frac{n^1}{N^1}\right).$$

After adding the similar result for the quantity $\frac{\partial}{\partial t}N^2H\left(\frac{n^2}{N^2}\right)$ and integration in x, we arrive at the result. □

PROOF OF THEOREM 4.1. Again these statements are direct consequences of the entropy inequality of Lemma 4.2, and we just indicate the choice of the entropy function $H(\,\cdot\,)$.

For the L^∞ bound we set (as in Section 2.1)

$$C = \max\left(\left\|\frac{n^1(t=0,\cdot)}{N^1(\,\cdot\,)}\right\|_{L^\infty}, \left\|\frac{n^2(t=0,\cdot)}{N^2(\,\cdot\,)}\right\|_{L^\infty}\right)$$

and the choice $H(u) = (u - C)_+$ concludes. For the L^1 contraction it is enough to use $H(u) = |u|$. And the long time convergence again requires standard compactness arguments that can be found for instance in [40, 39]. □

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