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Harnack Inequality for Solutions of Mixed Boundary Value Problems Containing Boundary Terms

ABSTRACT. — We prove Harnack inequality near the boundary for solutions of mixed boundary value problems for a class of divergence form elliptic equations with discontinuous and unbounded coefficients, in presence of boundary integrals.

1. - INTRODUCTION

In this note we prove Harnack inequality, near the boundary of Ω , of the solutions of a mixed problem for a class of divergence form elliptic equations, containing integral terms on the boundary.

Let Ω be an open subset Ω of \mathbb{R}^n , and V be the subspace of $H^1(\Omega)$ defined by

$$(1) \quad V := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_o \text{ in the sense of } H^1(\Omega)\}$$

where Γ_o is a closed (possibly empty) subset of $\partial\Omega$. Consider furthermore the bilinear form

$$(2) \quad a(u, v) := \int_{\Omega} \left\{ \sum_{i,j=1}^n a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n (b_i u_{x_i} v + d_i u v_{x_i}) + cuv \right\} dx + \int_{\Gamma} gw d\sigma$$

where $\Gamma := \partial\Omega \setminus \Gamma_o$.

Let $u \in V$ be a solution of the equation

$$(3) \quad a(u, v) = \int_{\Omega} \left\{ f_o v + \sum_{i=1}^n f_i v_{x_i} \right\} dx + \int_{\Gamma} hv d\sigma \quad \forall v \in V.$$

We can remark that, if the functions we consider are sufficiently regular (for example of

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class $C^1(\overline{\Omega})$, as well as Γ , then u is a solution of the following problem:

$$\begin{cases} Lu = f_o - \sum_{i=1}^n (f_i)_{x_i} & \text{in } \Omega, \\ \sum_{i,j=1}^n a_{ij}u_{x_i}N_j + \sum_{i=1}^n d_iN_iu + gu = b + \sum_{i=1}^n f_iN_i & \text{on } \Gamma, \\ u = 0 & \text{on } \Gamma_o \end{cases}$$

where N is the normal unit vector to Γ (oriented towards the exterior of Ω) and L the operator defined by

$$Lu := - \sum_{i,j=1}^n a_{ij}u_{x_i x_j} + \sum_{i=1}^n \left[b_i - d_i - \sum_{j=1}^n (a_{ij})_{x_j} \right] u_{x_i} + \left[c - \sum_{i=1}^n (d_i)_{x_i} \right] u$$

In a former work [1] we proved Hölder regularity up to the boundary for solutions of equation (3), under suitable hypotheses on the coefficients of the bilinear form $a(\cdot, \cdot)$ and known terms f_i ($i = 0, 1, 2, \dots, n$) and b .

In this note we prove Harnack inequality near the part Γ of the boundary for solutions of equation (3), following, as in [1], the classical work by Stampacchia [4]

2. - NOTATIONS AND HYPOTHESES

Let Ω be an open bounded subset of \mathbb{R}^n (with $n \geq 3$ for simplicity); for the definition of the spaces $H^{1,p}(\Omega)$ we refer for example to [2], [3].

Let us suppose $a_{ij} \in L^\infty(\Omega)$ ($i, j = 1, 2, \dots, n$), $\sum a_{ij}t_i t_j \geq v|t|^2 \forall t \in \mathbb{R}^n$ a.e. in Ω , with v a positive constant; $b_i \in L^n(\Omega)$, $d_i \in L^p(\Omega)$, ($i = 1, 2, \dots, n$), $c \in L^{p/2}(\Omega)$, $g \in \overline{L^p}(\Gamma)$ with $p > n$, $\overline{p} := p(n-1)/n$.

Let Γ_o be a closed (possibly empty) subset of $\partial\Omega$, and define $\Gamma := (\partial\Omega) \setminus \Gamma_o$. We suppose that Γ is representable locally as the graph of a Lipschitz function: for the precise hypotheses see [1], par. 3.

3. - MAIN RESULTS

THEOREM 3.1: *Suppose that all the hypotheses listed in the previous paragraph are verified; let $u \in V$ be a non negative solution of the equation*

$$a(u, v) = 0 \quad \forall v \in V$$

Then there exist two positive constants K_1 and \overline{r} , depending on the coefficients of $a(\cdot, \cdot)$ and on the regularity of Γ , such that for any $\overline{x} \in \Gamma$ and any r with $0 < r \leq \overline{r}$ it turns out

$$\text{ess sup}_{\Omega(\overline{x}, r)} u \leq K_1 \text{ess inf}_{\Omega(\overline{x}, r)} u$$

where $\Omega(\overline{x}, r) := \{x \in \Omega : |x_i - \overline{x}_i| < r, i = 1, 2, \dots, n\}$.

