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Some Remarks on the Compatibility Conditions in Elasticity

ABSTRACT. — We compare different forms of the Saint Venant's equations of compatibility and of the related Donati's theorem and we give an extension to L^p of Ting's version of this theorem. Using a general result on the completeness of the self-equilibrated Beltrami's functions we give a different proof of some results of P.G. Ciarlet and P. Ciarlet, Jr.

1. - INTRODUCTION

Let Ω be an open, connected and bounded domain in $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$. Relative to an orthonormal cartesian basis $\{\boldsymbol{e}_i\}$, $(i = 1, 2, 3)$, the coordinates of a generic point will be denoted by $\{x_1, x_2, x_3\}$, the components of a vector field $\boldsymbol{v}$ by v_i and the components of a second-order tensor field $\boldsymbol{S}$, by S_{ij}. Latin indices range in the set $\{1, 2, 3\}$. The summation convention with respect to the repeated indices is used.
Let $\boldsymbol{E}$ be a smooth symmetric second-order tensor field. It was discovered by A. J.C. B. de Saint Venant (1864) the following result:

THEOREM 1 (Saint Venant's compatibility theorem). *The strain field $\boldsymbol{E}$ corresponding to a class $C^3(\Omega)$ displacement vector field $\boldsymbol{v}$ satisfies the compatibility equations:*

$$rot\, rot\, \boldsymbol{E} = 0. \tag{1}$$

Conversely, if Ω is a simply-connected domain, and if $\boldsymbol{E}$ is a class $C^N(\Omega)$, $N \geq 2$, symmetric tensor field on Ω that satisfies the compatibility equations, then there exists a class $C^{N+1}(\Omega)$ vector field $\boldsymbol{v}$ satisfying the strain-displacement relations :

$$\boldsymbol{E} = \frac{1}{2}(\nabla \boldsymbol{v}^T + \nabla \boldsymbol{v}) \tag{2}$$

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We denote by $rot\boldsymbol{E}$ the tensor whose components are : $(rot\boldsymbol{E})_{ij} = \varepsilon_{ipk}E_{jk,p}$. The commas stand for partial differentiations with respect to x and ε_{ipk} denotes the alternator. We denote by M^3_{sym} the vector space of symmetric second-order tensors.
The first rigorous proof of sufficiency was given by E. Beltrami(1886). For more details and historical notes see [6]. In 1890, L. Donati has proved the following theorem:

THEOREM 2 (Donati's theorem). *Let $\boldsymbol{E}$ be a tensor field of class $C^2(\Omega; M^3_{sym})$, such that*

$$\int_{\Omega} \boldsymbol{E} : \boldsymbol{S}\, d\Omega = 0 \tag{3}$$

for every tensor field $\mathbf{S}$ in $C^\infty(\overline{\Omega}; M^3_{sym})$ that vanishes near the boundary $\partial\Omega$ and satisfies $div\boldsymbol{S} = 0$. Then $\boldsymbol{E}$ satisfies the equation of compatibility (1).

We denote by $\boldsymbol{E} : \boldsymbol{S}$ the scalar product of the symetric tensors $\boldsymbol{E}$ and $\boldsymbol{S}$.
An explicit solution was given by E. Cesàro(1906) and by V. Volterra (1907). More precisely, let P_0 be a fixed point in Ω and let define for each P in Ω the line integral:

$$u_i(\boldsymbol{x}) = \int_{P_0}^{P} E_{ij}(\boldsymbol{y})dy_j + \int_{P_0}^{P} (x_k - y_k)\big(E_{ij,k}(\boldsymbol{y}) - E_{kj,i}(\boldsymbol{y})\big)dy_j. \tag{4}$$

When Ω is simply-connected, the displacement vector $\boldsymbol{u}$ is independant of the path from P_0 to P. For multiply-connected domains the definition of $\boldsymbol{u}$ requires some cautions; they have been given by V. Volterra.
In 1974 T. W. Ting gives the following extension of the Donati's theorem [12]:

THEOREM 3 (Ting's theorem). *Let $\boldsymbol{E}$ be a second order symmetric tensor field in $L^2(\Omega; M^3_{sym})$, such that*

$$\int_{\Omega} \boldsymbol{E} : \boldsymbol{S}\, d\Omega = 0 \tag{5}$$

for every tensor field $\mathbf{S}$ in Σ_{ad} the closure in $L^2\left(\Omega; M^3_{sym}\right)$ of:

$$\mathcal{V} = \left\{\boldsymbol{S} \in \mathfrak{D}\left(\Omega; M^3_{sym}\right);\ div\boldsymbol{S} = 0 \ in\ \Omega\right\}.$$

Then there exists a vector $\boldsymbol{v} \in (H^1(\Omega))^3$ satisfying the strain-displacement relations (2).

