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On the Double Well Duffing Equation with a Small Bounded Forcing Term

ABSTRACT. — By using differential inequalities and a kind of approximate maximum principle, a smallness condition on the bounded forcing term f is shown to imply the existence of exactly 3 different bounded solution which asymptote any solution of the Duffing equation $u'' + cu' + u^3 - u = f(t)$ as t tends to infinity, at least for c large enough. For small values of c a weaker property is obtained when f is T-periodic, namely a smallness condition on f ensuring the existence of exactly 3 different T-periodic solutions.

INTRODUCTION

In this paper we consider the second order ODE

$$u'' + cu' + u^3 - u = f(t) \tag{1}$$

where $c > 0$ and $f \in L^\infty(\mathbb{R})$ which is a reduced adimensional form for the more general-looking equation

$$x'' + \alpha x' + \beta^2(x^2 - a^2)x = g(t)\,. \tag{2}$$

In fact the transformation

$$x(t) = au(a\beta t) \Longleftrightarrow u(t) = \frac{1}{a} x\left(\frac{t}{a\beta}\right)$$

reduces (2) to (1) with

$$c := \frac{\alpha}{a\beta}, \quad f(t) := \frac{1}{a\beta^2} g\left(\frac{t}{a\beta}\right)$$

The equation (2) has several interesting interpretations in mechanics, cf. for instance F. C. Moon and P. J. Holmes [11]. This equation has been intensively studied from both theoretical and applied points of view. A first easy remark is that any solution u of (1) for

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$t \geq t_0$ is bounded as well as u' and u''. However it is well known that the solutions of (1) may have a very complicated behavior. For instance when f is periodic with minimal period T, there may in general exist, in addition to some T-periodic orbits, also periodic solutions with minimal period nT where $n \geq 2$, called in the literature "subharmonic" solutions. This bad phenomenon even happens in the better case where the nonlinear term $u^3 - u$ is replaced by $u^3 + u$ (cf eg. Cartwright & Littlewood [4], Loud [9], Hale & Taboas [6], and also Chow, Hale&Mallet-Paret [5] for a related problem). An interesting question is to find sufficient conditions on f and c preventing subharmonic and chaotic behaviors. This is the main object of the present work, where we show that a smallness condition on f ensures a less complicated behavior of the system, at least for c large enough. The plan of the paper is as follows: Section 1 contains the statement of the main results, Section 2 is devoted to the statement and proof of optimal estimates on the norm of a certain inverse operator in the space of bounded functiond on the line. In Sections 3, 4, 5 and 6 we give the proofs of the main results together with some remarks on optimality of the results.

1. - Main results

We start with a local result valid for arbitrary bounded forcing terms.

Theorem 1.1: *Under the condition*

$$\|f\|_\infty < \frac{2}{3\sqrt{3}} \tag{1.1}$$

equation (1) *has a unique solution* $\omega_0 \in W^{2,\infty}(\mathbb{R})$ *such that*

$$\|\omega_0\|_\infty < \frac{1}{\sqrt{3}}\,. \tag{1.2}$$

If in addition $c \geq 2\sqrt{2}$, *under the additional assumption*

$$\|f\|_\infty < 2\left(\frac{5}{3}\sqrt{\frac{5}{3}} - 2\right) \tag{1.3}$$

(1) *has a unique solution* ω_+ *and a unique solution* ω_- *in* $W^{2,\infty}(\mathbb{R})$ *such that*

$$\|\omega_+ - 1\|_\infty < \sqrt{\frac{5}{3}} - 1, \qquad \|\omega_- + 1\|_\infty < \sqrt{\frac{5}{3}} - 1\,. \tag{1.4}$$

If $c \leq 2\sqrt{2}$, *assuming the additional smallness condition*

$$\|f\|_\infty < \left(\sqrt{1 + \frac{c}{3\sqrt{2}}} - 1\right)\left(\frac{2c}{3\sqrt{2}} + 1 - \sqrt{1 + \frac{c}{3\sqrt{2}}}\right) := \eta(c) \tag{1.5}$$

(1) *has a unique solution* ω_+ *and a unique solution* ω_- *in* $W^{2,\infty}(\mathbb{R})$ *such that*

$$\|\omega_+ - 1\|_\infty < \sqrt{1+\frac{c}{3\sqrt{2}}} - 1, \qquad \|\omega_- + 1\|_\infty < \sqrt{1+\frac{c}{3\sqrt{2}}} - 1\,. \tag{1.6}$$

Finally if $c \leq 1$, *assuming*

$$\|f\|_\infty < \left(\sqrt{1+\frac{c}{3}} - 1\right)\left(\frac{2c}{3} + 1 - \sqrt{1+\frac{c}{3}}\right) := \eta_1(c) \tag{1.7}$$

(1) *has a unique solution* ω_+ *and a unique solution* ω_- *in* $W^{2,\infty}(\mathbb{R})$ *such that*

$$\|\omega_+ - 1\|_\infty < \sqrt{1+\frac{c}{3}} - 1, \qquad \|\omega_- + 1\|_\infty < \sqrt{1+\frac{c}{3}} - 1. \tag{1.8}$$

In some cases the local result of Theorem 1.1 can be refined to a global one under an additional smallness restriction on f.

THEOREM 1.2: *Under the conditions*

$$c \geq 2\sqrt{2}, \qquad f \in C_b(\mathbb{R}), \quad \|f\|_\infty < \frac{1}{32}\frac{c}{\sqrt{1+c^2}} \tag{1.9}$$

any solution u *of* (1) *on some halfline* $J = (t_0, +\infty)$ *is asymptotic to one of the* 3 *solutions* ω_0, ω_+, ω_- *as* $t \to +\infty$.

COROLLARY 1.3: *Under the hypotheses of Theorem* 1.2, *if* f *is almost periodic,* (1) *has exactly* 3 *almost periodic solutions* $\omega_0, \omega_+, \omega_-$. *Moreover if* f *is T-periodic then so are* ω_0, ω_+, ω_-.

COROLLARY 1.4: *Under the hypotheses of Theorem* 1.2, *if* f *is T- periodic, then* (1) *has no subharmonic periodic solution.*

Our last main result is restricted to T-periodic solutions

THEOREM 1.5: *Let* f *be bounded and T-periodic. Under the condition*

$$\|f\|_\infty\left(1 + \frac{T}{c}\sqrt{\frac{31}{4} + 12\frac{c^2+1}{c^2}\|f\|_\infty^2}\right) < \frac{2}{3}\inf\left\{\sqrt{1+\frac{c}{3\sqrt{2}}} - 1, \sqrt{\frac{5}{3}} - 1\right\} \tag{1.10}$$

equation (1) *has at most* 3 *T-periodic solutions.*

COROLLARY 1.6: *Let* f *be bounded and T-periodic satisfying both smallness conditions of Theorem* 1.1 *and Theorem* 1.5. *Then equation* (1) *has exactly three T- periodic solutions.*

REMARK 1.7: Under the hypotheses of Theorem 1.5 or Corollary 1.6, we do not know if there can exist any subharmonic solution of the equation.

