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Completeness Theorems for the Dirichlet Problem for the Polyharmonic Equation

To the memory of Professor Luigi Amerio

ABSTRACT. — We prove a Plemelj type formula for general potentials in C^1 domains. By means of that we obtain completeness theorems in L^p norm for the Dirichlet problem for the polyharmonic equation $\Delta^m u = 0$.

Teoremi di completezza per il problema di Dirichlet per l'equazione iperarmonica

SUNTO. — Nella prima parte del presente lavoro viene dimostrata una formula tipo Plemelj per potenziali di tipo generale in domini di classe C^1. Per mezzo di questa si ottengono teoremi di completezza in norma L^p per il problema di Dirichlet per l'equazione iperarmonica $\Delta^m u = 0$.

1. - INTRODUCTION

Roughly speaking there are two different kinds of completeness theorems for systems of particular solutions of partial differential equations. Results of a first kind show that we can approximate in a certain norm a solution of a partial differential equation by a sequence of particular solutions of the same equation. For example, if we have a holomorphic function f of one complex variable, we may ask when f can be approximated in some norms by polynomials or by rational functions. The classical Theorems of Runge and Mergelyan are the main results in this direction.

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These problems have been widely studied and extended to general elliptic partial differential equations. Quite general theorems are contained in [7]. For a more detailed description of these results and for a complete bibliography we refer to the first section of [16] .

A second kind of results is much more sophisticated and is related not only to a partial differential equation, but also to a particular boundary value problem.

To describe these results, let us consider, for simplicity, the Dirichlet problem for Laplace equation in a bounded domain $\Omega \subset \mathbb{R}^n$: $\Delta u = 0$ in Ω, $u = f$ on $\Sigma \equiv \partial\Omega$. Let us denote by $\{\omega_k\}$ the system of harmonic polynomials.

The completeness of $\{\omega_k\}$ in $C^0(\Sigma)$ $(L^p(\Sigma))$ implies the possibility of approximating any $f \in C^0(\Sigma)$ $(L^p(\Sigma))$ in the corresponding norm by a sequence of harmonic polynomials. This sequence will converge in Ω to the solution u of the Dirichlet problem, because of known inequalities.

This idea is very old and goes back to Picone [24]. Fichera [15] was the first one to prove completeness theorems for the Dirichlet problem, the Neumann problem and the mixed problem for Laplace equation.

In proving such results, an important role is played by the class of functions $\mathscr{A}^p$ $(p \geq 1)$, which was introduced by Amerio [1]. This is the class of the functions $u \in L^p(\Omega)$ for which there exist $A, B \in L^p(\Sigma)$ such that

$$\int_\Omega u\, \Delta\psi\, dx = \int_\Sigma \left(A\, \frac{\partial\psi}{\partial\nu} - B\, \psi \right) d\sigma$$

for any $\psi \in C^\infty(\mathbb{R}^n)$.

In view of the Caccioppoli-Weyl Lemma, any function u in $\mathscr{A}^p$ is harmonic. Amerio [1, 3] proved also that, if $u \in \mathscr{A}^p$, then for almost $y \in \Sigma$ we have

$$\lim_{x \to y} u(x) = A(y), \qquad \lim_{x \to y} \frac{\partial u}{\partial \nu} = B(y)$$

where x tends to y on the internal normal at Σ in y.

As Fichera writes in [16], the Amerio result *is of remarkable technical and historical interest, since it is the first "regularization theorem" on the boundary for weak solutions of an elliptic equation. It appeared only a few years later of the results by Caccioppoli (1938) and by Weyl (1940) concerning the "interior regularization".*

While very general completeness theorems analogous to the Mergelyan one are known, this is not the case for the completeness theorems as proposed by Picone. There are several results which are connected to harmonic and biharmonic equation, to the elasticity system, to the heat equation, to general 2nd order elliptic equations and to higher order elliptic equations with constant coefficients in two variables. We refer again to [16] for the corresponding references, to which we would like to add [8, 9, 10, 11, 12].

These completeness results are interesting also from the numerical point of view. In fact, by means of the completeness for $p = 2$, we can determine an approximating sequence in two different ways.

The first is the classical least squares method, while the second consists in considering a Fischer-Riesz system. For instance, in the case of the Dirichlet problem for Laplace equation we have considered before, we can write

$$\int_{\Sigma} \omega_k \frac{\partial u}{\partial \nu} \, d\sigma = \int_{\Sigma} f \frac{\partial \omega_k}{\partial \nu} \, d\sigma \qquad k = 1, 2, \ldots$$

and consider this as a Fischer-Riesz system in the unknown $\partial u/\partial \nu$. The completeness of $\{\omega_k\}$ in $L^2(\Sigma)$ implies the possibility of constructing a sequence of harmonic polynomials approximating $\partial u/\partial \nu$ in L^2 norm. This leads to an explicit approximation of the solution u.

In papers [1, 2, 3] Amerio investigated the equivalence between several boundary value problems and such Fischer-Riesz systems. In particular, in [2], some boundary value problems for the polyharmonic equation were considered. The study of the polyharmonic equation is a classical problem, for which recently there is some renewed interest (see, e.g., [5, 27, 6, 18, 25, 4, 26, 19, 20]).

The aim of the present paper is to prove a completeness theorem for the Dirichlet problem for the polyharmonic equation of order m. Namely, if we denote by $\{\omega_k^{(m)}\}$ the sequence of polynomials solutions of the equation $\Delta^m u = 0$, we prove that the system

$$\{(\omega_k^{(m)}, \partial_\nu \omega_k^{(m)}, \ldots, \partial_\nu^{m-1} \omega_k^{(m)})\}$$

(∂_ν denotes the normal derivative) is complete in $[L^p(\Sigma)]^m$ $(1 \leq p < \infty)$.

So far such a result was known only for the n-dimensional harmonic and biharmonic problem (see references in [16]). Moreover the extension of the techniques used in these two cases to the general iterated Laplacian Δ^m does not lead to completeness theorems for the Dirichlet problem, but to completeness theorems for the boundary value problem in which the data on Σ are $\Delta^j u$ and $\partial_\nu \Delta^j u$ for $j = 0, \ldots, s-1$ if $m = 2s$, or $\Delta^j u$ for $j = 1, \ldots, s$ and $\partial_\nu \Delta^j u$ for $j = 0, \ldots, s-1$ if $m = 2s+1$.

We have to say that for $n = 2$ and $p = 2$ our theorem is contained in the results of [9], but the proof used there cannot be extended to n-dimensional problems.

We would like to stress the fact that we prove these completeness Theorems in a bounded domain of $\mathbb{R}^n$ whose boundary is merely required to be C^1. Usually results of this kind are proved for Lyapunov boundaries (see [16]).

While for $n = 2$ some results for non Lyapunov boundaries are known (see [14, 8, 9]), this is not the case for $n \geq 3$, as far as we know. Thus our result seems to provide a progress even in the simplest cases of harmonic and biharmonic equations.