Let us stress that in this theorem the assumption of simply-connectivity of Ω is not required.
J. J. Moreau gave in 1979 an other extension of Donati's theorem in general distribution framework [8].

THEOREM 4 (Moreau's theorem). *Let* $\boldsymbol{E}$ *be a second order symmetric tensor field in* $\mathfrak{D}'(\Omega; M^3_{sym})$*, such that*

$$\langle \boldsymbol{E}\,, \boldsymbol{S}\,\rangle = 0 \tag{6}$$

for every tensor field $\mathbf{S}$ *in* $\mathcal{V}$*. Then there exists a vector* $\boldsymbol{v} \in (\mathfrak{D}'(\Omega))^3$ *satisfying the strain-displacement relations (2).*

We denote by $\langle .\,, .\rangle$ the duality pairing between $\mathfrak{D}'(\Omega; M^3_{sym})$ and $\mathfrak{D}(\Omega; M^3_{sym})$.

More recently, P. G. Ciarlet and P. Ciarlet, Jr. revisited the Theorem 1 for $\boldsymbol{E} \in L^2\left(\Omega; M^3_{sym}\right)$ taking the compatibility relations (1) in $H^{-2}\left(\Omega; M^3_{sym}\right)$ and established for simply-connected domain the existence of $\boldsymbol{v} \in (H^1(\Omega))^3$ satisfying the strain-displacement relations (2), [2].

In this paper we will prove that an extension to L^p of the Theorem 3 can be deduced from some results of [4]. Then we will give a general formulation of the completeness of Beltrami's solution and we will deduce an extension of the results of P. G. Ciarlet and P. Ciarlet, Jr.

We also mention that similar results have also been simultaneously obtained by C. Amrouche, P. G. Ciarlet, L. Gratie and S. Kasavan, albeit by different proofs, [1].

2. - EXTENSIONS OF DONATI'S AND TING'S THEOREMS

We prove at first the following extension of Donati's theorem.

THEOREM 5: *Let be* $\boldsymbol{E} \in W^{-1,p}(\Omega; M^3_{sym})$ *a second order symmetric tensor field such that*

$$\langle \boldsymbol{E}\,, \boldsymbol{S}\,\rangle = 0 \tag{7}$$

for every tensor field $\boldsymbol{S} \in W^{1,p'}_0(\Omega; M^3_{sym})$*, with* $\frac{1}{p} + \frac{1}{p'} = 1$*, such that* $div\boldsymbol{S} = 0$ *in* Ω*. Then there exists a vector* $\boldsymbol{v} \in (L^p(\Omega))^3$ *satisfying the strain-displacement relations (2).*

PROOF: Let us define the linear and continuous map A from $(L^p(\Omega))^3$ into $W^{-1,p}(\Omega; M^3_{sym})$ by

$$\boldsymbol{v} \longmapsto \mathsf{A}\boldsymbol{v} = \frac{1}{2}(\nabla \boldsymbol{v}^T + \nabla \boldsymbol{v})$$

Its adjoint A^* is linear and continuous from $W^{1,p'}_0(\Omega; M^3_{sym})$ into $(L^{p'}(\Omega))^3$. Since $\mathsf{A}^*\boldsymbol{S} = div\boldsymbol{S}$ the statement will be proved if we can establish that $Im(\mathsf{A})$ is closed ; indeed then $Im(\mathsf{A}) = (Ker(\mathsf{A}^*))^o$. The closedness of $Im(\mathsf{A})$ follows from Peetre's lemma [7] and a general a priori estimate of Nečas [9] and Smith [11]. ■

For future use the following slight extension of Ting's theorem will be useful:

PROPOSITION 1: *Let be* $\boldsymbol{E} \in W^{-1,p}(\Omega; M^3_{sym})$ *a second order symmetric tensor field such that*

$$\langle \boldsymbol{E} , \boldsymbol{S} \rangle = 0$$

for every tensor field $\boldsymbol{S} \in \mathcal{V}$. *Then there exists a vector* $\boldsymbol{v} \in (L^p(\Omega))^3$ *satisfying the strain-displacement relations* (2).

