MAURIZIO CHICCO (*)

Harnack Inequality for Solutions of Mixed Boundary Value Problems Containing Boundary Terms

Abstract. — We prove Harnack inequality near the boundary for solutions of mixed boundary value problems for a class of divergence form elliptic equations with discontinuous and unbounded coefficients, in presence of boundary integrals.

1. Introduction

In this note we prove Harnack inequality, near the boundary of \( \Omega \), of the solutions of a mixed problem for a class of divergence form elliptic equations, containing integral terms on the boundary.

Let \( \Omega \) be an open subset \( \Omega \) of \( \mathbb{R}^n \), and \( V \) be the subspace of \( H^1(\Omega) \) defined by

\[
V := \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0 \text{ in the sense of } H^1(\Omega) \}
\]

where \( \Gamma_0 \) is a closed (possibly empty) subset of \( \partial \Omega \). Consider furthermore the bilinear form

\[
a(u, v) := \left\{ \sum_{i,j=1}^{n} a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} (b_i u_{x_i} v + d_i v_{x_i}) + c uv \right\} dx + \int_{\Gamma} g uv \, d\sigma
\]

where \( \Gamma := \partial \Omega \setminus \Gamma_0 \).

Let \( u \in V \) be a solution of the equation

\[
a(u, v) = \left\{ f_0 v + \sum_{i=1}^{n} f_i v_{x_i} \right\} dx + \int_{\Gamma} h v \, d\sigma \quad \forall \ v \in V.
\]

We can remark that, if the functions we consider are sufficiently regular (for example of

class \(C^1(\Omega)\), as well as \(\Gamma\), then \(u\) is a solution of the following problem:

\[
\begin{cases}
Lu = f_0 - \sum_{i=1}^{n} (f_i)_{x_i} \quad \text{in } \Omega, \\
\sum_{i,j=1}^{n} a_{ij} u_{x_i} N_j + \sum_{i=1}^{n} d_i N_i u + gu = b + \sum_{i=1}^{n} f_i N_i \quad \text{on } \Gamma, \\
\quad \quad \quad u = 0 \quad \text{on } \Gamma_0
\end{cases}
\]

where \(N\) is the normal unit vector to \(\Gamma\) (oriented towards the exterior of \(\Omega\)) and \(L\) the operator defined by

\[
Lu := - \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} \left[ b_i - d_i - \sum_{j=1}^{n} (a_{ij})_{x_j} \right] u_{x_i} + \left[ c - \sum_{i=1}^{n} (d_i)_{x_i} \right] u
\]

In a former work [1] we proved Hölder regularity up to the boundary for solutions of equation (3), under suitable hypotheses on the coefficients of the bilinear form \(a(.,.)\) and known terms \(f_i\) \((i = 0, 1, 2, \ldots, n)\) and \(b\).

In this note we prove Harnack inequality near the part \(\Gamma\) of the boundary for solutions of equation (3), following, as in [1], the classical work by Stampacchia [4]

2. - Notations and hypotheses

Let \(\Omega\) be an open bounded subset of \(\mathbb{R}^n\) (with \(n \geq 3\) for simplicity); for the definition of the spaces \(H^{1,p}(\Omega)\) we refer for example to [2], [3].

Let us suppose \(a_{ij} \in L^{\infty}(\Omega)\) \((i,j = 1, 2, \ldots, n)\), \(\sum a_{ij} t_i t_j \geq \nu |t|^2 \quad \forall t \in \mathbb{R}^n\) a.e. in \(\Omega\), with \(\nu\) a positive constant; \(b_i \in L^p(\Omega)\), \(d_i \in L^p(\Omega)\), \((i = 1, 2, \ldots, n)\), \(c \in L^{p/2}(\Omega)\), \(g \in L^\infty(\Gamma)\) with \(p > n\), \(p := p(n-1)/n\).

Let \(\Gamma_0\) be a closed (possibly empty) subset of \(\partial \Omega\), and define \(\Gamma := (\partial \Omega) \setminus \Gamma_0\). We suppose that \(\Gamma\) is representable locally as the graph of a Lipschitz function; for the precise hypotheses see [1], par. 3.

3. - Main results

**Theorem 3.1:** Suppose that all the hypotheses listed in the previous paragraph are verified; let \(u \in V\) be a non negative solution of the equation

\[
a(u, v) = 0 \quad \forall v \in V
\]

Then there exist two positive constants \(K_1\) and \(\overline{r}\), depending on the coefficients of \(a(.,.)\) and on the regularity of \(\Gamma\), such that for any \(\overline{\sigma} \in \Gamma\) and any \(r\) with \(0 < r \leq \overline{r}\) it turns out

\[
\text{ess sup}_{\Omega(\overline{\sigma}, r)} u \leq K_1 \text{ ess inf}_{\Omega(\overline{\sigma}, r)} u
\]

where \(\Omega(\overline{\sigma}, r) := \{x \in \Omega : |x_i - \overline{x}_i| < r, \ i = 1, 2, \ldots, n\}\).
Proof: As in [1], it is convenient to follow the proof e.g. of Theorem 8.2 of [4], remarking only the differences due to the presence of the new term \( \int g \nu \, d\sigma \) in the definition of the bilinear form \( a(\cdot, \cdot) \) (see (2)).