The main ingredient of our proof is a *Plemelj type formula* for certain derivatives of polyharmonic potentials on C^1 domains (see Theorem 6). This will be derived from some general results of Potential Theory in the spirit of [13], which seem to be interesting in itself.

2. - Some general results of Potential Theory

Let us introduce some notations. If h is a function defined in $\mathbb{R}^n \setminus \{0\}$, we say that h is homogeneous of degree α if $h(\rho x) = \rho^\alpha h(x)$ for any $x \in \mathbb{R}^n \setminus \{0\}$, $\rho > 0$. We say that h is *essentially homogeneous of degree* α is h is homogeneous in case $\alpha < 0$ or, if α is an integer ≥ 0, h has the form $h(x) = h_1(x) \log|x| + h_2(x)$, where h_1 is a homogeneous polynomial of degree α and h_2 is homogeneous of degree α.

Lemma 1: *If $K(x) \in C^1(\mathbb{R}^n \setminus \{0\})$ is a homogeneous function of degree $-m$ ($m \in \mathbb{N}$), there exists a constant Γ such that*

$$|K(x) - K(y)| \leq \Gamma\, |x - y| \sum_{h=0}^{m} |x|^{-1-h} |y|^{h-m} \qquad \forall\, x, y \in \mathbb{R}^n \setminus \{0\}.$$

For the proof, see [13, p.47].

By $\mathcal{H}(\mathbb{R}^{n-1})$ we denote the space of the functions φ which are bounded and measurable in $\mathbb{R}^{n-1}$, have a compact support and are continuous at 0.

Theorem 1: *Let $K(x;t) \in C^1(\mathbb{R}^n \setminus \{0\})$, where $x = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$. Let us suppose that K is odd and homogeneous of degree $1-n$.*

We have

(i) *for any $t \neq 0$ the following integral does exist*

$$\int_{\mathbb{R}^{n-1}} K(x;t)\, dx = \lim_{M \to \infty} \int_{|x|<M} K(x;t)\, dx;$$

(ii) *there exists $\gamma \in \mathbb{R}$ such that*

$$\int_{\mathbb{R}^{n-1}} K(x;t)\, dx = \gamma; \qquad \int_{\mathbb{R}^{n-1}} K(x;-t)\, dx = -\gamma \qquad \forall\, t > 0.$$

(iii) *for any $\delta > 0$*

$$\lim_{t \to 0} \int_{|x|>\delta} K(x;t)\, dx = 0.$$

Moreover, for any $\varphi \in \mathcal{H}(\mathbb{R}^{n-1})$, we have

$$\lim_{t \to 0^+} \int_{\mathbb{R}^{n-1}} \varphi(x)\, [K(x;t) - K(x;-t)]\, dx = 2\, \gamma\, \varphi(0). \tag{2.1}$$

Proof: Since K is odd, we have

$$\int_{|\eta|=1} K(\eta;0)\, d\eta = 0. \tag{2.2}$$

It is known that from (2.2) it follows (i) for $t > 0$, the first equality in (ii) and the limit relation in (iii) for $t \to 0^+$ (see [13, Th. I and II]). On the other hand we have

$$\int_{\mathbb{R}^{n-1}} K(x; -t)\, dx = \int_{\mathbb{R}^{n-1}} K(-x; -t)\, dx = -\int_{\mathbb{R}^{n-1}} K(x; t)\, dx$$

and then (i), (ii) and (iii) are true.

Set

$$A(t) = \int_{\mathbb{R}^{n-1}} \varphi(x)[K(x; t) - K(x; -t)]\, dx - 2\,\gamma\,\varphi(0).$$

Keeping in mind (ii), given $\varepsilon > 0$, we may write

$$A(t) = \int_{\mathbb{R}^{n-1}} [\varphi(x) - \varphi(0)]\, [K(x; t) - K(x; -t)]\, dx =$$

$$\int_{|x|<\delta} [\varphi(x) - \varphi(0)]\, [K(x; t) - K(x; -t)]\, dx +$$

$$\int_{|x|>\delta} \varphi(x)\, [K(x; t) - K(x; -t)]\, dx - \varphi(0) \int_{|x|>\delta} [K(x; t) - K(x; -t)]\, dx$$

where $\delta > 0$ is such that $|\varphi(x) - \varphi(0)| < \varepsilon$ for $|x| < \delta$.

Since the support of φ is compact, there exists $M > 0$ such that

$$\int_{|x|>\delta} \varphi(x)\, [K(x; t) - K(x; -t)]\, dx = \int_{\delta<|x|<M} \varphi(x)\, [K(x; t) - K(x; -t)]\, dx$$

and then

$$\lim_{t\to 0^+} \int_{|x|>\delta} \varphi(x)\, [K(x; t) - K(x; -t)]\, dx = \lim_{t\to 0^+} \int_{\delta<|x|<M} \varphi(x)\, [K(x; t) - K(x; -t)]\, dx =$$

$$\int_{\delta<|x|<M} \varphi(x)\, [K(x; 0) - K(x; 0)]\, dx = 0.$$

The limit relation (iii) implies

$$\lim_{t\to 0^+} \varphi(0) \int_{|x|>\delta} [K(x; t) - K(x; -t)]\, dx = 0.$$

Finally, since Lemma 1 shows that there exists a constant C such that

$$|K(x; t) - K(x; -t)| \leq C\, \frac{t}{(|x|^2 + t^2)^{n/2}} \qquad (x; t) \in \mathbb{R}^n \setminus \{0\},\ \ t > 0,$$

we have

$$\int_{|x|<\delta} |\varphi(x) - \varphi(0)|\,|K(x;t) - K(x;-t)|\,dx \leq C\,\varepsilon \int_{|x|<\delta} \frac{t}{(|x|^2 + t^2)^{n/2}}\,dx.$$

But

$$\int_{|x|<\delta} \frac{t}{(|x|^2 + t^2)^{n/2}}\,dx = \omega_{n-1} \int_0^d \frac{t\rho^{n-2}}{(\rho^2 + t^2)^{n/2}}\,d\rho \leq \omega_{n-1} \int_0^d \frac{t}{\rho^2 + t^2}\,d\rho \leq \frac{\pi}{2}\omega_{n-1} \tag{2.3}$$

(ω_{n-1} being the hypersurface measure of the unit sphere in $\mathbb{R}^{n-1}$) and then

$$\int_{|x|<\delta} |\varphi(x) - \varphi(0)|\,|K(x;t) - K(x;-t)|\,dx \leq C\,\frac{\pi}{2}\,\omega_{n-1}\,\varepsilon.$$