Its proof is based on a corollary of the estimate of Nečas and Smith :

PROPOSITION 2: *For every* $1 < p < \infty$, *the following property holds true. Let be* $\boldsymbol{v} \in (L^p_{loc}(\Omega))^3$ *with* $\frac{1}{2}(\nabla \boldsymbol{v}^T + \nabla \boldsymbol{v}) \in W^{-1,p}(\Omega; M^3_{sym})$ *then* $\boldsymbol{v} \in (L^p(\Omega))^3$.

Let us define for every $1 < p < \infty$

$$\Sigma^p = \left\{ \boldsymbol{S} \in L^p\left(\Omega; M^3_{sym}\right);\ div \boldsymbol{S} \in (L^p(\Omega))^3 \right\}$$

Following the well-known approach of Lions-Magenes [7], it has been proved in [4] that $\mathfrak{D}(\overline{\Omega}; M^3_{sym})$ is dense in Σ^p and that the map

$$\boldsymbol{S} \mapsto \Gamma_{\boldsymbol{n}}(\boldsymbol{S}) = (\boldsymbol{S}.\boldsymbol{n})|_{\partial\Omega} ,$$

well-defined for $\boldsymbol{S} \in \mathfrak{D}(\overline{\Omega}; M^3_{sym})$, can be extended to a linear and continuous map, still denoted $\Gamma_{\boldsymbol{n}}$, from Σ^p to $(W^{-1/p,p}(\partial\Omega))^3$. Moreover for every $\boldsymbol{S} \in \Sigma^p$ and every $\boldsymbol{v} \in (W^{1,p'}(\Omega))^3$ with $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, the following Green's formula holds:

$$\int_\Omega \boldsymbol{E}(\boldsymbol{v}) : \boldsymbol{S}\, d\Omega + \int_\Omega div \boldsymbol{S} . \boldsymbol{v}\, d\Omega = \langle \Gamma_{\boldsymbol{n}}(\boldsymbol{S}) , \boldsymbol{v} \rangle_{\partial\Omega} \tag{8}$$

where $\langle . , . \rangle_{\partial\Omega}$ denotes the duality pairing between $(W^{-1/p,p}(\partial\Omega))^3$ and $(W^{1-1/p',p'}(\partial\Omega))^3$ and $\boldsymbol{E}(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{v}^T + \nabla \boldsymbol{v})$. We assume that the boundary $\partial\Omega$ is partitioned in two open and disjoints subsets : $\partial_1\Omega$ and $\partial_2\Omega$ satisfying the condition : $\partial\Omega = \overline{\partial_1\Omega} \cup \overline{\partial_2\Omega}$. Let us define for $1 < p < \infty$:

$$\Sigma^p_{ad,(\partial_1\Omega)} = \left\{ \boldsymbol{S} \in L^p\left(\Omega; M^3_{sym}\right);\ div \boldsymbol{S} = 0 \text{ in } \Omega \text{ and } \Gamma_{\boldsymbol{n}}(\boldsymbol{S}) = 0 \text{ on } \partial_1\Omega \right\}.$$

We can now prove the following results :

PROPOSITION 3: *For every* $1 < p < \infty$, *the following properties hold true:*

1. $\mathfrak{D}(\Omega ; M^3_{sym})$ *is dense in*

$$Ker(\Gamma_{\boldsymbol{n}}) = \left\{ \boldsymbol{S} \in L^p\left(\Omega ; M^3_{sym}\right);\ div \boldsymbol{S} \in (L^p(\Omega))^3; \Gamma_{\boldsymbol{n}}(\boldsymbol{S}) = 0 \right\}$$

2. $\mathcal{V} = \mathfrak{D}(\Omega ; M^3_{sym}) \cap \Sigma^p_{ad,(\partial\Omega)} = \{ \boldsymbol{S} \in \mathfrak{D}(\Omega; M^3_{sym});\ div \boldsymbol{S} = 0 \textit{ in } \Omega \}$ *is dense in* $\Sigma^p_{ad,(\partial\Omega)}$.