2. - Norm estimates for the inverse of a second order operator

In this section we consider the second order differential operator defined on

$$X := L^\infty(\mathbb{R})$$

by

$$D(\mathcal{L}) = W^{2,\infty}(\mathbb{R}) = \{u \in C^1(\mathbb{R}), \quad u, u', u'' \in L^\infty(\mathbb{R})\} \tag{2.1}$$

$$\forall u \in D(\mathcal{L}), \quad \mathcal{L}u = u'' + cu' + \omega^2 u \tag{2.2}$$

for some positive constants c, ω. It is classical that for any $f \in X = L^\infty(\mathbb{R})$ there exists a unique solution u of $\mathcal{L}u = f$ which asymptotes exponentially fast all solutions of

$$u'' + cu' + \omega^2 u = f, \quad t \geq t_0$$

as $t \to +\infty$. We can therefore define

$$\mathcal{L}^{-1} : X \to D(\mathcal{L}) \subset X.$$

The estimate of the norm of $\mathcal{L}^{-1}$ in $L(X)$ is crucial for our purpose. For notational convenience we introduce

$$D(\partial) = W^{1,\infty}(\mathbb{R})$$

and

$$\forall u \in D(\partial), \quad \partial u = u'.$$

We have the following result.

Theorem 2.1: *For any positive constants c, ω, we have the following*

i) *If $c \geq 2\omega$, $\mathcal{L}^{-1}$ is order preserving and*

$$\|\mathcal{L}^{-1}\|_{L(X)} = \frac{1}{\omega^2} \tag{2.3}$$

ii) *If $c \leq 2\omega$,*

$$\frac{4}{\pi c\omega} \leq \|\mathcal{L}^{-1}\|_{L(X)} = \frac{1}{\omega^2} \coth\left\{\frac{c\pi}{2\sqrt{4\omega^2 - c^2}}\right\} \leq \frac{2}{c\omega} \tag{2.4}$$

and these inequalities are optimal for arbitrary $c \leq 2\omega$.

Proof: i) The positivity of $\mathcal{L}^{-1}$ in this case is well-known, cf. e.g.[1, 10]. Actually it is sufficient to remark that since

$$\mathcal{L} = (\partial + \beta I)(\partial + \alpha I)$$

with

$$\alpha = \frac{c + \sqrt{c^2 - 4\omega^2}}{2} \qquad \beta = \frac{c - \sqrt{c^2 - 4\omega^2}}{2}$$

we have

$$\mathcal{L}^{-1} = (\partial + aI)^{-1}(\partial + \beta I)^{-1}.$$

Then for any $f \in X$ we have

$$-\|f\|_X \le f \le \|f\|_X$$

therefore

$$-\frac{1}{\omega^2}\|f\|_X = -\mathcal{L}^{-1}\|f\|_X \le \mathcal{L}^{-1}f \le \mathcal{L}^{-1}\|f\|_X = \frac{1}{\omega^2}\|f\|_X$$

and i) is proved.

ii) In this case $\mathcal{L}^{-1}$ is no longer positive and the complex factorization

$$\mathcal{L}^{-1} = (\partial + aI)^{-1}(\partial + \beta I)^{-1}$$

only provides the weak estimate $\|\mathcal{L}^{-1}\|_{L(X)} \le \frac{4}{c^2}$ which is quite bad when $\frac{\omega}{c}$ is large. In order to evaluate the norm $\|\mathcal{L}^{-1}\|_{L(X)}$, we simply compute the operator $\mathcal{L}^{-1}$ in the following way. First the general solution of the homogeneous equation

$$v'' + cv' + \omega^2 v = 0$$

is given by the explicit formula

$$v(t) = C_1 e^{(-\gamma+i\delta)t} + C_2 e^{(-\gamma-i\delta)t}$$

with

$$\gamma = \frac{c}{2}, \quad \delta = \frac{1}{2}\sqrt{4\omega^2 - c^2}$$

and

$$C_1 = \frac{1}{2}\left(v(0) - i\frac{v'(0) + \gamma v(0)}{\delta}\right) \quad C_2 = \frac{1}{2}\left(v(0) + i\frac{v'(0) + \gamma v(0)}{\delta}\right).$$

The bounded solution of

$$u'' + cu' + \omega^2 u = f$$

is then given (cf. e.g. [7]) by the formula

$$(u(t), u'(t)) = \int_0^\infty T(\tau)(0, f(t-\tau)d\tau$$

with $T(t)$ the group of operators solving the homogeneous equation. Since the first component of

$$T(\tau)(0, f(t-\tau))$$

is

$$C_1(0, f(t-\tau))\, e^{(-\gamma+i\delta)\tau} + C_2(0, f(t-\tau))\, e^{(-\gamma-i\delta)\tau}$$

an easy calculation yields

$$[\mathcal{L}^{-1}f](t) = u(t) = \frac{1}{\delta}\int_0^\infty e^{-\gamma\tau}\sin(\delta\tau)f(t-\tau)d\tau \tag{2.5}$$

and in particular

$$\|\mathcal{L}^{-1}\|_{L(X)} \le \frac{1}{\delta}\int_0^\infty e^{-\gamma\tau}|\sin(\delta\tau)|d\tau.$$

Moreover the special choice

$$f(s) = -sgn(\sin(\delta s))$$

gives

$$u(0) = \frac{1}{\delta}\int_0^\infty e^{-\gamma\tau}|\sin(\delta\tau)|d\tau$$

and finally

$$\|\mathcal{L}^{-1}\|_{L(X)} = \frac{1}{\delta}\int_0^\infty e^{-\gamma\tau}|\sin(\delta\tau)|d\tau.$$

The computation of this integral is easy. Indeed

$$\int_0^\infty e^{-\gamma\tau}|\sin(\delta\tau)|d\tau = \sum_0^\infty \int_{\frac{n\pi}{\delta}}^{\frac{(n+1)\pi}{\delta}} e^{-\gamma\tau}|\sin(\delta\tau)|d\tau =$$

$$= \sum_0^\infty e^{-\frac{n\pi\gamma}{\delta}}\int_0^{\frac{\pi}{\delta}} e^{-\gamma t}\sin(\delta t)dt = \frac{1}{1-e^{\frac{-\pi\gamma}{\delta}}}\int_0^{\frac{\pi}{\delta}} e^{-\gamma t}\sin(\delta t)dt =$$

$$= \frac{1}{1-e^{\frac{-\pi\gamma}{\delta}}}\frac{1}{\delta}\int_0^\pi e^{\frac{-\gamma s}{\delta}}\sin s\, ds = \frac{1}{\delta\left(1+\frac{\gamma^2}{\delta^2}\right)}\frac{1+e^{\frac{-\pi\gamma}{\delta}}}{1-e^{\frac{-\pi\gamma}{\delta}}}$$

and therefore

$$\|\mathcal{L}^{-1}\|_{L(X)} = \frac{1}{\delta}\int_0^\infty e^{-\gamma\tau}|\sin(\delta\tau)|d\tau = \frac{1}{\gamma^2+\delta^2}\times\frac{1+e^{\frac{-\pi\gamma}{\delta}}}{1-e^{\frac{-\pi\gamma}{\delta}}}. \tag{2.6}$$