We have thus proved that

$$\limsup_{t\to 0^+}\ |A(t)| \leq C\,\frac{\pi}{2}\,\omega_{n-1}\,\varepsilon$$

and this implies (2.1). □

Theorem 2: *Let $h \in C^2(\mathbb{R}^n \setminus \{0\})$ be even and essentially homogeneous of degree $2-n$. Set $K(x;t) = \partial h(x;t)/\partial t$, $K_j(x;t) = \partial h(x;t)/\partial x_j$ $(j = 1, \dots, n-1)$. For any $\varphi \in \mathcal{H}(\mathbb{R}^{n-1})$ we have*

$$\lim_{t\to 0^+} \int_{\mathbb{R}^{n-1}} \varphi(x)\,[K(x;t) - K(x;-t)]\,dx = 2\gamma\,\varphi(0),$$

$$\lim_{t\to 0^+} \int_{\mathbb{R}^{n-1}} \varphi(x)\,[K_j(x;t) - K_j(x;-t)]\,dx = 0 \qquad (j = 1, \dots, n-1).$$

The constant γ is given by

$$\gamma = \begin{cases} \pi h_1 - \dfrac{1}{2} \displaystyle\int_{|\xi|=1} \Delta h_2(\xi)\,\log|\xi_2|\,d\sigma_\xi & \text{if } n = 2 \\ \dfrac{1}{2} \displaystyle\int_{|\xi|=1} [(2-n)h(\xi) - \Delta h(\xi)\,\log|\xi_n|]\,d\sigma_\xi & \text{if } n \geq 3. \end{cases}$$

Proof: It is clear that the kernels K and K_j satisfy the hypothesis of Theorem 1. The expression of γ for K and the fact that $\gamma_j = 0$ for K_j was proved in [13]. □

From now on Ω is a bounded domain of $\mathbb{R}^n$ with the boundary Σ of class C^1.

THEOREM 3: *Let $h \in C^2(\mathbb{R}^n \setminus \{0\})$ be even and essentially homogeneous of degree $2-n$. If $\varphi \in L^1(\Sigma)$ and x_0 is a Lebesgue point for φ, then*

$$\lim_{x\to x_0}\left(\int_\Sigma \varphi(y)\frac{\partial}{\partial \nu_{x_0}}h(x-y)d\sigma_y - \int_\Sigma \varphi(y)\frac{\partial}{\partial \nu_{x_0}}h(x'-y)d\sigma_y\right) = 2\gamma(x_0)\varphi(x_0) \tag{2.4}$$

where x is a point on the inner normal to Σ at x_0, x' is its symmetric with respect to x_0 and $\gamma(x_0)$ is given by

$$\gamma(x_0) = \begin{cases} \pi h_1 - \dfrac{1}{2}\displaystyle\int_{|\xi|=1} \Delta h_2(\xi)\log|\xi\cdot\nu_{x_0}|d\sigma_\xi & \text{if } n=2 \\ \dfrac{1}{2}\displaystyle\int_{|\xi|=1}[(2-n)h(\xi) - \Delta h(\xi)\log|\xi\cdot\nu_{x_0}|]d\sigma_\xi & \text{if } n\geq 3. \end{cases} \tag{2.5}$$

PROOF: We first prove (2.4) in the case $\varphi \in C^0(\Sigma)$.

Let $(\tau_1,\dots,\tau_{n-1},\nu_{x_0})$ be an orthonormal system and let us consider the coordinate system $(\eta;t) = (\eta_1,\dots,\eta_{n-1},t)$ with the origin in x_0 corresponding to the basis $(\tau_1,\dots,\tau_{n-1},\nu_{x_0})$. We denote by B_d the ball $\{\eta\in\mathbb{R}^{n-1} \mid |\eta|\leq d\}$. Given $\varepsilon>0$, let Σ_d be the part of Σ which admits the representation $t=\gamma(\eta)$ with $\gamma\in C^1(B_d)$, $\gamma(0)=0$, $\nabla\gamma(0)=0$, $|\nabla\gamma(\eta)|\leq\varepsilon$ for $|\eta|\leq d$. We remark that in B_d we have

$$|\gamma(\eta)| \leq \varepsilon|\eta|. \tag{2.6}$$

We shall suppose also $\varepsilon < 1/2$.

If we write $x=(0;\delta)$, we have $x'=(0;-\delta)$ and

$$x-y = -\sum_{h=1}^{n-1}\eta_h\tau_h + [\delta-\gamma(\eta)]\,\nu_{x_0}\,, \quad x'-y = -\sum_{h=1}^{n-1}\eta_h\tau_h - [\delta+\gamma(\eta)]\,\nu_{x_0}\,.$$

Setting

$$h_{x_0}(\eta;t) = h\left[-\sum_{h=1}^{n-1}\eta_h\tau_h + t\nu_{x_0}\right], \qquad K(\eta;t) = \frac{\partial}{\partial t}h_{x_0}(\eta;t) \tag{2.7}$$

we have

$$\frac{\partial}{\partial\nu_{x_0}}h(x-y) = K(\eta;\delta-\gamma(\eta)), \qquad \frac{\partial}{\partial\nu_{x_0}}h(x'-y) = K(\eta;-\delta-\gamma(\eta)).$$

Therefore we can write

$$\int_{\Sigma_d}\varphi(y)\left[\frac{\partial}{\partial\nu_{x_0}}h(x-y) - \frac{\partial}{\partial\nu_{x_0}}h(x'-y)\right]d\sigma_y = \int_{\mathbb{R}^{n-1}}\Phi(\eta)\,[K(\eta;\delta-\gamma(\eta)) - K(\eta;-\delta-\gamma(\eta))]\,d\eta$$

where $\Phi(\eta) = \varphi\left[x_0 + \sum_{h=1}^{n-1}\eta_h\tau_h + \gamma(\eta)\nu_{x_0}\right](1+|\nabla\gamma(\eta)|^2)^{1/2}$ if $|\eta|\leq d$ and $\Phi(\eta)=0$ if $|\eta|>d$.

If we set

$$\mathscr{H}(\eta;t) = K(\eta;t-\gamma(\eta)) - K(\eta;-t-\gamma(\eta)) - K(\eta;t) + K(\eta;-t)$$

we have

$$\int_{\Sigma_d} \varphi(y)\left[\frac{\partial}{\partial \nu_{x_0}} h(x-y) - \frac{\partial}{\partial \nu_{x_0}} h(x'-y)\right] d\sigma_y =$$

$$= \int_{\mathbb{R}^{n-1}} \Phi(\eta)\ \mathscr{H}(\eta;\delta)\, d\eta + \int_{\mathbb{R}^{n-1}} \Phi(\eta)\,[K(\eta;\delta) - K(\eta;-\delta)]\, d\eta.$$

Fix $\eta \neq 0$ and set $F(v) = K(\eta;\delta - v) - K(\eta;-\delta - v)$; we have

$$|\mathscr{H}(\eta;\delta)| = |F(\gamma(\eta)) - F(0)| = |F'(\sigma\,\gamma(\eta))|\,|\gamma(\eta)|$$

($\sigma \in (0,1)$). Since $F'(v) = -K_t(\eta;\delta - v) + K_t(\eta;-\delta - v)$, Lemma 1 shows that

$$|F'(\sigma\,\gamma(\eta))| \leq 2\,\Gamma\,\delta \sum_{h=0}^{n} [|\eta|^2 + (\delta - \sigma\,\gamma(\eta))^2]^{-(1+h)/2}[|\eta|^2 + (\delta + \sigma\,\gamma(\eta))^2]^{(h-n)/2}.$$

