Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
123° (2005), Vol. XXIX, fasc. 1, pagg. 123-148

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Global Attractors for Non-Autonomous Ginzburg–Landau Equation with Singularly Oscillating Terms (**)

Abstract. — We study the global attractor $\mathcal{A}^\varepsilon$ of the non-autonomous complex Ginzburg–Landau (G.–L.) equation with singularly oscillating external force of the form $g_0(x,t) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right)$, $t \geq 0$, $x \in \Omega \Subset \mathbb{R}^n$, $n \geq 3$, $0 < \rho \leq 1$. We assume that the dispersion coefficient $\beta(t)$ in the G.–L. equation satisfies the inequality $|\beta(t)| \leq \sqrt{3}$, $t \geq 0$. In this case, the Cauchy problem for the G.–L. equation has a unique solution in a weak sense and the corresponding semiprocess $\{U_\varepsilon(t,\tau),\ t \geq \tau \geq 0\}$ acting in the space $\boldsymbol{H} = L_2(\Omega; \mathbb{C})$ has the global attractor $\mathcal{A}^\varepsilon$ such that $\|\mathcal{A}^\varepsilon\|_{\boldsymbol{H}} \leq C\varepsilon^{-\rho}$ for $\varepsilon > 0$. Along with this G.–L. equation, we consider its "limit" equation with external force $g_0(x,t)$. We assume that the function $g_1(z,t)$ has the following divergence presentation: $g_1(z,t) = \Sigma_{i=1}^n \partial_{z_1} G_i(z,t)$ $(z = (z_1, \dots, z_n) \in \mathbb{R}^n_z)$, where the norms of the functions $G_i(z,t)$ are bounded in the space $C_b(\mathbb{R}_+; \boldsymbol{Z})$, $\boldsymbol{Z} = L_2^b(\mathbb{R}^n_z; \mathbb{C})$ (see (50) and (51)).

We have found the estimate for the deviation (in $\boldsymbol{H}$) of the solutions of the original G.–L. equation from the solutions of the corresponding "limit" equation with the same initial data.

When the coefficients and the external force of the G.–L. equation are almost periodic (a.p.) functions in time $t \in \mathbb{R}$, we consider the family of G.–L. equations whose coefficients and external forces $\left(\hat{\alpha}(t), \hat{\beta}(t), \hat{R}(t), \hat{g}_0(x,t) + \frac{1}{\varepsilon^\rho} \hat{g}_1\left(\frac{x}{\varepsilon}, t\right)\right)$ belong to the hull of the initial coefficients and external force. We have proved that if all the functions $\hat{g}_1(z,t)$ belonging to the hull of the initial function $g_1(z,t)$ admits a divergence presentation described above and the exponent ρ in the amplitude $\frac{1}{\varepsilon^\rho}$ is sufficiently small, then the global attractors $\mathcal{A}^\varepsilon$ are uniformly (with respect to $\varepsilon, 0 < \varepsilon \leq 1$) bounded in $\boldsymbol{H}$: $\|\mathcal{A}^\varepsilon\|_{\boldsymbol{H}} \leq C(\rho)$, where $C(\rho)$ is independent of ε.

We have also studied the case where the global attractor $\mathcal{A}^0$ of the "limit" G.–L. equation is exponential. In such a situation, we have proved the estimate for the deviation of the global attractor $\mathcal{A}^\varepsilon$ from $\mathcal{A}^0$: $\text{dist}_{\boldsymbol{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leq C(\rho)\varepsilon^{1-\rho}$ for all ε, $0 < \varepsilon \leq 1$, $0 < \rho < 1$, where the constant $C(\rho)$ is independent of ε.

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(**) The work is partially supported by the Russian Foundation of Basic Researches (project no. 05-01-00390), the Civilian Research & Development Foundation (CRDF) Grant no. RUMI-2654-MO-05, and by the Russian Science Support Foundation.