Since

$$\gamma^2+\delta^2 = \omega^2, \quad \frac{\gamma}{\delta} = \frac{c}{\sqrt{4\omega^2-c^2}}$$

we end up with

$$\|\mathcal{L}^{-1}\|_{L(X)} = \frac{1}{\omega^2} \times \frac{1 + e^{\frac{-c\pi}{\sqrt{4\omega^2 - c^2}}}}{1 - e^{\frac{-c\pi}{\sqrt{4\omega^2 - c^2}}}} = \frac{1}{\omega^2} \coth\left\{ \frac{c\pi}{2\sqrt{4\omega^2 - c^2}} \right\}. \tag{2.7}$$

Now we have

$$c\omega\|\mathcal{L}^{-1}\|_{L(X)} = g(t) := t \coth\left\{ \frac{\pi t}{2\sqrt{4 - t^2}} \right\}$$

where

$$t = \frac{c}{\omega}$$

the change of variable

$$\theta = \frac{t}{\sqrt{4 - t^2}} \Longleftrightarrow t = \frac{2\theta}{\sqrt{1 + \theta^2}}$$

gives

$$c\omega\|\mathcal{L}^{-1}\|_{L(X)} = h(\theta) := \frac{2\theta}{\sqrt{1 + \theta^2}} \coth \frac{\pi\theta}{2}.$$

A rather straightforward calculation shows that

$$h'(\theta) = \frac{k(\theta)}{(1 + \theta^2)^{\frac{3}{2}} sh^2\left(\frac{\pi\theta}{2}\right)}$$

with

$$k(\theta) = sh(\pi\theta) - \pi(\theta + \theta^3) \geq \frac{1}{6}(\pi\theta)^3 - \pi\theta^3 = \pi\theta^3\left(\frac{1}{6}\pi^2 - 1\right) > 0.$$

Hence h is increasing on $(0, +\infty)$ and therefore

$$\frac{4}{\pi} = \lim_{\theta \to 0} h(\theta) \leq 2c\omega\|\mathcal{L}^{-1}\|_{L(X)} \leq \lim_{\theta \to \infty} h(\theta) = 2.$$

REMARK 2.2: The proof of

$$\|\mathcal{L}^{-1}\|_{L(X)} \leq \frac{2}{c\omega} \tag{2.8}$$

from the exact formula is rather involved. It is is also possible to establish it directly by relying on a technique from [8, lemme 3.2.6] refined to deal with bounded solutions rather than exponential decay. For any solution u of (1) we have

$$\frac{d}{dt}(u'^2 + \omega^2 u^2 + cuu') = 2(u'' + \omega^2 u)u' + cu'^2 + cuu'' =$$

$$= 2u'(f - cu') + cu'^2 + cu(f - cu' - \omega^2 u) = -c(u'^2 + \omega^2 u^2 + cuu') + f(2u' + cu).$$

On the other hand the condition $c \leq 2\omega$ yields the inequality

$$(2u' + cu)^2 = 4u'^2 + c^2u^2 + 4cuu' \leq 4(u'^2 + \omega^2u^2 + cuu')$$

and we deduce

$$f(2u' + cu) \leq \frac{c}{8}(2u' + cu)^2 + \frac{2}{c}f^2 \leq \frac{c}{2}(u'^2 + \omega^2u^2 + cuu') + \frac{2}{c}f^2$$

hence

$$\frac{d}{dt}(u'^2 + \omega^2u^2 + cuu') \leq -\frac{c}{2}(u'^2 + \omega^2u^2 + cuu') + \frac{2}{c}f^2 .$$

Assuming u to be bounded on $\mathbb{R}$, classically u', u'' are also bounded and solving for the above differential inequality, we obtain with

$$\Phi(t) := (u'^2 + \omega^2u^2 + cuu')(t)$$

$$\forall t \geq s, \quad \Phi(t) \leq \exp\Big(-\frac{c}{2}(t-s)\Big)\Phi(s) + \frac{4}{c^2}\|f\|_\infty^2 .$$

Letting s tend to $-\infty$ we derive

$$\sup_{t\in R} \Phi(t) \leq \frac{4}{c^2}\|f\|_\infty^2 .$$

Finally let us consider an "asymptotically maximizing" sequence t_n such that

$$\lim_{n\to\infty} u^2(t_n) = \sup_{t\in R} u^2(t) .$$

Assuming this limit to be positive, since u'' is bounded for $t \geq 0$ it is clear that $\lim\limits_{n\to\infty} u'(t_n) = 0$, consequently we have

$$\omega^2 \lim_{n\to\infty} u^2(t_n) \leq \frac{4}{c^2}\|f\|_\infty^2 \Longrightarrow \sup_{t\in R} u^2(t) \leq \frac{4}{\omega^2c^2}\|f\|_\infty^2$$

which is equivalent to ii).

3. - Existence of 3 bounded solutions for f small

First we establish the existence of the "small" solution. We introduce the operator Λ such that

$$D(\Lambda) = W^{2,\infty}(\mathbb{R}) = \{u \in C^1(\mathbb{R}), \quad u, u', u'' \in L^\infty(\mathbb{R})\}$$

and

$$\forall u \in D(\Lambda), \quad \Lambda u = u'' + cu' - u$$

so that a bounded solution u of (1) is just a solution of

$$\Lambda u = u'' + cu' - u = f - u^3$$

Since $-\Lambda$ is an elliptic operator, it is clearly invertible on X and we have

$$\|\Lambda^{-1}\|_{L(X)} = 1\,.$$

We write the previous equation as

$$u = \Lambda^{-1}(f - u^3)\,.$$

Now the mapping

$$\mathcal{T}(v) = \Lambda^{-1}(f - v^3)$$

leaves invariant the ball

$$B_r = \{v \in X, \|v\|_X \leq r\}$$

as soon as

$$\|f\|_X + r^3 \leq r\,.$$

This is satisfied for some positive r whenever

$$\|f\|_X < \sup_{r>0}\,(r - r^3) = \frac{2}{3\sqrt{3}}\,.$$

Since the supremum is achieved for $r = r_0 := \dfrac{1}{\sqrt{3}}$ under the condition above there is $r < r_0$ such that

$$\mathcal{T}B_r \subset B_r\,.$$

Now since $r < \dfrac{1}{\sqrt{3}}$, on B_r the map $v \to v^3$ is a uniform X-contraction and so is $\mathcal{T}$. So there is a unique fixed point u of $\mathcal{T}$ in B_r, which is the solution of our problem. In addition we have $\|u\|_X < \dfrac{1}{\sqrt{3}}$.