Since $|2\,\sigma\,\delta\,\gamma(\eta)| \leq 2\,\delta\,|\gamma(\eta)| \leq 2\,\varepsilon\,\delta\,|\eta| \leq (1/2)(|\eta|^2 + \delta^2)$, we have

$$|\eta|^2 + (\delta \pm \sigma\,\gamma(\eta))^2 = |\eta|^2 + \delta^2 \pm 2\,\sigma\,\delta\,\gamma(\eta) + \sigma^2\gamma^2(\eta) \geq \frac{1}{2}\,(|\eta|^2 + \delta^2)$$

and then, recalling (2.6),

$$|\mathscr{H}(\eta;\delta)| \leq C\,\frac{\delta\,|\gamma(\eta)|}{(|\eta|^2 + \delta^2)^{(n+1)/2}} \leq C\,\varepsilon\,\frac{\delta}{(|\eta|^2 + \delta^2)^{n/2}}\ .$$

As in (2.3), this inequality leads to

$$\left|\int_{\mathbb{R}^{n-1}} \Phi(\eta)\ \mathscr{H}(\eta;\delta)\, d\eta\right| \leq \widetilde{C}\,\varepsilon \int_{B_d} \frac{\delta}{(|\eta|^2 + \delta^2)^{n/2}}\, d\eta \leq \widetilde{C}\,\frac{\pi}{2}\,\omega_{n-1}\,\varepsilon\,.$$

Theorem 2 shows that

$$\lim_{\delta \to 0^+} \int_{\mathbb{R}^{n-1}} \Phi(\eta)\,[K(\eta;\delta) - K(\eta;-\delta)]\, d\eta = 2\,\gamma(x_0)\,\Phi(0) = 2\,\gamma(x_0)\,\varphi(x_0)$$

where

$$\gamma(x_0) = \begin{cases} \pi h_1 - \dfrac{1}{2}\displaystyle\int_{|\xi|=1} \Delta h_{x_0 2}(\xi)\,\log|\xi_2|\,d\sigma_\xi & \text{if } n = 2 \\[2ex] \dfrac{1}{2}\displaystyle\int_{|\xi|=1} [(2-n)h_{x_0}(\xi) - \Delta h_{x_0}(\xi)\,\log|\xi_n|]\,d\sigma_\xi & \text{if } n \geq 3. \end{cases}$$

A simple substitution shows that $\gamma(x_0)$ can be written as in (2.5) (see [13, p. 50]). We have proved that

$$\limsup_{x\to x_0}\left|\int_{\Sigma_d}\varphi(y)\left[\frac{\partial}{\partial\nu_{x_0}}h(x-y)-\frac{\partial}{\partial\nu_{x_0}}h(x'-y)\right]d\sigma_y-2\,\gamma(x_0)\,\varphi(x_0)\right|\le\widetilde{C}\,\frac{\pi}{2}\,\omega_{n-1}\,\varepsilon$$

which, for the arbitrariness of ε, implies

$$\lim_{x\to x_0}\int_{\Sigma_d}\varphi(y)\left[\frac{\partial}{\partial\nu_{x_0}}h(x-y)-\frac{\partial}{\partial\nu_{x_0}}h(x'-y)\right]d\sigma_y=2\,\gamma(x_0)\,\varphi(x_0)$$

This proves the Theorem for $\varphi\in C^0(\Sigma)$, because

$$\lim_{x\to x_0}\int_{\Sigma\setminus\Sigma_d}\varphi(y)\left[\frac{\partial}{\partial\nu_{x_0}}h(x-y)-\frac{\partial}{\partial\nu_{x_0}}h(x'-y)\right]d\sigma_y=0.$$

Let us suppose now that $\varphi\in L^1(\Sigma)$ and x_0 is a Lebesgue point. Since we know that (2.4) is true for $\varphi\equiv 1$ and that

$$\lim_{x\to x_0}\int_{\Sigma\setminus\Sigma_d}[\varphi(y)-\varphi(x_0)]\left[\frac{\partial}{\partial\nu_{x_0}}h(x-y)-\frac{\partial}{\partial\nu_{x_0}}h(x'-y)\right]d\sigma_y=0$$

(where Σ_d have the same meaning as before), we have only to show that

$$\lim_{x\to x_0}\int_{\Sigma_d}[\varphi(y)-\varphi(x_0)]\left[\frac{\partial}{\partial\nu_{x_0}}h(x-y)-\frac{\partial}{\partial\nu_{x_0}}h(x'-y)\right]d\sigma_y=0. \tag{2.8}$$

We have

$$\int_{\Sigma_d}|\varphi(y)-\varphi(x_0)|\left|\frac{\partial}{\partial\nu_{x_0}}h(x-y)-\frac{\partial}{\partial\nu_{x_0}}h(x'-y)\right|d\sigma_y=$$

$$=\int_{B_d}|\Psi(\eta)|\,|K(\eta;\delta-\gamma(\eta))-K(\eta;-\delta-\gamma(\eta))|\,d\eta$$

where now $\Psi(\eta)=\left[\varphi[x_0+\sum_{h=1}^{n-1}\eta_h\tau_h+\gamma(\eta)\nu_{x_0}\right]-\varphi(x_0)](1+|\nabla\gamma(\eta)|^2)^{1/2}$.

Lemma 1 shows that

$$|K(\eta;\delta-\gamma(\eta))-K(\eta;-\delta-\gamma(\eta))|\le 2\,\Gamma\,\frac{\delta}{(|\eta|^2+\delta^2)^{n/2}}$$

and then

$$\int_{B_d}|\Psi(\eta)|\,|K(\eta;\delta-\gamma(\eta))-K(\eta;-\delta-\gamma(\eta))|\,d\eta\le 2\,\Gamma\int_0^d\frac{\delta\rho^{n-2}}{(\rho^2+\delta^2)^{n/2}}\,d\rho\int_{|\xi|=1}|\Psi(\rho\xi)|\,d\sigma_\xi.$$

We can now use a known argument (see the Appendix of [17]) which we repeat here for the reader convenience. Set

$$G(r) = \int_0^r \rho^{n-2} d\rho \int_{|\xi|=1} |\Psi(\rho\xi)|\, d\sigma_\xi .$$