Introduction

The complex Ginzburg–Landau equation plays an important role in the theory of superconductivity and in the nonlinear optic. It appears in many mathematical models of various pattern formation systems in mechanics, physics, and chemistry. This equation also describes the finite amplitude evolution of instability waves in large variety of dissipative systems in the points of transition from regular to turbulence behaviour, for example, in the theory of hydrodynamic instability: the Reyleigh–Bérnard convection, the Poiseuille and the Taylor–Couette problems, etc. (see the review [1] and the literature cited therein). The mathematical questions related to the Ginzburg–Landau equation such as well-posedness of solutions, the properties of special solution classes, the study of global attractors of the corresponding semigroup, the number of degrees of freedom of the related dynamical systems, etc., were studied in many papers and books (see, for example, [2]–[14]).

Some problems related to the homogenization and averaging of global attractors of evolution equations of mathematical physics with rapidly (non-singularly) oscillating coefficients and terms were studied in [15]–[24].

In the paper, we study the non-autonomous Ginzburg–Landau (G.–L.) equation with singularly oscillating external force of the form

$$(1) \quad \partial_t u = (1 + i\alpha(t))\Delta u + R(t)u - (1 + i\beta(t))|u|^2 u + g_0(x,t) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right), \; u|_{\partial\Omega} = 0,$$

where $u = u_1(x,t) + iu_2(x,t)$ is a complex function, $x \in \Omega \Subset \mathbb{R}^n$, $0 \in \Omega$, $t \geq 0$. The coefficients $\alpha(t), \beta(t)$, and $R(t)$ are real functions belonging to the space $C_b(\mathbb{R}_+)$. We assume that

$$|\beta(t)| \leq \sqrt{3}, \; t \in \mathbb{R}_+.$$

In (1), ρ is a positive parameter such that $0 < \rho \leq \rho_0 \leq 1$. The value of ρ_0 will be given explicitly. We shall use the spaces $\boldsymbol{H} = L_2(\Omega; \mathbb{C})$ and $\boldsymbol{Z} = L_2^{\mathrm{b}}(\mathbb{R}^n; \mathbb{C})$. We assume that $g_0(x,t) \in L_2^{\mathrm{b}}(\mathbb{R}_+; \boldsymbol{H})$, that is,

$$(2) \qquad \|g_0(\cdot,\cdot)\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;\boldsymbol{H})} := \sup_{\tau\in\mathbb{R}_+} \int_\tau^{\tau+1} \|g_0(\cdot,s)\|^2_{\boldsymbol{H}} ds < +\infty,$$

and the function $g_1(z,t) \in L_2^{\mathrm{b}}(\mathbb{R}_+; \boldsymbol{Z})$ $(z = (z_1, z_2, \ldots, z_n))$, where $\boldsymbol{Z} = L_2^{\mathrm{b}}(\mathbb{R}_z^n; \mathbb{C})$. The norm in the space $\boldsymbol{Z}$ is defined in Section 1 (see (19)).

For every $u_0(\,\cdot\,) \in \boldsymbol{H}$, the Cauchy problem for equation (1) with initial data $u|_{t=0} = u_0(x)$ has a unique solution $u(t) := u(x,t)$ belonging to the space

$$(3) \qquad C_{\mathrm{b}}(\mathbb{R}_+; \boldsymbol{H}) \cap L_2^{\mathrm{b}}(\mathbb{R}_+; \boldsymbol{V}) \cap L_4^{\mathrm{b}}(\mathbb{R}_+; \boldsymbol{L}_4),$$

where $\boldsymbol{V} = H_0^1(\Omega; \mathbb{C})$, $\boldsymbol{L}_4 = L_4(\Omega; \mathbb{C})$. The function $u(t)$ satisfies equation (1) in the weak distribution sense (see Section 1).

In Section 1, we prove that the semiprocess $\{U_\varepsilon(t,\tau), t \geq \tau \geq 0\}$ corresponding to (1) (recall that $U_\varepsilon(t,\tau)u_\tau = u(t),\ t \geq \tau$, where $u(t)$ is a weak solution of (1) with initial data $u_\tau \in \boldsymbol{H}$) has the compact absorbing set $B_{1,\varepsilon}$:

$$B_{1,\varepsilon} = \{v \in \boldsymbol{V} \mid \|v\|_{\boldsymbol{V}} \leq C(1 + \|g_0\|_{L_2^b(\mathbb{R}_+;\boldsymbol{H})} + \varepsilon^{-\rho}\|g_1\|_{L_2^b(\mathbb{R}_+;\boldsymbol{Z})})\}. \tag{4}$$

We note that the norm $|B_{1,\varepsilon}|_{\boldsymbol{H}} \leq C\varepsilon^{-\rho},\ 0 < \varepsilon \leq 1$, and $|B_{1,\varepsilon}|_{\boldsymbol{H}}$ may tend to infinity as $\varepsilon \to 0+$.