For the two other solutions, due to the odd character of the non-linearity, by changing f to $(-f)$ we just need to study the existence of a second bounded solution close to 1. Setting u= 1+v, we are reduced to consider the equation

$$v'' + cv' + 2v = f - 3v^2 - v^3$$

that we rewrite in the form

$$v = \mathcal{L}^{-1}(f - 3v^2 - v^3)$$

where

$$\mathcal{L} = \partial^2 + c\partial + 2I$$

is the operator defined by (2.1)-(2.2) with $\omega = \sqrt{2}$. Then we distinguish 3 cases.

case 1: $c \geq 2\sqrt{2}$

Then $\|\mathcal{L}^{-1}\|_{L(X)} = \dfrac{1}{2}$ and therefore

$$\mathcal{T}(v) = \mathcal{L}^{-1}(f - 3v^2 - v^3)$$

leaves invariant the ball

$$B_r = \{v \in X, \|v\|_X \le r\}$$

as soon as

$$\frac{1}{2}(\|f\|_X + 3r^2 + r^3) \le r\,.$$

This is satisfied for some positive r whenever

$$\|f\|_X \le \sup_{r>0}\,(2r - 3r^2 - r^3) := M$$

An easy calculation shows that

$$M = 2r_0 - 3r_0^2 - r_0^3; \quad r_0 = \sqrt{\frac{5}{3}} - 1$$

In addition, since $6r_0 + 3r_0^2 = 2$ we have

$$\forall v \in (-r_0, r_0), \quad |(3v^2 + v^3)'| = |6v + 3v^2| < 2.$$

Therefore for any $r \in (0, r_0)$ such that

$$\|f\|_X \le (2r - 3r^2 - r^3)$$

we have $\mathcal{T}B_r \subset B_r$ and $\mathcal{T} : B_r \to B_r$ is a uniform contraction. The fixed point of $\mathcal{T}$ is the positive bounded solution we looked for. To finish the proof in this case, two additional remarks are necessary.

1) We have

$$M = r_0\left(\frac{4}{3} - r_0\right) = 2\left(\frac{5}{3}\sqrt{\frac{5}{3}} - 2\right) < \frac{2}{3\sqrt{3}}\,.$$

2) The solution near 0 and the solution near 1 are distinct since the second one is greater than $1 - \left(\sqrt{\frac{5}{3}} - 1\right) = 2 - \sqrt{\frac{5}{3}} > \frac{1}{\sqrt{3}}$.

case 2: $c \le 2\sqrt{2}$

Then $\|\mathcal{L}^{-1}\|_{L(X)} \le \dfrac{2}{c\omega} = \dfrac{\sqrt{2}}{c}$ and therefore

$$\mathcal{T}(v) = \mathcal{L}^{-1}(f - 3v^2 - v^3)$$

leaves invariant the ball

$$B_r = \{v \in X, \|v\|_X \le r\}$$

as soon as

$$\frac{\sqrt{2}}{c}(\|f\|_X + 3r^2 + r^3) \le r.$$

A computation similar to case 1 gives now the condition

$$\|f\|_X < \frac{c}{\sqrt{2}}r_1 - (3r_1^2 + r_1^3)$$

where

$$r_1 = \sqrt{1 + \frac{c}{3\sqrt{2}}} - 1 \leq r_0 \,.$$

Moreover $\mathcal{T}$ is still a contraction on B_r for $r < r_1$ and the positive solution, still strictly greater than the small one. The final condition on f in this case is

$$\|f\|_X < \left(\sqrt{1 + \frac{c}{3\sqrt{2}}} - 1\right)\left(\frac{2c}{3\sqrt{2}} + 1 - \sqrt{1 + \frac{c}{3\sqrt{2}}}\right) := \eta(c)\,.$$

case 3: $c \leq 1$

Using the exact formula, we prove

$$\|\mathcal{L}^{-1}\|_{L(X)} \leq \frac{1}{c}\,.$$

Indeed for ω fixed, $c\|\mathcal{L}^{-1}\|_{L(X)}$ is an increasing function of c, hence for $c \leq 1$

$$c\|\mathcal{L}^{-1}\|_{L(X)} \leq \frac{1}{2} \times \frac{1 + e^{\frac{-\pi}{\sqrt{7}}}}{1 - e^{\frac{-\pi}{\sqrt{7}}}} < 1$$

since $\dfrac{\pi}{\sqrt{7}} > \ln 3$. The conclusion follows as in case 2.

Remark 3.1: It is immediate to check that, as $c \to 0$

$$\eta_1(c) \sim \frac{c^2}{12}$$

whereas for $c = 1$, we obtain the sufficient condition

$$\|f\|_X \leq 0.08\,.$$

4. - Ultimate bound of the general solution.

In this section we derive a general, probably not optimal, estimate of the ultimate bound of the general solution of (1).

Proposition 4.1: *For any solution u of* (1) *we have*

$$\overline{\lim_{t\to\infty}}\, u^2(t) \leq 1 + \sqrt{1 + \frac{c^2 + 16}{4c^2}\|f\|_\infty^2}\,. \tag{4.1}$$

Proof: We introduce the energy $E(t) = \frac{1}{2}u'^2 + \frac{1}{4}u^4 - \frac{1}{2}u^2$. We have

$$E'(t) = fu' - cu'^2 \leq -\frac{3c}{4}u'^2 + \frac{f^2}{c}$$

and

$$(uu')' = u'^2 + uu'' = u'^2 + u(f - cu' - u^3 + u)$$

hence

$$\frac{d}{dt}\left(E + \frac{c}{4}uu'\right) \le -\frac{c}{2}u'^2 - \frac{c^2}{4}uu' - \frac{c}{4}u^4 + \frac{c}{4}u^2 + \frac{c}{4}uf + \frac{f^2}{c} =$$

$$= -c\left(E + \frac{c}{4}uu'\right) - \frac{c}{4}u^2 + \frac{c}{4}uf + \frac{f^2}{c} \le -c\left(E + \frac{c}{4}uu'\right) + \left(\frac{c}{16} + \frac{1}{c}\right)f^2 .$$

Let $M \ge 0$ be such that $E(t) + \frac{c}{4}uu' + M \ge 0$ and introduce $F(t) = E(t) + \frac{c}{4}uu' + M$. Then

$$F'(t) \le -cF(t) + cM + \left(\frac{c}{16} + \frac{1}{c}\right)\|f\|_\infty^2$$

hence

$$\overline{\lim_{t\to\infty}} F(t) \le M + \frac{c^2+16}{16c^2}\|f\|_\infty^2$$

which provides

$$\overline{\lim_{t\to\infty}}(E(t) + \frac{c}{4}uu') \le \frac{c^2+16}{16c^2}\|f\|_\infty^2 .$$

Now for any $\varepsilon > 0$ we have for t large enough

$$\left(E(t) + \frac{c}{4}uu'\right) \le \frac{c^2+16}{16c^2}\|f\|_\infty^2 + \varepsilon$$

finally let us consider an "asymptotically maximizing" sequence t_n such that

$$\lim_{n\to\infty} u^2(t_n) = \overline{\lim_{t\to\infty}} u^2(t) .$$

Assuming this limit to be positive, since u'' is bounded for $t \ge 0$ it is clear that $\lim_{n\to\infty} u'(t_n) = 0$, consequently for n large enough

$$\frac{1}{4}u^4(t_n) - \frac{1}{2}u^2(t_n) \le E(t_n) + \frac{\varepsilon}{2} \le \frac{c^2+16}{16c^2}\|f\|_\infty^2 + 2\varepsilon \Longrightarrow$$

$$\Longrightarrow (u^2(t_n) - 1)^2 \le 1 + \frac{c^2+16}{4c^2}\|f\|_\infty^2 + 2\varepsilon .$$

Now either $u^2(t_n) \le 1$ for an infinite number of integers n , in which case $\limsup u^2 \le 1$, or for n large enough we have $u^2(t_n) \ge 1$. In the second case the inequality above implies

$$\overline{\lim_{t\to\infty}} u^2(t) \le 1 + \sqrt{1 + \frac{c^2+16}{4c^2}\|f\|_\infty^2 + 2\varepsilon}$$

and since ε is arbitrary we obtain (4.1).