Since x_0 is a Lebesgue point, given $\varepsilon > 0$, we may choose d in such a way $r^{1-n}G(r) < \varepsilon$ for $0 < r < d$. An integration by parts gives

$$\int_{|\xi|=1} d\sigma_\xi \int_0^d |\Psi(\rho\xi)| \frac{\delta\rho^{n-2}}{(\rho^2+\delta^2)^{n/2}}\, d\rho = \int_0^d G'(\rho) \frac{\delta}{(\rho^2+\delta^2)^{n/2}}\, d\rho \leq$$

$$\leq d^{1-n}G(d) + n\int_0^d G(\rho) \frac{\delta\rho}{(\rho^2+\delta^2)^{(n+2)/2}}\, d\rho \leq$$

$$\leq \varepsilon\left(1 + n\int_0^d \frac{\delta\rho^n}{(\rho^2+\delta^2)^{(n+2)/2}}\, d\rho\right) \leq \varepsilon\left(1 + n\int_0^d \frac{\delta}{\rho^2+\delta^2}\, d\rho\right) \leq \varepsilon\Big(1 + n\,\frac{\pi}{2}\Big).$$

From this (2.8) follows and the Theorem is proved. □

Consider now the tangential operators:

$$M_{x_0}^{ik} = \nu_i(x_0)\frac{\partial}{\partial x_k} - \nu_k(x_0)\frac{\partial}{\partial x_i} \qquad (i, k = 1, \ldots, n).$$

Theorem 4: *If $h(x)$ satisfies the hypothesis of Theorem 3, $\varphi \in L^1(\Sigma)$ and x_0 is a Lebesgue point for φ, we have*

$$\lim_{x\to x_0}\left(\int_\Sigma \varphi(y)\, M_{x_0}^{ik} h(x-y)\, d\sigma_y - \int_\Sigma \varphi(y)\, M_{x_0}^{ik}(x'-y)\, d\sigma_y\right) = 0$$

where x is a point on the inner normal to Σ at x_0 and x' is its symmetric with respect to x_0.

Proof: If $i = k$ or $\nu_i(x_0) = \nu_k(x_0) = 0$, the Theorem is true. Otherwise let $(\tau_1, \ldots, \tau_{n-1}, \nu_{x_0})$ be an orthonormal system, where $\tau_1 = (\tau_{11}, \ldots, \tau_{1n})$ is given by $\tau_{1k} = -\nu_i(x_0)$, $\tau_{1i} = \nu_k(x_0)$, $\tau_{1j} = 0$ if $j \neq i$. The result can be proved by means of the same arguments used in Theorem 3, because

$$M_{x_0}^{ik}[h(x-y)] = K(\eta; t - \gamma(\eta))$$

where

$$K(\eta; t) = -\frac{\partial}{\partial \eta_1} h_{x_0}(\eta; t)$$

and h_{x_0} is given by (2.7).

The next Theorem shows that the function γ can be expressed my means of the Fourier transform.

THEOREM 5: *Under the same hypothesis of Theorems 2 or 3, we have*

$$\gamma(x) = \frac{1}{2} \mathscr{F}(\Delta h)(v_x) = -2\pi^2 \mathscr{F}(h)(v_x) \tag{2.9}$$

where the Laplacian Δ has to be understood in the sense of distributions and $\mathscr{F}$ denotes the Fourier transform.

PROOF: Let us first consider the case $n > 2$. Since h is homogeneous of degree $2 - n$, its second derivatives are homogeneous of degree $-n$ and we may write

$$\Delta h(x) = \frac{\Omega(x')}{|x|^n} \qquad x \in R^n \setminus \{0\}.$$

Let us prove that

$$\Delta h = \frac{\Omega(x')}{|x|^n} + \left((2 - n) \int_{|\xi|=1} h(\xi)\, d\sigma_\xi \right) \delta \tag{2.10}$$

in the sense of distributions, where δ is the Dirac delta. Let $\varphi \in \mathring{C}^\infty(\mathbb{R}^n)$. We have

$$\begin{aligned} \langle \Delta h, \varphi \rangle &= \int_{\mathbb{R}^n} h \Delta\varphi\, dx = \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} h\, \Delta\varphi\, dx = \lim_{\varepsilon \to 0^+} \left(\int_{|x|>\varepsilon} \varphi \Delta h dx + \right. \\ &\left. + \varepsilon \int_{|\xi|=1} h(\xi) \frac{\partial \varphi}{\partial \nu}(\varepsilon\xi) d\sigma_\xi + (2 - n) \int_{|\xi|=1} h(\xi) \varphi(\varepsilon\xi) d\sigma_\xi \right) = \\ &= \lim_{\varepsilon \to 0^+} \left(\int_{|x|>\varepsilon} \varphi(x) \Omega(x') |x|^{-n} dx \right) + \left((2 - n) \int_{|\xi|=1} h(\xi) d\sigma_\xi \right) \varphi(0). \end{aligned} \tag{2.11}$$

This shows that, for any $\varphi \in \mathring{C}^\infty(\mathbb{R}^n)$, the limit

$$\lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} \varphi(x)\, \Omega(x')/|x|^n\, dx \tag{2.12}$$

exists and is finite. It follows that

$$\int_{|\xi|=1} \Omega(\xi)\, d\sigma_\xi = 0$$

and then $\Omega(x')/|x|^n$ is a singular convolution kernel which defines the tempered distribution (2.12) (see, e.g., [23, 157-162]). We denote this distribution by the same symbol $\Omega(x')/|x|^n$.

Formula (2.10) follows from (2.11).

Keeping in mind that Ω is even, we have

$$\mathscr{F}(\Omega(x')/|x|^n) = -\int_{|\xi|=1} \Omega(\xi)\, \log|\xi \cdot x'|\, d\sigma_\xi$$

(see, e.g., [23, p.98]) and since $\mathscr{F}(\delta) = 1$, (2.10) implies the first equality in (2.9).

The second follows form the very well known equality

$$\mathscr{F}(\Delta T)(x) = -4\pi^2|x|^2 \mathscr{F}(T)(x)$$

which holds for any temperate distribution T.

If $n = 2$, h is essentially homogeneous of degree 0, i.e.

$$h(x) = h_1 \log|x| + h_2(x)$$

where h_1 is a real constant and $h_2(x)$ is a homogeneous function of degree 0.

Therefore

$$\Delta h = \Delta h_2 + 2\pi h_1 \delta$$

and (2.9) follows as in the case $n > 2$. □

3. - On Polyharmonic Potentials

In this section we shall apply the results of the previous section to the polyharmonic potentials

$$\int_\Sigma \varphi(y)\, D_y^\alpha F_m(x-y)\, d\sigma_y, \qquad |\alpha| = 2m-1,$$

where

$$F_m(x) = \frac{(-1)^m \Gamma\left(\frac{n}{2} - m\right)}{2^{2m}\pi^{n/2}\Gamma(m)}\, |x|^{2m-n}$$

for odd n and for even $n > 2m$, and

$$F_m(x) = \frac{(-1)^{(n-2)/2}}{2^{2m-1}\pi^{n/2}(m-1)!\,(m-n/2)!}\, |x|^{2m-n} \log|x|$$

for even $n \leq 2m$.

It is well known that $F_m(x-y)$ is a fundamental solution for the iterated Laplacian Δ^m (see [21, p. 43-44]).