Along with equation (1), we consider the "limit" equation of the form

$$\partial_t u^0 = (1 + i\alpha(t))\Delta u^0 + R(t)u^0 - (1 + i\beta(t))|u^0|^2 u^0 + g_0(x,t),\ u^0|_{\partial\Omega} = 0. \tag{5}$$

This equation also generates a semiprocess $\{U_0(t,\tau), t \geq \tau \geq 0\}$ acting in $\boldsymbol{H}$ and having the compact absorbing set $B_{1,0} = \{v \in \boldsymbol{V} \mid \|v\|_{\boldsymbol{V}} \leq C(1 + \|g_0\|_{L_2^b(\mathbb{R}_+;\boldsymbol{H})})\}$.

We assume that the function $g_1(z,t), z \in \mathbb{R}^n, t \geq 0$, in the right-hand side of (1) satisfies the following condition (see also (50) and (51)).

Condition I. There exist functions $G_j(z,t) \in C_b(\mathbb{R}_+;\boldsymbol{Z})$ with $\dfrac{\partial G_j}{\partial z_j} \in L_2^{loc}(\mathbb{R}_+;\boldsymbol{Z})$ for $j = 1,2,\ldots,n$ ($z = (z_1,\ldots,z_n)$), such that

$$\sum_{j=1}^{n} \frac{\partial G_j}{\partial z_j}(z,t) = g_1(z,t),\ t \in \mathbb{R}_+. \tag{6}$$

Let $u(x,t), t \geq 0$,and $u^0(x,t), t \geq 0$, be solutions of equations (1) and (5) respectively with common initial data

$$u|_{t=0} = u_0(x),\ u^0|_{t=0} = u_0(x),\ u_0(\cdot) \in \boldsymbol{H}.$$

We set $\bar{R} = \sup_{t\in\mathbb{R}_+} R(t)$. In Section 2, we prove that the deviation $w(t) = u(\cdot,t) - u^0(\cdot,t)$ of the solutions satisfies the following estimate:

$$\|w(t)\|_{\boldsymbol{H}} = \|u(\cdot,t) - u^0(\cdot,t)\|_{\boldsymbol{H}} \leq C\varepsilon^{1-\rho}e^{rt},\ \forall t \geq 0, \tag{7}$$

where $r = 0$ for $\bar{R} < \lambda_1$ and $r = \bar{R} - \lambda_1 + \delta$ for $\bar{R} \geq \lambda_1$ ($\delta > 0$ is arbitrary small, and $C = C(\delta)$). Here λ_1 is the first eigenvalue of the minus Laplace operator $\{-\Delta u,\ u|_{\partial\Omega} = 0\}$. Notice that, in some sense, estimate (7) is a generalization of the known estimates of N. N. Bogolubov (see [25]).

In Section 4, we study the global attractor $\mathcal{A}^\varepsilon$ of the non-autonomous G.–L. equation (1). We assume that the coefficients $\alpha(t),\ \beta(t)$, and $R(t)$ are defined for all $t \in \mathbb{R}$ and they are almost periodic (a.p.) functions with values in $\mathbb{R}$. We also assume that the functions $g_0(x,t)$ and $g_1(z,t)$ are defined for $t \in \mathbb{R}$ and they are a.p. with values in $\boldsymbol{H} = L_2(\Omega;\mathbb{C})$ and $\boldsymbol{Z} = L_2^b(\mathbb{R}^n;\mathbb{C})$, respectively (see Section 3).We recall that a.p. functions with values in function spaces were introduced in the works of S.Bochner and L.Amerio (see [26]).