5. - A PRECISE ESTIMATE FOR c LARGE

When $c \geq 2\sqrt{2}$, the inequality (4.1) and the positivity preserving property of $\mathcal{L}^{-1}$ allow a more precise estimate on u for t large.

PROPOSITION 5.1: *For any $c \geq 2\sqrt{2}$, we have*

$$\overline{\lim_{t\to\infty}} |u(t)| \leq 1 + \frac{1}{2}\|f\|_\infty \tag{5.1}$$

valid whenever $\|f\|_\infty \leq 2$.

PROOF: When $c \geq 2\sqrt{2}$, according to section 2, the operator $\mathcal{L} = \partial^2 + c\partial + 2I$ has positive inverse on L^∞. In addition the estimate (4.1) here provides

$$\overline{\lim_{t\to\infty}} u^2(t) \leq 1 + \sqrt{1 + \frac{3}{4}\|f\|_\infty^2}\,.$$

In particular if we assume

$$\|f\|_\infty \leq 2$$

then we find

$$\overline{\lim_{t\to\infty}} u^2(t) \leq 3\,.$$

Now if u is any solution of (1) we set $u = 1 + v$ so that

$$v'' + cv' + 2v + 3v^2 + v^3 = f\,.$$

Since $u \geq -2$ for t large we have $3v^2 + v^3 = v^2(3 + v) \geq 0$ for t large. We claim that

$$\overline{\lim_{t\to\infty}} u(t) \leq 1 + \frac{1}{2}\|f\|_\infty\,.$$

Assuming that this inequality is false, we can select $\delta > 0$ and t_n some sequence tending to $+\infty$ such that

$$u(t_n) \geq 1 + \frac{1}{2}\|f\|_\infty + \delta\,.$$

Now replace v by $v(t + t_n)$ and f by $f(t + t_n)$. We can then pass to the limit along a subsequence, for which the sequence of translates of f converges in L^2 weak, the limit being of course in the same ball of L^∞. We can also assume that the translates of v converge in C^1, then the limiting function fulfills the limiting equation. Finally we are reduced to consider the case where v is bounded on $\mathbb{R}$. In this case since

$$v'' + cv' + 2v \leq \|f\|_\infty$$

we obtain

$$v \leq \frac{1}{2}\|f\|_\infty$$

in contradiction with

$$v(0) \geq \frac{1}{2}\|f\|_\infty + \delta$$

thereby proving the claim. By considering in the same way $w = 1 + u$ and deriving the lower estimate for w we conclude

$$\overline{\lim_{t\to\infty}}|u(t)| \leq 1 + \frac{1}{2}\|f\|_\infty$$

valid whenever

$$c \geq 2\sqrt{2}, \qquad \|f\|_\infty \leq 2\,.$$

REMARK 5.2: The result of Proposition 5.1 is no longer true for small values of c. Indeed for ε small , let us consider the solution u of

$$u'' + u^3 - u = 0$$

with

$$u(0) = 1 + \varepsilon, \quad u'(0) = 0\,.$$

It is well-known that for ε small enough u is periodic and in particular

$$\overline{\lim_{t\to\infty}}|u(t)| = \|u\|_\infty \geq 1 + \varepsilon\,.$$

On the other hand u is positive and the maximum of $|u'|$ is achieved when $u'' = u - u^3 = 0$, equivalent to $u = 1$. By the energy identity

$$u'^2 + \frac{1}{2}u^4 - u^2 = \frac{1}{2}u_0^4 - u_0^2$$

we find

$$\|u'\|_\infty^2 = \frac{1}{2}(u_0^4 - 1) - (u_0^2 - 1) \sim 2\varepsilon^2$$

hence

$$\|u'\|_\infty \sim \varepsilon\sqrt{2}\,.$$

Now u is also a solution of

$$u'' + u^3 - u + cu' = cu' := f$$

and

$$\|f\|_\infty \sim c\varepsilon\sqrt{2}\,.$$

So we see that for any $c < \sqrt{2}$ the result of Proposition 5.1 is not valid. In addition, no estimate of the form

$$\overline{\lim_{t\to\infty}}|u(t)| \leq 1 + C\|f\|_\infty$$

is valid with C fixed as c tends to 0.

REMARK 5.3: When this paper is written we do not know whether an estimate of the form

$$\overline{\lim_{t\to\infty}}|u(t)| \le 1 + C(c)\|f\|_\infty$$

is valid for c small. This would be enough to generalize Theorem 1.2 (under relevant smallness assumptions) for c small , whereas at the present time the author can only achieve that for values of c slightly smaller than $c_0 = 2\sqrt{2}$.

6. - PROOF OF THEOREM 1.2 AND ITS COROLLARIES

One of the main ingredients of the proof is a precise formulation of the asymptotic stability of the bounded solutions ω_+, ω_-. Of course, changing u and f to their opposites it is sufficient to consider ω_+. In this case we have

LEMMA 6.1: *Assume*

$$\|f\|_X < \frac{1}{32}\frac{c}{\sqrt{1+c^2}} \,. \tag{6.1}$$

Then for any $\delta < \frac{1}{16}$, *there exists* $\eta > 0$ *such that the conditions*

$$|u(t_0) - 1| \le \delta \quad \textit{and} \quad |u'(t_0)| \le \eta$$

imply

$$\forall t \ge t_0, \quad |u(t) - 1| \le \frac{1}{4} \tag{6.2}$$

and

$$\overline{\lim_{t\to+\infty}}|u(t) - 1| \le \frac{4\sqrt{2}\sqrt{1+c^2}}{c}\|f\|_X \,. \tag{6.3}$$

In addition if $c \ge 2\sqrt{2}$, *under the same assumptions we have*

$$\lim_{t\to+\infty} (|u(t) - \omega_+(t)| + |u'(t) - \omega'_+(t)|) = 0 \,. \tag{6.4}$$