PROOF: As in [1], it is convenient to follow the proof e.g. of Theorem 8.2 of [4], remarking only the differences due to the presence of the new term $\int_{\Gamma} guv d\sigma$ in the definition of the bilinear form $a(.,.)$ (see (2)).

Let us begin with the proof of Lemma 8.1 of [4]. Proceeding in the same manner, we find in the right member of the last inequality (at the end of page 239) the new term $|p| \int_{\Gamma} |g| \alpha^2 v^2 d\sigma$. By using the inequality (33) of [1] we get at once

$$(5) \quad \int_{\Gamma \cap Q} |g| \alpha^2 v^2 d\sigma \leq K_4^2 \|g\|_{L^{n-1}(\Gamma \cap Q)} \| (av)_x \|_{L^2(\Omega \cap Q)}^2$$

where we have denoted for brevity $Q := \{x \in \mathbb{R}^n : |x_i - \bar{x}_i| < r, i = 1, 2, \dots, n\}$ and $a \in C_o^1(Q)$. Note that in (5) the quantity $\|g\|_{L^{n-1}(\Gamma \cap Q)}$ can be made as small as we like by a suitable choice of \bar{r} . Therefore, combining (5) and the proof of Lemma 8.1 in [4], we can bring the term $\|av_x\|_{L^2(\Omega \cap Q)}^2$ to the left member of the last inequality of page 239 of [4], so the conclusion follows as in [4].

As what concerns Lemma 8.2 of [4], even in this case we find in the inequality (8.8) the new term $\int_{\Gamma} |g| \alpha^2 d\sigma$. In order to evaluate this integral, we can use (5) where we put $v = 1$. Since it is $a(x) = 0$ if $x \notin Q(\bar{x}, 2\rho)$, $|a_x| \leq 2/\rho$ (see [4]), we easily deduce

$$\int_{\Gamma} |g| \alpha^2 d\sigma \leq K_o \|g\|_{L^{n-1}(\Gamma \cap Q)} \rho^{n-2}$$

where K_o depends only on K_4 and n ; the rest of the proof continues as in [4].

Also the remaining part of the proof of Theorem 8.1 of [4] can be followed in our new situation. There are few differences, for example in the proof of Lemma 8.4 (page 241 at the end) we must use Theorem 2 of [1] instead of the Remark 5.1 of [4]. The simple details can be left to the reader. \square

COROLLARY 1. *Suppose that all the hypotheses listed in the previous paragraph are verified; let $u \in V$ be a non negative solution of the equation (3). Then there exist two positive constants K_2 and \bar{r} , depending on the coefficients of $a(.,.)$ and on the regularity of Γ , such that for any $\bar{x} \in \Gamma$ and any r with $0 < r \leq \bar{r}$ it turns out*

$$\begin{aligned} \text{ess sup}_{\Omega(\bar{x}, r)} u &\leq K_2 \text{ess inf}_{\Omega(\bar{x}, r)} u + \\ &+ K_2 \left(\sum_{i=1}^n \|f_i\|_{L^p(\Omega)} + \|f_o\|_{L^{np/(n+p)}(\Omega)} + \|b\|_{L^{\bar{p}}(\Gamma)} \right) r^{1-n/p} \end{aligned}$$

PROOF: We can proceed again as in [4] (Theorem 8.2), where we must use Lemma 3 of [1] instead of Theorem 5.2 of [4]. \square

COROLLARY 2. *Suppose that all the hypotheses listed in the previous paragraph are verified; let $u \in V$ be a non negative solution of the equation*

$$a(u, v) = 0 \quad \forall v \in V$$

Let G be an open subset of Ω such that $\overline{G} \cap \partial\Omega \subset \Gamma$. Then there exists a positive constant K_3 , depending on G , Γ and the coefficients of $a(\cdot, \cdot)$, such that it turns out

$$\operatorname{ess\,sup}_G u \leq K_3 \operatorname{ess\,inf}_G u$$

PROOF: As in [4] it is possible to deduce our assertion from the previous Theorem, by covering \overline{G} by a finite number of sufficiently small cubes. In particular we must cover $\overline{G} \cap \partial\Omega = \overline{G} \cap \Gamma$ by small cubes with centres on Γ , in such a way as to apply the Theorem to each of them, while we can apply Theorem 8.1 of [4] to the cubes contained in Ω . The details of the proof can be left to the reader. \square

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