THEOREM 6: *Let $\varphi \in L^p(\Sigma)$ and $x_0 \in \Sigma$ be a Lebesgue point for φ. For any multi-index α with $|\alpha| = 2m-1$, we have*

$$\lim_{x\to x_0}\left(\int_\Sigma \varphi(y)D_y^\alpha F_m(x-y)d\sigma_y - \int_\Sigma \varphi(y)D_y^\alpha F_m(x'-y)d\sigma_y\right) = -\nu^\alpha(x_0)\varphi(x_0) \tag{3.1}$$

where x is a point on the inner normal to Σ at x_0 and x' is its symmetric with respect to x_0.

PROOF: Let us write $\alpha = \alpha_0 + \alpha_1$, with $|\alpha_0| = 1$, $|\alpha_1| = 2m-2$. We may write

$$\int_\Sigma \varphi(y)\, D_y^\alpha F_m(x-y)\, d\sigma_y = -D_x^{\alpha_0}\int_\Sigma \varphi(y)\, D_y^{\alpha_1} F_m(x-y)\, d\sigma_y\,;$$

Theorems 3, 4 and 5 give

$$\lim_{x\to x_0}\left(\int_\Sigma \varphi(y)\, D_y^\alpha F_m(x-y)\, d\sigma_y - \int_\Sigma \varphi(y)\, D_y^\alpha F_m(x'-y)\, d\sigma_y\right) =$$

$$-\lim_{x\to x_0}\left(\int_\Sigma \varphi(y)\, D_x^{\alpha_0} D_y^{\alpha_1} F_m(x-y)\, d\sigma_y - \int_\Sigma \varphi(y)\, D_x^{\alpha_0} D_y^{\alpha_1} F_m(x'-y)\, d\sigma_y\right) =$$

$$-\,2\,\nu^{\alpha_0}(x_0)\,\gamma_{\alpha_1}(x_0)\,\varphi(x_0)$$

where

$$\gamma_{\alpha_1}(x) = -2\pi^2 \mathscr{F}(D^{\alpha_1}F_m)(\nu_x).$$

On the other hand

$$\mathscr{F}(D^{\alpha_1}F_m)(x) = (-2\pi i)^{2m-2} x^{\alpha_1}\mathscr{F}(F_m)(x)$$

and since [1]

$$\mathscr{F}(F_m)(x) = \frac{(-1)^m}{4^m\pi^{2m}}\,\frac{1}{|x|^{2m}}$$

we get

$$\gamma_{\alpha_1}(x) = \frac{1}{2}\,\nu^{\alpha_1}(x).$$

□

[1] From $\Delta^m F_m = \delta$, we deduce $\mathscr{F}(\Delta^m F_m) = \mathscr{F}(\delta)$, i.e. $(-4\pi^2|x|^2)^m \mathscr{F}(F_m) = 1$.

LEMMA 2: *Let K be a kernel such that*

$$K(x, y) = \mathcal{O}(|x - y|^{1-n})$$

for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ *with* $x \neq y$. *If* $\varphi \in L^p(\Sigma)$, *the potential*

$$u(x) = \int_\Sigma \varphi(y)\, K(x, y)\, d\sigma_y$$

belongs to $L^p_{\text{loc}}(\mathbb{R}^n)$.

PROOF: Let s be such that $(n-1)/n < s < 1$ and set $\alpha = sn/p$, $\beta = n - 1 - \alpha$. We have

$$|u(x)|^p \leq C_0 \left(\int_\Sigma \frac{|\varphi(y)|}{|y - x|^{\alpha+\beta}}\, d\sigma_y \right)^p \leq$$

$$\leq C_0 \left(\int_\Sigma \frac{|\varphi(y)|^p}{|y - x|^{\alpha p}}\, d\sigma_y \right) \left(\int_\Sigma \frac{d\sigma_y}{|y - x|^{\beta q}} \right)^{p/q} \qquad (q = p/(p-1)).$$

Let us fix a compact $H \subset \mathbb{R}^n$. Since $\beta q < n - 1$, we have that

$$\int_\Sigma \frac{d\sigma_y}{|y - x|^{\beta q}} \leq C_1$$

for any $x \in H$. Therefore

$$\int_H |u(x)|^p dx \leq C_2 \int_\Sigma |\varphi(y)|^p d\sigma_y \int_H \frac{dx}{|x - y|^{\alpha p}} \leq C_3 \|\varphi\|^p_{L^p(\Sigma)}$$

and the result is proved. □

COROLLARY 1: *Let* α *be a multi-index with* $|\alpha| = m - 1$. *Let us denote by u the potential*

$$u(x) = \int_\Sigma \varphi(y)\, D^\alpha_y [F_m(y - x)]\, d\sigma_y\,.$$

If $\varphi \in L^p(\Sigma)$, *then* $u \in W^{m,p}_{\text{loc}}(\mathbb{R}^n)$.

PROOF: If $x \notin \Sigma$ we have

$$D^\beta u(x) = \int_\Sigma \varphi(y)\, D^\beta_x D^\alpha_y [F_m(y - x)]\, d\sigma_y\,. \tag{3.2}$$

Lemma 2 shows that the potential (3.2) belongs to $L^p_{\text{loc}}(\Omega)$, provided $|\beta| \leq m - 1$. To prove the result, we have to verify that (3.2) is a weak derivative. Indeed we have

$$\int_{\mathbb{R}^n} u(x)\, D^\beta \psi(x)\, dx = \int_{\mathbb{R}^n} D_x^\beta \psi(x)\, dx \int_\Sigma \varphi(y)\, D_y^\alpha [F_m(y-x)]\, d\sigma_y =$$

$$= \int_\Sigma \varphi(y)\, d\sigma_y \int_{\mathbb{R}^n} D_x^\beta \psi(x)\, D_y^\alpha [F_m(y-x)]\, dx =$$

$$= (-1)^{|\beta|} \int_\Sigma \varphi(y)\, d\sigma_y \int_{\mathbb{R}^n} \psi(x)\, D_x^\beta D_y^\alpha [F_m(y-x)]\, dx =$$

$$= (-1)^{|\beta|} \int_{\mathbb{R}^n} \psi(x)\, dx \int_\Sigma \varphi(y)\, D_x^\beta D_y^\alpha [F_m(y-x)]\, d\sigma_y$$

for any $\psi \in \mathring{C}^\infty(\mathbb{R}^n)$ and for any β with $|\beta| \leq m-1$. We could integrate by parts because $D_x^\beta D_y^\alpha [F_m(y-\cdot)]$ is locally integrable for any fixed y, provided $|\beta| \leq m-1$. The last equality follows from the Fubini and Tonelli Theorems, because

$$\int_{\mathbb{R}^n} |\psi(x)|\, dx \int_\Sigma |\varphi(y)|\, |D_x^\beta D_y^\alpha [F_m(y-x)]|\, d\sigma_y \leq C\, \|\varphi\|_{L^p(\Sigma)}\, \|\psi\|_{L^q(\mathbb{R}^n)}$$

in view of Lemma 2. □

4. - Completeness Theorems

We begin this section by recalling a known existence and uniqueness result. Let P be the operator

$$Pu = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u) \tag{4.1}$$

where $a_{\alpha\beta}$ are complex valued functions belonging to $C^0(\overline{\Omega})$.