PROOF: By setting $u = 1 + v$ we obtain the equation for v

$$v'' + cv' + 2v + 3v^2 + v^3 = f \,.$$

Introducing

$$P(v) = v^2 + v^3 + \frac{v^4}{4} = v^2\left(\frac{v}{2} + 1\right)^2 .$$

We remark that

$$|v| \le \frac{1}{4} \Longrightarrow \frac{1}{2}v^2 \le P(v) \le \frac{3}{2}v^2$$

we now introduce

$$F(t) = \frac{1}{2} v'^2(t) + P(v)(t)$$

and

$$\Phi(t) = F(t) + avv'(t)$$

where $a > 0$ will be chosen later. First we notice that if $a < \frac{1}{4}$, we have

$$|avv'| \leq \left|\frac{1}{4} vv'\right| \leq \frac{1}{8}(v^2 + v'^2)$$

so that

$$\frac{1}{4}(v^2 + v'^2) \leq \Phi(t) = F(t) + avv'(t) \leq 2(v^2 + v'^2)$$

whenever the condition

$$|v| \leq \frac{1}{4}$$

is fulfilled. Let

$$T = \sup\left\{ t \geq t_0,\ |v(t)| \leq \frac{1}{4} \right\}$$

and $J := [t_0, T)$. We now derive a sequence of estimates valid for $t \in J$. We have

$$F'(t) = -cv'^2 + fv' \leq -\frac{c}{2} v'^2 + \frac{1}{2c} f^2$$

$$(vv')' = v'^2 + vv'' = v'^2 - cvv' - (2v^2 + 3v^3 + v^4) + fv\,.$$

Since $v \geq -\frac{1}{4}$ on J we find

$$(vv')' \leq v'^2 - cvv' - \left(\left(1 + \frac{1}{4}\right)v^2 + v^4\right) + fv \leq v'^2 - cvv' - v^2 - v^4 + f^2$$

and by using

$$-cvv' \leq \frac{1}{2} v^2 + \frac{c^2}{2} v'^2$$

we deduce

$$\Phi' \leq \left(a + \frac{1}{2c}\right) f^2 + \left(-\frac{c}{2} + a + a\frac{c^2}{2}\right) v'^2 - \frac{a}{2} v^2.$$

We select

$$a = \frac{c}{4 + 2c^2}$$

so that $a\left(1+\frac{c^2}{2}\right)=\frac{c}{4}$ and

$$\Phi' \le \left(a+\frac{1}{2c}\right)f^2 - \frac{c}{4}v'^2 - \frac{a}{2}v^2 \le -\frac{a}{2}(v^2+v'^2) + \left(a+\frac{1}{2c}\right)f^2$$

therefore we find

$$\forall t \in J, \quad \Phi' \le -\frac{a}{4}\Phi + \left(a+\frac{1}{2c}\right)f^2$$

which is easily integrated to give

$$\forall t \in J, \quad \Phi(t) \le \exp\left(-\frac{a}{4}(t-t_0)\right)\Phi(t_0) + \frac{4a+\frac{2}{c}}{a}\|f\|_\infty^2 .$$

In order to achieve $T=\infty$, we need to ensure $|v|<\frac{1}{4}$ on J, which is satisfied as soon as

$$\Phi(t_0) + \frac{4a+\frac{2}{c}}{a}\|f\|_\infty^2 \le \frac{1}{4}\left(\frac{1}{4}\right)^2 = \frac{1}{64} .$$

To achieve this condition it is sufficient to ensure

$$\Phi(t_0) \le \frac{1}{128} \quad \text{and} \quad \frac{4a+\frac{2}{c}}{a}\|f\|_\infty^2 \le \frac{1}{128} .$$

The first condition is satisfied whenever $2v^2(t_0) < \frac{1}{128}$ and $2v'^2(t_0) \le \frac{1}{128} - 2v^2(t_0)$ which corresponds to our hypothesis. The second condition is equivalent to

$$\|f\|_\infty^2 \le \frac{\frac{c}{4+2c^2}}{128\left(\frac{2}{c}+\frac{4c}{4+2c^2}\right)} = \frac{c^2}{1024(c^2+1)}$$

or

$$\|f\|_X \le \frac{1}{32}\frac{c}{\sqrt{1+c^2}} .$$

Under these conditions we have $T=\infty$ and

$$\forall t \ge t_0, \quad \Phi(t) \le \exp\left(-\frac{a}{4}t\right)\Phi(t_0) + \frac{4a+\frac{2}{c}}{a}\|f\|_\infty^2$$

and the inequality

$$|u(t)-1| \le 2\Phi(t)^{1/2}$$

together with

$$\frac{4a + \frac{2}{c}}{a} = 4 + \frac{2}{ca} = 4 + \frac{2}{c}\frac{4 + 2c^2}{c} = 8\frac{c^2 + 1}{c^2}$$

gives the final estimate. To prove the second part, we observe that the asymptotic distance between u and 1 is less than

$$\frac{1}{4} < \sqrt{\frac{5}{3}} - 1 = r_0 \,.$$

Then we claim that u asymptotes ω_+ as t tends to infinity and since u'' is bounded in the Stepanov space S^2 it will follow that u' asymptotes ω'_+. We now use the translation-compactness method developed in the almost periodic setting by Amerio [2] and Biroli [3]. Assuming, by contradiction, the existence of a_n tending to infinity with

$$\lim_{n\to\infty} |u(a_n) - \omega_+(a_n)| = \eta > 0$$

we can replace a_n by a subsequence, still denoted a_n for convenience, such that

$$u(a_n + t), \quad \omega_+(a_n + t), \quad f(a_n + t)$$

converge respectively to v, w and g on $\mathbb{R}$, uniformly on compacta for the first two functions, in local L^2 weak for the third. Then v, w are two bounded solutions of

$$z'' + cz' + z^3 - z = g$$

with

$$Max\,\{\|v - 1\|_\infty, \|w - 1\|_\infty\} < r_0 \,.$$

In particular $v = w$ and for $t = 0$ we obtain a contradiction with

$$\lim_{n\to\infty} |u(a_n) - \omega_+(a_n)| = \eta > 0 \,.$$

This contradiction proves the claim and completes the proof of Lemma 6.1.

In order to complete the proof of Theorem 1.2, we need the following result

LEMMA 6.2: *Let $J = (a, +\infty)$ and $u \in C^2(J)$ be such that $u \le M$ on J. Let*

$$U := \overline{\lim_{t\to+\infty}}\, u(t) \,.$$

Then there exists a sequence of reals $t_n \in J$ such that $t_n \to +\infty$ and

$$\overline{\lim_{n\to+\infty}}\, u''(t_n) \le 0, \quad \lim_{n\to\infty} u(t_n) = U$$

PROOF: It is obviously enough to prove that

$$\forall \varepsilon > 0, \forall T \ge a, \quad \exists t \ge T \quad \text{with} \quad u(t) \ge U - \varepsilon \quad \text{and} \quad u''(t) \le \varepsilon \,.$$

Assume on the contrary for some T and $\varepsilon > 0$

$$\forall t \ge T,\ u(t) \ge U - \varepsilon \Longrightarrow u''(t) > \varepsilon \,.$$

Pick any $t_0 \geq T$ such that $u(t_0) \geq U - \varepsilon$. If $u'(t_0) \geq 0$, since $u''(t_0) > 0$ we have $u > U - \varepsilon$ on $[t_0, t_0 + \delta]$ for some $\delta > 0$. We claim that $u > U - \varepsilon$ on $[t_0, +\infty]$. Assuming the contrary let

$$T = \inf\{t \geq t_0, \quad u(t) \leq U - \varepsilon\}.$$

Then since $u'' > 0$ on $[t_0, T]$ we have $u' > 0$ on $[t_0, T]$ and in particular $u(T) \geq u(t_0 + \delta) > U - \varepsilon$, contradicting the definition of T. Now since $u > U - \varepsilon$ on $[t_0, +\infty]$, u is increasing and strictly convex on $[t_0, +\infty]$. Therefore u tends to $+\infty$ with t, contrary to the assumption . As a consequence

$$\forall t \geq T,\ u(t) \geq U - \varepsilon \Longrightarrow u''(t) > \varepsilon,\ u'(t) < 0.$$