Along with equation (1), we consider the family of equations

$$\partial_t\hat{u}^\varepsilon = (1 + i\hat{\alpha}(t))\Delta\hat{u}^\varepsilon + \hat{R}(t)\hat{u}^\varepsilon - (1 + i\hat{\beta}(t))|\hat{u}^\varepsilon|^2\hat{u}^\varepsilon + \hat{g}_0(x,t) + \frac{1}{\varepsilon^\rho}\hat{g}_1\left(\frac{x}{\varepsilon},t\right), \hat{u}^\varepsilon|_{\partial\Omega} = 0. \tag{8}$$

The function $\hat{\sigma}^\varepsilon(t) = (\hat{\alpha}(t),\hat{\beta}(t),\hat{R}(t),\hat{g}_0(x,t) + \frac{1}{\varepsilon^\rho}\hat{g}_1\left(\frac{x}{\varepsilon},t\right)), t \in \mathbb{R}$, is called the *symbol* of this equation. The symbols $\hat{\sigma}^\varepsilon(t)$ of equations (8) belong to the hull $\mathcal{H}(\sigma^\varepsilon(t))$ of the symbol $\sigma^\varepsilon(t) = (\alpha(t), \beta(t), R(t), g_0(x,t) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right))$ of the initial equation (1). The hull $\mathcal{H}(\sigma^\varepsilon(t))$ is taken in the space $C_b(\mathbb{R}; \mathbb{R}^3 \times \boldsymbol{H})$ and it is compact in this space since the function $\sigma^\varepsilon(t)$ is a.p. with values in $\mathbb{R}^3 \times \boldsymbol{H}$ (see Section 4). The family of equations (8) with symbols $\hat{\sigma}^\varepsilon \in \mathcal{H}(\sigma^\varepsilon)$ generates the family of processes $\{U_{\hat{\sigma}^\varepsilon}(t,\tau),\ t \geq \tau, \tau \in \mathbb{R}\}$, $\hat{\sigma}^\varepsilon \in \Sigma^\varepsilon$, acting in $\boldsymbol{H}$. In Section 4, we prove that this family has the compact absorbing set $B_{1,\varepsilon}$ (see (4)). Moreover, the family of processes $\{U_{\hat{\sigma}^\varepsilon}(t,\tau)\}$, $\hat{\sigma}^\varepsilon \in \Sigma^\varepsilon$, has the global attractor $\mathcal{A}^\varepsilon$ such that $\mathcal{A}^\varepsilon \subset B_{1,\varepsilon}$. It follows from (4) that

$$\|\mathcal{A}^\varepsilon\| \leq C\varepsilon^{-\rho},\ 0 < \rho \leq 1. \tag{9}$$

In Section 4, we present the condition that provides the uniform boundedness (w.r.t. $\varepsilon \in (0,1]$) of the global attractors $\mathcal{A}^\varepsilon$.

Condition $\hat{\mathrm{I}}$. For every $\hat{g}_1(z,t) \in \mathcal{H}(g_1)$, there exist functions $\hat{G}_j(z,t) \in C_b(\mathbb{R}; \boldsymbol{Z})$ with $\partial_{z_j} \hat{G}_j \in L_2^{loc}(\mathbb{R}; \boldsymbol{Z})$ for $j = 1, 2, \ldots, n$ $(z = (z_1, \ldots, z_n))$, such that

$$\sum_{j=1}^{n} \frac{\partial \hat{G}_j}{\partial z_j}(z,t) = \hat{g}_1(z,t),\ \forall z \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{10}$$

and

$$\|\hat{G}_j(\cdot,\cdot)\|_{C_b(\mathbb{R};\boldsymbol{Z})} \leq M, \tag{11}$$

where the constant M is independent of $\hat{g}_1(z,t)$ (recall that the function $g_1(z,t)$ is a.p. with values in $\boldsymbol{Z}$).