Theorem 7: *If P satisfies the Gårding inequality*

$$\mathscr{Re} \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^\alpha u \overline{D^\beta u}\, dx \geq C\, \|u\|^2_{W^{m,2}(\Omega)} \qquad \forall\, u \in \mathring{C}^\infty(\Omega), \tag{4.2}$$

there exists one and only one solution $u \in W^{m,p}(\Omega)$ of the Dirichlet problem

$$Pu = f, \qquad u - g \in \mathring{W}^{m,p}(\Omega)$$

where $f \in W^{-m,p}(\Omega)$, $g \in W^{m,p}(\Omega)$ are given.

This result is a particular case of a general existence and uniqueness theorem given in [22, p. 303]. We remark that Theorem 7 can be obtained by means of the usual variational methods for $p=2$, but for $p \neq 2$ is very delicate, especially if we do not assume the boundary Σ to be smooth. According to [22], Theorem 7 holds if Ω satisfies the so called $N_p^{1-1/p}$ condition (see [22, p. 287]). This is a mild condition and is certainly satisfied if $\Sigma \in C^1$.

Theorem 8: *There exists one and only one solution $u \in W^{m,p}(\Omega)$ of the Dirichlet problem*

$$\Delta^m u = f, \qquad u - g \in \overset{\circ}{W}{}^{m,p}(\Omega)$$

where $f \in W^{-m,p}(\Omega)$, $g \in W^{m,p}(\Omega)$ are given.

Proof: Define

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\, \xi^\alpha \eta^\beta \begin{cases} = \xi \cdot \eta\, |\xi|^{2s} |\eta|^{2s} & \text{if } m = 2s+1; \\ = |\xi|^{2s} |\eta|^{2s} & \text{if } m = 2s. \end{cases}$$

The corresponding operator (4.1) is Δ^m. By Theorem 7, it suffices to prove the Gårding inequality (4.2).

We have

$$\int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\, D^\alpha u\, D^\beta u\, dx \begin{cases} = \int_\Omega |\nabla(\Delta^s u)|^2 dx & \text{if } m = 2s+1; \\ = \int_\Omega |\Delta^s u|^2 dx & \text{if } m = 2s. \end{cases}$$

Integrating by parts we get

$$\int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\, D^\alpha u\, D^\beta u\, dx = \int_\Omega |\nabla_m u|^2 dx \qquad \forall\, u \in \overset{\circ}{C}{}^\infty(\Omega)$$

(where ∇_m denotes the gradient of order m) and (4.2) follows from Poincaré's inequality. □

Let us denote by $\{\omega_k^{(m)}\}$ a complete system of polyharmonic polynomials. This means that any polynomial solution of the equation $\Delta^m u = 0$ can be written as a finite linear combination of polynomials $\omega_k^{(m)}$. Such a system can be obtained in the following way.

If $\{Y_{hs}\}$ $(s = 1, \ldots, p_{nh}, h = 0, 1, \ldots)$ is a complete system of ultra-spherical harmonics, where $p_{nh} = (2h+n-2)(h+n-3)!/((n-2)!h!)$, the system

$$|x|^h Y_{hs}\left(\frac{x}{|x|}\right) \qquad (s = 1, \ldots, p_{nh}, h = 0, 1, \ldots)$$

provides a complete system of harmonic polynomials $\{\omega_k\}$.

A classical theorem of Almansi (see, e.g., [5]) states that u is a solution of the equation $\Delta^m u = 0$ in a star-shaped domain if, and only if, there exist m harmonic functions

$u_0, \dots, u_{m-1}$ such that

$$u(x) = \sum_{j=0}^{m-1} |x|^{2j} u_j(x).$$

It is easily seen that if u is a polynomial then u_j are polynomials. Hence a complete system of polyharmonic polynomials $\{\omega_k^{(m)}\}$ is given by

$$|x|^{h+2j} Y_{hs}\left(\frac{x}{|x|}\right) \qquad (j = 0, \dots, m-1, s = 1, \dots, p_{nh}, h = 0, 1, \dots).$$

Theorem 9: *Let Ω be a bounded domain of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected. Let $1 \le p < \infty$. The system*

$$\{(\omega_k^{(m)}, \partial_\nu \omega_k^{(m)}, \dots, \partial_\nu^{m-1} \omega_k^{(m)})\}$$

is complete in $[L^p(\Sigma)]^m$.

Proof: Let $1 < p < \infty$ and $q = p/(p-1)$. Let $(\varphi_1, \dots, \varphi_m) \in [L^q(\Sigma)]^m$ be such that

$$\int_\Sigma (\varphi_1 \omega_k^{(m)} + \ldots + \varphi_m \partial_\nu^{m-1} \omega_k^{(m)})\, d\sigma = 0 \qquad \forall\ \omega_k^{(m)} \in \{\omega_k^{(m)}\}. \tag{4.3}$$

If we prove that $\varphi_1 = \ldots = \varphi_m = 0$, the assertion follows.

We first show that there exists $R > 0$ such that, for any x with $|x| > R$, we have

$$F_m(y - x) = \sum_{|\alpha|=0}^{\infty} c_\alpha^{(m)}(x)\, w_\alpha^{(m)}(y) \tag{4.4}$$

uniformly for $y \in \overline{\Omega}$, where $w_\alpha^{(m)}$ are m-polyharmonic polynomials ([2]).

Since $F_m(t)$ is analytical for $t \neq 0$, for any fixed ξ such that $|\xi| = 1$, there exists $r_\xi > 0$ such that

$$F_m(t - \xi) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{1}{\alpha!} [D^\alpha F_m(v)]_{v=-\xi} t^\alpha \tag{4.5}$$

uniformly for $|t| \le r_\xi$. From the compactness of the unit sphere $S = \{\xi \in \mathbb{R}^n \mid |\xi| = 1\}$ it follows easily that we can choose $r > 0$ independent of ξ such that (4.5) holds uniformly for $|t| \le r$, for any $\xi \in S$.

Let us fix $x = |x|\xi$, $\xi \in S$.

Let us suppose n odd or n even with $n > 2m$. We have

$$F_m(y - x) = |x|^{2m-n} F_m(y/|x| - \xi) = |x|^{2m-n} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{1}{\alpha!} [D^\alpha F_m(v)]_{v=-\xi} (y/|x|)^\alpha$$

([2]) Our proof of (4.4) hinges on a idea due to G. Fichera (private communication).

uniformly for $|y/|x|| \leq r$. Since the function F_m is homogeneous of degree $2m-n$, we may write

$$F_m(y-x) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{1}{\alpha!} [D^\alpha F_m(v)]_{v=-x} y^\alpha \tag{4.6}$$

uniformly for $|y| \leq r|x|$.