Now let $T_1 > T$ be such that $u(T_1) \geq U - \varepsilon$ and so large that

$$\frac{\varepsilon}{2}(T_1 - T)^2 + U - \varepsilon > M.$$

A simple continuation argument as above shows that

$$\forall t \in [T, T_1], \quad u(t) \geq U - \varepsilon, \quad u'(t) < 0.$$

As a consequence

$$\forall t \in [T, T_1], \quad u''(t) > \varepsilon.$$

By integrating

$$\forall t \in [T, T_1], \quad u'(t) \leq -\varepsilon(T_1 - t)$$

and by integrating once more

$$u(t) \geq u(T_1) + \frac{\varepsilon}{2}(T_1 - t)^2.$$

Taking $t = T$ we obtain

$$u(T) \geq u(T_1) + \frac{\varepsilon}{2}(T_1 - T)^2 \geq U - \varepsilon + \frac{\varepsilon}{2}(T_1 - T)^2 > M,$$

a contradiction showing that our initial assumption cannot be satisfied and concludes the proof of Lemma 6.2.

In the proof of theorem 1.2, we shall use the following simple lemma

Lemma 6.3: *For any $\varepsilon > 0$, the inequality $u - u^3 \leq \varepsilon$ implies either $u \leq \frac{3\varepsilon}{2}$ or $u \geq 1 - \frac{4\varepsilon}{3}$*

Proof: If $u \leq 0$ there is nothing to prove. If $u > 0$ we distinguish 2 cases

i) If $u < \frac{1}{\sqrt{3}}$, then $1 - u^2 > \frac{2}{3}$ and therefore

$$u - u^3 = u(1 - u^2) \leq \varepsilon \Longrightarrow \frac{2}{3}u \leq \varepsilon \Longrightarrow u \leq \frac{3}{2}\varepsilon$$

ii) If $u \geq \frac{1}{\sqrt{3}}$, then $u > \frac{1}{2}$ and therefore

$$u - u^3 = (u + u^2)(1 - u) \leq \varepsilon \Longrightarrow \frac{3}{4}(1 - u) \leq \varepsilon \Longrightarrow u \geq 1 - \frac{4}{3}\varepsilon.$$

Proof of Theorem 1.2: Let u be a solution of (1.2) on $\mathbb{R}$ and introduce

$$M = \overline{\lim_{t\to+\infty}}\, u(t), \quad m = \liminf_{t\to+\infty} u(t), \quad \varepsilon = \|f\|_\infty .$$

As a consequence of Lemma 6.2, there exists a sequence of reals t_n such that

$$\lim_{n\to+\infty} u''(t_n) \leq 0, \quad \lim_{n\to\infty} u(t_n) = M .$$

Since u'' is bounded, it follows easily that

$$\lim_{n\to\infty} u'(t_n) = 0 .$$

Now we have

$$(u - u^3)(t_n) = -f(t_n) + u''(t_n) + u'(t_n)$$

and therefore

$$\overline{\lim_{n\to\infty}}\, (u - u^3)(t_n) \leq \varepsilon .$$

As a consequence of Lemma 6.2, for n large enough we have either

$$u(t_n) \leq 2\varepsilon$$

or

$$u(t_n) \geq 1 - 2\varepsilon .$$

In the first case we conclude

$$M \leq 2\varepsilon .$$

In the second case we have in fact, by virtue of Section 5

$$1 - 2\varepsilon \leq u(t_n) \leq 1 + \frac{\varepsilon}{2} .$$

As a consequence of Lemma 6.1, since $2\varepsilon < \dfrac{1}{16}$ and since $\lim_{n\to\infty} u'(t_n) = 0$, we conclude that u is asymptotic to ω_+ at $+\infty$. In this case the proof is over.

Coming back to the first case, we now consider a sequence s_n such that

$$u''(s_n) \geq 0, \quad \lim_{n\to\infty} u(s_n) = m$$

and by the same argument as above we conclude that either u is asymptotic to ω_- at $+\infty$, or

$$m \geq -2\varepsilon .$$

In this second and last case we have

$$\overline{\lim_{t\to+\infty}}\, |u(t)| \leq 2\varepsilon$$

and by our hypothesis on f this implies

$$\lim_{t\to\infty} |u(t) - \omega_0(t)| = 0 .$$

We prove this last property using again the translation method of Amerio-Biroli. Indeed assuming, on the contrary, the existence of a_n tending to infinity with

$$\lim_{n\to\infty} |u(a_n) - \omega_0(a_n)| = \eta > 0$$

we can replace a_n by a subsequence, still denoted a_n for convenience, such that

$$u(a_n + t), \quad \omega_0(a_n + t), \quad f(a_n + t)$$

converge respectively to v, w and g on $\mathbb{R}$, uniformly on compacta for the first two functions, in local L^2 weak for the third. Then v, w are two bounded solutions of

$$z'' + cz' + z^3 - z = g$$

with

$$Max\{\|v\|_\infty, \|w\|_\infty\} \le 2\varepsilon.$$

In particular $v = w$ and for $t = 0$ we obtain a contradiction with

$$\lim_{n\to\infty} |u(a_n) - \omega_0(a_n)| = \eta > 0\,.$$

This contradiction proves the claim and completes the proof of the Theorem 1.2.

Proof of Corollary 1.3: Any almost periodic solution is asymptotic to one of the 3 solutions $\omega_0, \omega_+, \omega_-$ as $t \to +\infty$. The result follows then from the fact that an almost periodic function tending to 0 at infinity is identically 0. The periodicity statement comes from the fact that $\omega_0, \omega_+, \omega_-$ are T-periodic.

Proof of Corollary 1.4: Immediate from Corollary 1.3.

7. - Proof of Theorem 1.5

First we show that for a *fixed period* T, T-periodic solutions u are such that $\|u^3 - u\|_\infty$ tends to 0 with $\|f\|_\infty$.