In Section 4, we prove the following main result of the paper. *Let the number ρ satisfy the inequality*

$$0 < \rho < \rho_0,$$

where $\rho_0 = 1$ for $\bar{R} < \lambda_1$ and $\rho_0 = \lambda_1/\bar{R}$ for $\bar{R} \geq \lambda_1$ ($\bar{R} = \sup_{t \in \mathbb{R}} R(t)$). Then, under Condition $\hat{I}$, the global attractors A^ε of equations (8) *are uniformly (w.r.t. $\varepsilon \in]0,1]$) bounded in $\boldsymbol{H}$, that is,*

$$\|\mathcal{A}^\varepsilon\|_{\boldsymbol{H}} \leq C(\rho),\ \forall \rho,\ 0 < \rho \leq 1,$$

where the constant $C(\rho)$ is independent of ε (compare with (9)).

In Sections 5 and 6, we consider the "limit" family of G.–L. equations corresponding to the family of equations (8):

$$\partial_t \hat{u}^0 = (1 + i\hat{\alpha}(t))\Delta \hat{u}^0 + \hat{R}(t)\hat{u}^0 - (1 + i\hat{\beta}(t))|\hat{u}^0|^2 \hat{u}^0 + \hat{g}_0(x,t), \hat{u}^0|_{\partial\Omega} = 0, \tag{12}$$

with symbols $\hat{\sigma}^0(t) = (\hat{\alpha}(t), \hat{\beta}(t), \hat{R}(t), \hat{g}_0(t)) \in \mathcal{H}(\sigma^0)$, where $\mathcal{H}(\sigma^0)$ is the hull of the symbol $\sigma^0(t) = (\alpha(t), \beta(t), R(t), g_0(t)), t \in \mathbb{R}$, whose terms are taken from the original equation (1). Recall that the function $\sigma(t), t \in \mathbb{R}$, is a.p. with values in $\mathbb{R}^3 \times \boldsymbol{H}$. Then the family of processes $\{U_{\hat{\sigma}^0}(t,\tau)\}$, $\hat{\sigma}^0 \in \mathcal{H}(\sigma^0)$, corresponding to equations (12) has the global attractor $\mathcal{A}^0$ such that the set $\mathcal{A}^0$ is compact in $\boldsymbol{H}$ and $\mathcal{A}^0 \subset B_{1,0}$ (see Section 5).

We now assume that

$$R(t) \le \lambda_1 - \kappa, \ \forall t \in \mathbb{R}, \ (\kappa > 0). \tag{13}$$

In this case in Section 5, we prove that the global attractor $\mathcal{A}^0$ attracts bounded sets of initial data with exponential rate. Finally, under Condition $\hat{I}$ and inequality (13) we prove that the Hausdorff distance (in $\boldsymbol{H}$) from the global attractor $\mathcal{A}^\varepsilon$ of the original equations (8) to the global $\mathcal{A}^0$ of the "limit" equations (12) satisfies the estimate

$$\mathrm{dist}_{\boldsymbol{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \le C(\rho)\varepsilon^{1-\rho}, \ \forall \varepsilon, \ 0 < \varepsilon \le 1, \ \forall \rho, \ 0 < \rho < 1, \tag{14}$$

where $C(\rho)$ is independent of ε.

In this paper, we have developed the method that can also be successfully applied to the study of various reaction-diffusion type systems with singularly oscillating terms. The similar results can be obtained for many other non-autonomous partial differential equations and systems arising in the problems of mathematical physics.

1. - Complex Ginzburg–Landau equation with singularly oscillating external force

We consider the following non-autonomous Ginzburg–Landau (G.–L.) equation:

$$\partial_t u = (1 + i\alpha(t))\Delta u + R(t)u - (1 + i\beta(t))|u|^2 u + g_0(x,t) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right), \ u|_{\partial\Omega} = 0. \tag{15}$$

Here $u = u_1(x,t) + iu_2(x,t)$ is an unknown complex function of the arguments $x \in \Omega \Subset \mathbb{R}^n, 0 \in \Omega, \ t \ge 0$. The real functions $\alpha(t), \beta(t), R(t) \in C_b(\mathbb{R}_+)$ are given and

$$|\beta(t)| \le \sqrt{3}, \ t \in \mathbb{R}_+. \tag{16}$$

In equation (15), ρ is a positive parameter, $0 < \rho \le \rho_0 \le 1$. The value ρ_0 will be given below. We set $\boldsymbol{H} = L_2(\Omega; \mathbb{C})$ and $\boldsymbol{Z} = L_2^{\mathrm{b}}(\mathbb{R}^n; \mathbb{C})$. The norm in $\boldsymbol{H}$ is denoted by $\|\cdot\|_{\boldsymbol{H}}$. A function $f(z) \in \boldsymbol{Z} = L_2^{\mathrm{b}}(\mathbb{R}^n_z; \mathbb{C})$ $(z = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n)$ if