PROOF:

1. The proof is an extension of some results obtained for vectors in the case p = 2 by V. Girault et P. A. Raviart [5]. Let be $L(\boldsymbol{S})$ be a linear and continuous functional on $Ker(\Gamma_{\boldsymbol{n}})$; then there exist $\widehat{\boldsymbol{S}} \in L^{p'}\left(\Omega\,; M^3_{sym}\right)$ and $\widehat{\boldsymbol{T}} \in (L^{p'}(\Omega))^3$ such that:

$$L(\boldsymbol{S}) = \int_\Omega \boldsymbol{S} : \widehat{\boldsymbol{S}}\, d\Omega + \int_\Omega div\boldsymbol{S}.\widehat{\boldsymbol{T}} d\Omega \tag{9}$$

Let us suppose that $L(\boldsymbol{\Phi}) = 0$ for all $\boldsymbol{\Phi} \in \mathfrak{D}(\Omega\,; M^3_{sym})$. In order to prove the density we have to prove that then $L(\boldsymbol{S}) = 0$ for all $\boldsymbol{S} \in Ker(\Gamma_{\boldsymbol{n}})$.

The assumption $L(\boldsymbol{\Phi}) = 0$ means that:

$$\int_\Omega \boldsymbol{\Phi} : \widehat{\boldsymbol{S}}\, d\Omega + \int_\Omega div\, \boldsymbol{\Phi}.\widehat{\boldsymbol{T}} d\Omega = 0$$

and hence:

$$\frac{1}{2}(\widehat{T}_{i,j} + \widehat{T}_{j,i}) = \widehat{S}_{ij} \in L^{p'}(\Omega)$$

Thank to the Korn's inequality it follows that $\widehat{\boldsymbol{T}} \in (W^{1,p'}(\Omega))^3$. We can now apply the Green formula (8) to (9) and we find for all $\boldsymbol{S} \in Ker(\Gamma_{\boldsymbol{n}})$:

$$L(\boldsymbol{S}) = \int_\Omega \boldsymbol{S} : \boldsymbol{E}(\widehat{\boldsymbol{T}})\, d\Omega + \int_\Omega div\boldsymbol{S}.\widehat{\boldsymbol{T}} d\Omega = 0$$

2. The proof follows the same path applying proposition 1. ■

As a consequence of proposition 3 it follows that $\Sigma^2_{ad,(\partial\Omega)} = \Sigma_{ad}$ and we can now interprete the following theorem prooved in [4] as a further extension of Donati's theorem:

THEOREM 6: *Let* $1 < p, p' < \infty$ *with* $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. *Let* $\boldsymbol{E}$ *be a second order symmetric tensor field in* $L^p(\Omega; M^3_{sym})$, *such that*

$$\int_\Omega \boldsymbol{E} : \boldsymbol{S}\, d\Omega = 0 \tag{10}$$

for every tensor field $\mathbf{S}$ *in* $\Sigma^{p'}_{ad,(\partial_1\Omega)}$.

Then there exists a vector $\boldsymbol{v} \in (W^{1,p}(\Omega))^3$ *satisfying the strain-displacement relations* (2) *and* $\boldsymbol{v} = 0$ *on* $\partial_2\Omega$.

3. - COMPLETENESS OF BELTRAMI'S SOLUTION

Let us recall that a regular symmetric tensor field $\boldsymbol{A}$ is said a stress function when there exists a differential operator $\mathcal{L}$ such that $\boldsymbol{S} = \mathcal{L}(\boldsymbol{A}) = \boldsymbol{S}^T$ verifies $div\boldsymbol{S} = 0$. In the two-dimensional case, one can express the solution of equations of equilibrium in terms of the so-called Airy's scalar function. Beltrami observed that this two-dimensional case and all the generalisations to the three-dimensional case are special choices of now called Beltrami's solution defined by the following proposition:

Proposition 4: *Let* $\boldsymbol{A}$ *a tensor field of class* $C^3(\Omega; M^3_{sym})$ *and let* $\boldsymbol{S} = rot\, rot \boldsymbol{A}$ *; then* $div \boldsymbol{S} = 0$ *and* $\boldsymbol{S} = \boldsymbol{S}^T$.