Let us begin with the proof of Lemma 8.1 of [4]. Proceeding in the same manner, we find in the right member of the last inequality (at the end of page 239) the new term \( |p| \int |g| a^2 \nu^2 \, d\sigma \). By using the inequality (33) of [1] we get at once

\[
(5) \quad \int_{\mathcal{T} \cap Q} |g| a^2 \nu^2 \, d\sigma \leq K_4^2 \|g\|_{L^{s-1}(\mathcal{T} \cap Q)}^2 \|(a\nu)_x\|_{L^2(\mathcal{T} \cap Q)}^2
\]

where we have denoted for brevity \( Q := \{ x \in \mathbb{R}^n : |x_i - x_\tau| < r, \ i = 1, 2, \ldots, n \} \) and \( a \in C^1_0(Q) \). Note that in (5) the quantity \( \|g\|_{L^{s-1}(\mathcal{T} \cap Q)} \) can be made as small as we like by a suitable choice of \( \tau \). Therefore, combining (5) and the proof of Lemma 8.1 in [4], we can bring the term \( \|a\nu_x\|_{L^2(\mathcal{T} \cap Q)}^2 \) to the left member of the last inequality of page 239 of [4], so the conclusion follows as in [4].

As what concerns Lemma 8.2 of [4], even in this case we find in the inequality (8.8) the new term \( \int |g| a^2 \nu \, d\sigma \). In order to evaluate this integral, we can use (5) where we put \( \nu = 1 \). Since it is \( a(x) = 0 \) if \( x \notin Q(\bar{x}, 2\rho) \), \( |a_x| \leq 2/\rho \) (see [4]), we easily deduce

\[
\int_{\mathcal{T}} |g| a^2 \, d\sigma \leq K_6 \|g\|_{L^{s-1}(\mathcal{T} \cap Q)} \rho^{n-2}
\]

where \( K_6 \) depends only on \( K_4 \) and \( n \); the rest of the proof continues as in [4].

Also the remaining part of the proof of Theorem 8.1 of [4] can be followed in our new situation. There are few differences, for example in the proof of Lemma 8.4 (page 241 at the end) we must use Theorem 2 of [1] instead of the Remark 5.1 of [4]. The simple details can be left to the reader. \( \square \)

Corollary 1. Suppose that all the hypotheses listed in the previous paragraph are verified; let \( u \in V \) be a non negative solution of the equation (3). Then there exist two positive constants \( K_2 \) and \( \tau \), depending on the coefficients of \( a(\cdot, \cdot) \) and on the regularity of \( \mathcal{T} \), such that for any \( \bar{x} \in \mathcal{T} \) and any \( r \) with \( 0 < r \leq \tau \) it turns out

\[
\text{ess} \sup_{\mathcal{T} \cap r} u \leq K_2 \text{ess inf}_{\mathcal{T} \cap r} u + K_2 \left( \sum_{i=1}^{n} \|f_i\|_{L^1(\mathcal{T})} + \|f_0\|_{L^{n/(n+1)\rho}(\mathcal{T})} + \|b\|_{L^1(\mathcal{T})} \right)^{1-n/p}
\]

Proof: We can proceed again as in [4] (Theorem 8.2), where we must use Lemma 3 of [1] instead of Theorem 5.2 of [4]. \( \square \)
Corollary 2. Suppose that all the hypotheses listed in the previous paragraph are verified; let \( u \in V \) be a non negative solution of the equation

\[
a(u, v) = 0 \quad \forall v \in V
\]

Let \( G \) be an open subset of \( \Omega \) such that \( \overline{G} \cap \partial \Omega \subset \Gamma \). Then there exists a positive constant \( K_3 \), depending on \( G \), \( \Gamma \) and the coefficients of \( a(., .) \), such that it turns out

\[
\text{ess sup}_G u \leq K_3 \text{ ess inf}_G u
\]

Proof: As in [4] it is possible to deduce our assertion from the previous Theorem, by covering \( \overline{G} \) by a finite number of sufficiently small cubes. In particular we must cover \( \overline{G} \cap \partial \Omega = \overline{G} \cap \Gamma \) by small cubes with centres on \( \Gamma \), in such a way as to apply the Theorem to each of them, while we can apply Theorem 8.1 of [4] to the cubes contained in \( \Omega \). The details of the proof can be left to the reader. \( \square \)

REFERENCES