$$\|f(\cdot)\|^2_{\boldsymbol{Z}} = \|f(\cdot)\|^2_{L_2^{\mathrm{b}}(\mathbb{R}^n_z;\mathbb{C})} := \sup_{z \in \mathbb{R}^n} \int_{z_1}^{z_1+1} \cdots \int_{z_n}^{z_n+1} |f(\zeta_1, \dots, \zeta_n)|^2 d\zeta_1 \cdots d\zeta_n < +\infty. \tag{17}$$

We assume that the function $g_0(x,t) = g_{01}(x,t) + ig_{02}(x,t), \ x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, belongs to the space $L_2^{\mathrm{b}}(\mathbb{R}_+; \boldsymbol{H})$ and the function $g_1(z,t) = g_{11}(z,t) + ig_{12}(z,t), z = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n$, belongs to $L_2^{\mathrm{b}}(\mathbb{R}_+; \boldsymbol{Z})$, i.e. the following norms of these functions are finite:

$$\|g_0(\cdot,\cdot)\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;\, \boldsymbol{H})} := \sup_{\tau \in \mathbb{R}_+} \int_\tau^{\tau+1} \|g_0(\cdot, s)\|^2_{\boldsymbol{H}}\, ds = \sup_{\tau \in \mathbb{R}_+} \int_\tau^{\tau+1} \left(\int_\Omega |g_0(x,s)|^2 dx \right) ds < +\infty, \tag{18}$$

$$\text{(19)}\quad \|g_1(\cdot,\cdot)\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;\boldsymbol{Z})} := \sup_{\tau\in\mathbb{R}_+}\int\limits_{\tau}^{\tau+1}\|g_1(\cdot,s)\|^2_{\boldsymbol{Z}}\,ds =$$

$$= \sup_{\tau\in\mathbb{R}_+}\int\limits_{\tau}^{\tau+1}\left(\sup_{z\in\mathbb{R}^n}\int\limits_{z_1}^{z_1+1}\cdots\int\limits_{z_n}^{z_n+1}|g_1(\zeta_1,\ldots,\zeta_n,s)|^2 d\zeta_1\cdots d\zeta_n\right)ds < +\infty,$$

where $z = (z_1, z_2, \ldots, z_n)$.

Equation (15) is equivalent to the following system of two equations for the vector-function $\boldsymbol{u} = (u_1, u_2)^\top$:

$$\text{(20)}\quad \partial_t\boldsymbol{u} = \begin{pmatrix}1 & -\alpha(t)\\ \alpha(t) & 1\end{pmatrix}\Delta\boldsymbol{u} + R\boldsymbol{u} - \begin{pmatrix}1 & -\beta(t)\\ \beta(t) & 1\end{pmatrix}|\boldsymbol{u}|^2\boldsymbol{u} + \boldsymbol{g}_0(x,t) + \frac{1}{\varepsilon^\rho}\,\boldsymbol{g}_1\Big(\frac{x}{\varepsilon},t\Big),$$

where $\boldsymbol{g}_0 = (g_{01}, g_{02})^\top$ and $\boldsymbol{g}_1 = (g_{11}, g_{12})^\top$.

Under the above assumption for every fixed ε, $0 < \varepsilon \le 1$, the Cauchy problem for equation (15) with initial data

$$\text{(21)}\qquad u|_{t=0} = u_0(x),\ u_0(\,\cdot\,) \in \boldsymbol{H},$$

has a unique weak solution $u(t) := u(x,t)$ such that

$$\text{(22)}\qquad \begin{gathered} u(\,\cdot\,) \in C_{\mathrm{b}}(\mathbb{R}_+;\boldsymbol{H}),\ u(\,\cdot\,) \in L_2^{\mathrm{b}}(\mathbb{R}_+;\boldsymbol{V}),\ u(\,\cdot\,) \in L_4^{\mathrm{b}}(\mathbb{R}_+;\boldsymbol{L}_4),\\ \boldsymbol{V} = H_0^1(\Omega;\mathbb{C}),\ \boldsymbol{L}_4 = L_4(\Omega;\mathbb{C}),\end{gathered}$$