Proposition 7.1: *Let f be bounded, T-periodic and let $u \in C^2(\mathbb{R})$ be a T-periodic solution of (1) . Then we have the estimate*

$$\|u^3 - u\|_\infty \le \|f\|_\infty \left(1 + \frac{T}{c}\sqrt{\frac{31}{4} + 12\frac{c^2+1}{c^2}\|f\|_\infty^2}\right) \tag{7.1}$$

Proof: By integrating (1) on $J := (0, T)$ we find

$$\int_J (u^3 - u)dt = \int_J f dt$$

in particular

$$\left|\frac{1}{T}\int_J (u^3 - u)dt\right| \le \|f\|_\infty .$$

Then multiplying (1) by u' and integrating on J we get

$$c\int_J u'^2\,dt = \int_J fu'\,dt \Longrightarrow c\int_J u'^2\,dt \le \left(\int_J u'^2\,dt\right)^{1/2}\left(\int_J f^2\,dt\right)^{1/2}$$

$$\int_J u'^2\,dt \le \frac{1}{c^2}\int_J f^2\,dt \le \frac{T}{c^2}\|f\|_\infty^2 .$$

Hence

$$\|u'\|_2 \le \frac{\sqrt{T}}{c}\|f\|_\infty .$$

Next multiplying (1) by u and integrating on J

$$\int_J (u^4 - u^2)dt = \int_J fudt + \int_J u'^2 dt \Longrightarrow \int_J (u^4 - u^2)dt \le \int_J fudt + \frac{T}{c^2}\|f\|_\infty^2 .$$

Hence

$$\int_J (u^4 - \frac{5}{4}u^2)dt \le \int_J f^2 dt + \frac{T}{c^2}\|f\|_\infty^2 \le T\left(\frac{c^2+1}{c^2}\right)\|f\|_\infty^2 .$$

On the other hand by Cauchy-Schwarz

$$\int_J u^2 dt \le \frac{1}{3}\int_J u^4 dt + \frac{3T}{4} \Longrightarrow \frac{3}{4}\int_J u^2 dt \le \frac{1}{4}\int_J u^4 dt + \frac{9T}{16} .$$

By addition we find

$$\frac{3}{4}\int_J u^4 dt - \frac{1}{2}\int_J u^2 dt \le T\left(\frac{c^2+1}{c^2}\right)\|f\|_\infty^2 + \frac{9T}{16}$$

multiplying by 12

$$9\int_J u^4 dt - 6\int_J u^2 dt \le 12T\left(\frac{c^2+1}{c^2}\right)\|f\|_\infty^2 + \frac{27T}{4}$$

adding T

$$\|3u^2 - 1\|_2^2 \le 12T\left(\frac{c^2+1}{c^2}\right)\|f\|_\infty^2 + \frac{31T}{4} .$$

Finally we find

$$\|u^3 - u\|_\infty \le \|f\|_\infty + \|(u^3 - u)'\|_1 \le \|f\|_\infty + \|3u^2 - 1\|_2\|u'\|_2$$

$$\|u^3 - u\|_\infty \le \|f\|_\infty\left(1 + \frac{\sqrt{T}}{c}\sqrt{12T\left(\frac{c^2+1}{c^2}\right)\|f\|_\infty^2 + \frac{31T}{4}}\right).$$

And (7.1) follows.

In order to prove Theorem 1.5, the following simple lemma is useful.

PROPOSITION 7.2: *For any $\varepsilon > 0$, the inequality $|u^3 - u| \le \varepsilon$ implies*

$$\inf\{|u|,\ |1-u|,\ |1+u|\} \le \frac{3\varepsilon}{2}.$$

PROOF: i) If $|u| < \frac{1}{\sqrt{3}}$, then $1 - u^2 > \frac{2}{3}$ and therefore

$$|u^3 - u| = |u||1-u^2| \le \varepsilon \Longrightarrow \frac{2}{3}|u| \le \varepsilon \Longrightarrow |u| \le \frac{3}{2}\varepsilon$$

ii) If $|u| \ge \frac{1}{\sqrt{3}}$, then $|u| > \frac{1}{2}$ and therefore

$$|1-u^2| = |1-|u|^2| = |1+|u|||1-|u|| \le 2\varepsilon \Longrightarrow \frac{3}{2}|1-|u|| \le 2\varepsilon$$

hence

$$|1-|u|| \le \frac{4}{3}\varepsilon \le \frac{3}{2}\varepsilon$$

and the result follows since

$$|1-|u|| = \inf\{|1-u|,\ |1+u|\}.$$

PROOF OF THEOREM 1.5: Under the hypothesis (1.10), as a consequence of Proposition 7.1 and lemma 7.2, any T-periodic solution u of (1) satisfies, for each t,

$$\inf\{|u(t)|\ ,|1-u(t)|\ ,|1+u(t)|\} \le \inf\left\{\sqrt{1+\frac{c}{3\sqrt{2}}} - 1,\ \sqrt{\frac{5}{3}} - 1\right\}.$$

Since u is continuous and the 3 closed intervals centered at $0, 1, -1$ with radius

$$\rho = \inf\left\{\sqrt{1+\frac{c}{3\sqrt{2}}} - 1,\ \sqrt{\frac{5}{3}} - 1\right\}$$

are disjoint, we have either

$$\|u-1\|_\infty \leq \inf\left\{\sqrt{1+\frac{c}{3\sqrt{2}}}-1, \sqrt{\frac{5}{3}}-1\right\}$$

in which case $u=\omega_+$ or

$$\|u+1\|_\infty \leq \inf\left\{\sqrt{1+\frac{c}{3\sqrt{2}}}-1, \sqrt{\frac{5}{3}}-1\right\}$$

in which case $u=\omega_-$, or

$$\|u\|_\infty \leq \inf\left\{\sqrt{1+\frac{c}{3\sqrt{2}}}-1, \sqrt{\frac{5}{3}}-1\right\} < \frac{1}{\sqrt{3}}$$

in which case $u=\omega_0$.

REFERENCES

[1] J.M. Alonso - J. Mawhin - R. Ortega, *Bounded solutions of second order semilinear evolution equations and applications to the telegraph equation*, J. Math. Pures Appl., 78 (1999), 49-63.
[2] L. Amerio, *Soluzioni quasi periodiche, o limitate, di sistemi differenziali non lineari quasi periodici, o limitati*, Ann. Mat. Pura Appl. 39 (1955), 97-119.
[3] M. Biroli, *Sur les solutions bornées et presque périodiques des équations et inéquations d'évolution*, Ann. Mat. Pura Appl. 93 (1972), 1-79.
[4] M.L. Cartwright - J.E. Littlewood, *On non-linear differential equations of the second order*, Ann. Math. 48 (1947), 472-494.
[5] S.N. Chow - J.K. Hale - J. Mallet-Paret, *An example of bifurcation to homoclinic orbits*, J.D.E. 37 (1980), 351-373.
[6] J.K. Hale - P.Z. Taboas, *Interaction of damping and forcing in a second order evolution equation*, Nonlinear analysis, T.M.A. 2, 1 (1978), 77-84.
[7] A. Haraux, *Nonlinear evolution equations: Global behavior of solutions*, Lecture Notes in Math. 841, Springer (1981)
[8] A. Haraux, *Systèmes dynamiques dissipatifs et applications, R.M.A.*, 17, P.G. Ciarlet et J.L. Lions (eds.), Masson, Paris, 1991.
[9] W.S. Loud, *Periodic solutions of x''+cx' +g(x) = f(t)*, Mem. Amer. Math. Soc., 31, 1959, 1-57.
[10] J. Mawhin - R. Ortega - A.M. Robles-Perez, *A maximun principle for bounded solutions of the telegraph equations and applications to nonlinear forcings*, J. Math. Anal. Appl. 251 (2000), 695-709.
[11] F.C. Moon - P.J. Holmes, *A magnetoelastic strange attractor, Journal of Sound and Vibration*, 65, 2 (1979), 275-296.

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