Let us remark that there exist stress fields $\boldsymbol{S}$ that do not admit a representation as a Beltrami's solution. Therefore it may be of interest to find sufficient conditions on $\boldsymbol{S}$ in order that such a representation be true; such conditions are called completeness conditions. Gurtin [6] proves that for a smooth domain Ω a sufficient condition is the nullity of the resultant force and of the moment on each closed regular surface contained in Ω. A global condition of nullity was given for Airy's stress function in general Lipschitz domain in [3]. This condition is extended to the general Beltrami's solution in :

Theorem 7 (Beltrami's completeness). *Let be* γ_p *the connected components of* $\partial\Omega$, $p = 0, \dots, P$.

1. *The two following statements are equivalent:*
   (i) *Let be* $\boldsymbol{S} \in L^2\left(\Omega; M^3_{sym}\right)$ *satisfying* $div \boldsymbol{S} = 0$, *and for* $i = 1, 2, 3$, $p = 0, \dots, P$

(11) $$\left\langle \Gamma_{\boldsymbol{n}}(\boldsymbol{S}), \boldsymbol{e}^i \right\rangle_{\gamma_p} = 0$$

(12) $$\left\langle \Gamma_{\boldsymbol{n}}(\boldsymbol{S}), \boldsymbol{P}^i \right\rangle_{\gamma_p} = 0$$

   where the components of the vector $\boldsymbol{P}^i$ *are :* $P^i_j = -\varepsilon_{ijk} x_k$
   (ii) $\boldsymbol{S} = rot\, rot \boldsymbol{A}$, *where* $\boldsymbol{A} \in H^2\left(\Omega; M^3_{sym}\right)$.
2. *Let* Ω *be simply connected, if moreover* $\boldsymbol{S} \in \Sigma_{ad}$, *then one can choose* $\boldsymbol{A} \in H^2_0(\Omega; M^3_{sym})$ *and conversely.*

Let us remark that (11) and (12) are global conditions of nullity of the resultant force and of the moment. The proof is based on some results of [5].

Following a remark of P. P. Podio Guidugli [10], the completeness of the Beltrami's solution is used to give the following proof of the results of P.G. Ciarlet and P. Ciarlet, Jr. [2].

Theorem 8: *Let* Ω *be simply connected. Let* $\boldsymbol{E}$ *be a second order symmetric tensor field in* $L^2(\Omega; M^3_{sym})$ *satisfying the compatibility relations* (1) *in* $H^{-2}(\Omega; M^3_{sym})$. *Then there exists a vector* $\boldsymbol{v} \in (H^1(\Omega))^3$ *satisfying the strain-displacement relations* (2).

Proof: Using the derivation in distribution sense, the compatibility relations (1) in $H^{-2}(\Omega; M^3_{sym})$ mean, that:

(13) $$0 = \langle rot\, rot \boldsymbol{E}, \boldsymbol{A} \rangle = \int_\Omega \boldsymbol{E} : rot\, rot \boldsymbol{A}\, d\Omega$$

for every $\boldsymbol{A} \in H_0^2(\Omega; M_{sym}^3)$ where $\langle ., . \rangle$ denotes the duality pairing between $H^{-2}(\Omega; M_{sym}^3)$ and $H_0^2(\Omega; M_{sym}^3)$. Let be $\boldsymbol{S} = rot\, rot \boldsymbol{A}$, then $\boldsymbol{S} \in \{\boldsymbol{S} \in L^2(\Omega; M_{sym}^3);\ div \boldsymbol{S} = 0\}$. From the density of $\mathfrak{D}(\Omega; M_{sym}^3)$ in $H_0^2(\Omega; M_{sym}^3)$ it follows that $\boldsymbol{S} \in \Sigma_{ad}$. One can then deduce from theorem 6, that there exists a vector $\boldsymbol{v} \in (H^1(\Omega))^3$ satisfying the strain-displacement relations (2) if one can prove :

$$rot\, rot\, (H_0^2(\Omega; M_{sym}^3)) = \Sigma_{ad}$$

This is exactly the second statement of theorem 7. ■

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