and the function $u(t)$ satisfies equation (15) in the sense of distributions of the space $\mathcal{D}'(\mathbb{R}_+;\boldsymbol{H}^{-r})$, where $\boldsymbol{H}^{-r} = H^{-r}(\Omega;\mathbb{C})$ and $r = \max\{1, n/4\}$ (recall that $n = \dim(\Omega)$). In particular, $\partial_t u(\,\cdot\,) \in L_2(0,M;\boldsymbol{H}^{-1}) + L_{4/3}(0,M;\boldsymbol{L}_{4/3})$ for any $M > 0$. The existence of such solution $u(t)$ is proved, for example, using the Galerkin approximation method (see, e.g. [2, 28, 3]). The proof of the uniqueness theorem is also standard (see Section 2 and, e.g., [3, pages 42, 118]) and relays on inequality (16). (We note that, if (16) does not hold, the uniqueness theorem for $n \ge 3$ and for arbitrary values of the dispersion parameters α and β is not proved yet, see [1, 10, 12] for important partial uniqueness results).

For brevity, we set $\|\cdot\| := \|\cdot\|_{\boldsymbol{H}}$. Any solution $u(t), t \ge 0$, of equation (15) satisfies the following differential identity:

$$\text{(23)}\qquad \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|^4_{\boldsymbol{L}_4} - R(t)\|u(t)\|^2 = \langle g^\varepsilon(t), u(t)\rangle,\ \forall t \ge 0,$$

where we denote $g^\varepsilon(t) := g_0(x,t) + \varepsilon^{-\rho}g_1\Big(\frac{x}{\varepsilon},t\Big)$. The function $\|u(t)\|^2$ is absolutely continuous for $t \ge 0$.

The proof of (23) is analogous to the proof of the corresponding identity for weak solutions of the reaction-diffusion systems considered in [3, 27] (see also [14]). For reader's convenience, we sketch the main idea of this reasoning.

A weak solution $u(t)$ of problem (15) and (21) can be written in the form

$$\text{(24)}\qquad \partial_t u(t) = w(t) + h(t),$$

where

(25) $$w(t) = (1 + i\alpha(t))\Delta u(t) + g^{\varepsilon}(t),\ h(t) = R(t)u(t) - (1 + i\beta(t))|u(t)|^2 u(t).$$

It follows from (22) and (25) that $w(t) \in L_2(0, M; \boldsymbol{H}^{-1})$ and $h(t) \in L_{4/3}(0, M; \boldsymbol{L}_{4/3})$ for all $M > 0$. Then we use the following natural generalization of the theorem proved in [29, Ch.III, Sec.1.2].

THEOREM 1.1: *Let H be a Hilbert space and let V, E, X be Banach spaces such that:*

$$V \subseteq H \subseteq V' \subseteq X \text{ and } E \subseteq H \subseteq E' \subseteq X.$$

Assume that $u(\,\cdot\,) \in L_2(0, M; V) \cap L_p(0, M; E)$ $(p > 1)$ and the distribution $\partial_t u(\,\cdot\,) \in \mathcal{D}'(0, M; X)$ is presented in the form $\partial_t u(t) = w(t) + h(t)$, where $w(\,\cdot\,) \in L_2(0, M; V')$ and $h(\,\cdot\,) \in L_q(0, M; E')$ $(1/p + 1/q = 1)$. Then $u(\,\cdot\,) \in C(\mathbb{R}_+; \boldsymbol{H})$, the function $\|u(t)\|^2_{\boldsymbol{H}}$ is absolutely continuous on $[0, M]$, and

(26) $$\frac{d}{dt}\|u(t)\|^2_{\boldsymbol{H}} = 2\langle w(t), u(t)\rangle + 2\langle h(t), u(t)\rangle,\ \forall t \in [0, M].$$

To apply Theorem 1.1 to a solution $u(t)$ of problem (15) and (21) we observe that $\langle w(t), u(t)\rangle = \langle (1 + i\alpha(t))\Delta u(t) + g^{\varepsilon}(t), u(t)\rangle = -\|\nabla u(t)\|^2 + \langle g^{\varepsilon}(t), u(t)\rangle$ and $\langle h(t), u(t)\rangle = R(t)\|u(t)\|^2 - \|u(t)\|^4_{\boldsymbol{L}_4}$ (see (25)). Thus, (26) implies (23) (see also [3, 14]).