Let us now $R_1^{(k)}(x)$, ... $R_{n_k}^{(k)}(x)$ be a basis for the functions $[D^\alpha F_m(v)]_{v=-x}$ ($|\alpha| = k$). Then (4.6) can be written in the form

$$F_m(y-x) = \sum_{k=0}^{\infty} \sum_{j=1}^{n_k} R_j^{(k)}(x)\, P_j^{(k)}(y) \tag{4.7}$$

where $P_j^{(k)}$ are homogeneous polynomials of degree k.

If n is even with $n \leq 2m$, since we may write

$$\log|x-y| = \log|x| + \log|y/|x| - \xi|,$$

we find

$$F_m(y-x) = q(x,y) + |x|^{2m-n} F_m(y/|x| - \xi)$$

$q(x,y)$ being a polynomial of degree $2m-n$ in y. Therefore

$$F_m(y-x) = q_m(x,y) + \sum_{k=0}^{\infty} \sum_{|\alpha|=k} |x|^{2m-n} [D^\alpha F_m(v)]_{v=-\xi} (y/|x|)^\alpha$$

uniformly for $|y| \leq r|x|$. This shows that also in this case the expansion (4.7) holds uniformly for $|y| \leq r|x|$.

This expansion can be derived term by term and since $\Delta^m F_m = 0$, we find that $\Delta^m P_j^{(k)} = 0$ and (4.4) is proved.

Setting $d = \max_{x\in\overline{\Omega}} |x|$, $R = d/r$, we have that (4.4) holds uniformly for $y \in \overline{\Omega}$, provided $|x| \geq R$.

For any $x \in \mathbb{R}^n$ with $|x| \geq R$, we have

$$\sum_{j=1}^{m} \int_\Sigma \varphi_j(y)\, \partial_{\nu_y}^{j-1} F_m(y-x)\, d\sigma_y = \sum_{|\alpha|=0}^{\infty} \sum_{j=1}^{m} c_\alpha^{(m)}(x) \int_\Sigma \varphi_j\, \partial_\nu^{j-1} w_\alpha^{(m)}\, d\sigma_y = 0$$

in view of conditions (4.3).

The potential

$$u(x) = \sum_{j=1}^{m} \int_\Sigma \varphi_j(y)\, \partial_{\nu_y}^{j-1} F_m(y-x)\, d\sigma_y$$

is analytic in $\mathbb{R}^n \setminus \Sigma$. Since $\mathbb{R}^n \setminus \overline{\Omega}$ is connected, we see that

$$u(x) = 0 \qquad \forall\, x \in \mathbb{R}^n \setminus \overline{\Omega}$$

On the other hand, in view of Corollary 1, the function u belongs to $W^{m,p}_{\text{loc}}(\mathbb{R}^n)$. Since Ω satisfies the restricted cone hypothesis and $u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$, we can find a sequence

$u_n \in \overset{\circ}{C}^\infty(\Omega)$ such that u_n tends to u in $W^{m,p}(\Omega)$ (see [7, p.148-149]). This means that u belongs to$\overset{\circ}{W}^{m,p}(\Omega)$ and Theorem 8 shows that $u = 0$ in Ω.

Let us suppose now m even, i.e. $m = 2s$. We have in particular

$$\Delta^s u(x) = 0 \qquad \forall\, x \in \mathbb{R}^n \setminus \Sigma$$

i.e.

$$\sum_{j=1}^{2s} \int_\Sigma \varphi_j(y)\, \partial_{\nu_y}^{j-1} \Delta_x^s [F_{2s}(y-x)]\, d\sigma_y = 0 \qquad \forall\, x \in \mathbb{R}^n \setminus \Sigma. \tag{4.8}$$

If n is even and $n \leq 2m$ we have

$$\Delta^s(F_{2s}(x)) = a_{n,s} F_s(x) + b_{n,s}|x|^{2s-n}$$

for some constants $a_{n,s}, b_{n,s}$ ($a_{n,s} \neq 0$). If n is even with $n > 2m$ or n is odd, we have

$$\Delta^s(F_{2s}(x)) = a_{n,s} F_s(x)$$

with $a_{n,s} \neq 0$. In any case, from (4.8), we find that

$$\lim_{x \to x_0} \sum_{j=1}^{2s} \int_\Sigma \varphi_j(y) \Big(\partial_{\nu_y}^{j-1} F_s(y-x) - \partial_{\nu_y}^{j-1} F_s(y-x') \Big)\, d\sigma_y = 0$$

where x is a point on the inner normal to Σ at x_0 and x' is its symmetric with respect to x_0.

On the other hand, keeping in mind (3.1), we have almost everywhere

$$\lim_{x \to x_0} \sum_{j=1}^{2s-1} \int_\Sigma \varphi_j(y) \Big(\partial_{\nu_y}^{j-1} F_s(y-x) - \partial_{\nu_y}^{j-1} F_s(y-x') \Big)\, d\sigma_y = 0$$

$$\lim_{x \to x_0} \int_\Sigma \varphi_{2s}(y) \Big(\partial_{\nu_y}^{2s-1} F_s(y-x) - \partial_{\nu_y}^{2s-1} F_s(y-x') \Big)\, d\sigma_y = -\varphi_m(x_0).$$

Therefore $\varphi_m = 0$ almost everywhere on Σ.

If m is odd, say $m = 2s+1$, we write again $\Delta^s u = 0$, i.e.

$$\sum_{j=1}^{2s+1} \int_\Sigma \varphi_j(y)\, \partial_{\nu_y}^{j-1} \Delta_x^s [F_{2s+1}(y-x)]\, d\sigma_y = 0 \qquad \forall\, x \in \mathbb{R}^n \setminus \Sigma. \tag{4.9}$$

By means of an argument similar to the previous one, we find that (4.9) implies

$$\lim_{x \to x_0} \sum_{j=1}^{2s+1} \int_\Sigma \varphi_j(y) \Big(\partial_{\nu_{x_0}} \partial_{\nu_y}^{j-1} F_{s+1}(y-x) - \partial_{\nu_{x_0}} \partial_{\nu_y}^{j-1} F_{s+1}(y-x') \Big)\, d\sigma_y = 0.$$

In view of (3.1), we find again $\varphi_m = 0$ almost everywhere on Σ.

If $m = 1$, the completeness is proved. Otherwise, since we have shown that (4.3) leads to

$$\sum_{j=1}^{m-1} \int_\Sigma \varphi_j\, \partial_\nu^{j-1} \omega_k^{(m)} d\sigma = 0 \qquad \forall\, \omega_k^{(m)} \in \{\omega_k^{(m)}\},$$

and then, in particular,

$$\sum_{j=1}^{m-1} \int_{\Sigma} \varphi_j \, \partial_\nu^{j-1} \omega_k^{(m-1)} d\sigma = 0 \qquad \forall \, \omega_k^{(m-1)} \in \{\omega_k^{(m-1)}\},$$

an induction argument gives the result.

Finally the completeness for $p = 1$ follows easily from the completeness for $p > 1$.
□

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