Using the standard transformations and the Gronwall lemma, we deduce from (23) that any weak solution $u(t)$ of equation (15) satisfies the inequality

(27) $$\|u(t)\|^2 \le \|u(0)\|^2 e^{-2\lambda_1 t} + C_0^2 + C_1^2 \varepsilon^{-2\rho},\ \forall t \ge 0.$$

where λ_1 is the first eigenvalue of the operator $\{-\Delta u,\ u|_{\partial\Omega} = 0\}$, the constant C_0 depends on $\|R\|_{C_b} = \sup_{t\ge 0}|R(t)|$ and $\|g_0\|_{L_2^{\mathrm{b}}(\mathbb{R}_+; \boldsymbol{H})}$ and the constant C_1 depends on $\|g_1\|_{L_2^{\mathrm{b}}(\mathbb{R}_+; \boldsymbol{Z})}$ (see (18) and (19)). We also use the following inequality:

(28) $$\int_0^t \int_\Omega \left|g_1\left(\frac{x}{\varepsilon}, s\right)\right|^2 e^{-\lambda_1(t-s)} dx ds \le C\|g_1\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+; \boldsymbol{Z})},$$

where C is independent of ε. Indeed,

$$\int_0^t \int_\Omega \left|g_1\left(\frac{x}{\varepsilon}, s\right)\right|^2 e^{-\lambda_1(t-s)} dx ds = \int_0^t e^{-\lambda_1(t-s)} \left(\varepsilon^n \int_{\varepsilon^{-1}\Omega} |g_1(z, s)|^2 dz\right) ds \le$$

$$\le C' \int_0^t e^{-\lambda_1(t-s)} \left(\sup_{z\in\mathbb{R}^n} \int_{z_1}^{z_1+1} \cdots \int_{z_n}^{z_n+1} |f(\zeta_1, \ldots, \zeta_n, s)|^2 d\zeta_1 \cdots d\zeta_n\right) ds \le$$

$$\le C''(\lambda_1)\|g_1\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+; \boldsymbol{Z})},$$

since we can cover the domain $\varepsilon^{-1}\Omega$ by $C'\varepsilon^{-n}$ unit boxes and, therefore, (28) is proved (see [3] for more details).

Clearly inequality of the type (27) holds with 0 and t being replaced by any τ and $t+\tau$, respectively, i.e.,

$$\|u(t+\tau)\|^2 \le \|u(\tau)\|^2 e^{-2\lambda_1 t} + C_0^2 + C_1^2 \varepsilon^{-2\rho}, \ \forall t \ge \tau \ge 0. \tag{29}$$

Integrating (23) in t and using (27), we obtain that

$$\frac{1}{2}\|u(t)\|^2 + \int_0^t \left(\|\nabla u(s)\|^2 + \|u(t)\|^4_{\boldsymbol{L}_4}\right) ds \le \le \frac{1}{2}\|u(0)\|^2 + \|R\|_{C_b} \int_0^t \|u(s)\|^2 ds + \int_0^t \|g^{\varepsilon}(s)\| \cdot \|u(s)\| ds, \tag{30}$$

$$\int_0^t \left(\|\nabla u(s)\|^2 + \|u(t)\|^4_{\boldsymbol{L}_4}\right) ds \le \le \frac{1}{2}\|u(0)\|^2 + C_2(t+1) + C_3\left(\|g_0\|^2_{L_2^b(\mathbb{R}_+;\boldsymbol{H})} + \varepsilon^{-2\rho}\|g_1\|^2_{L_2^b(\mathbb{R}_+;\boldsymbol{Z})}\right)t,$$

(see (18) and (19)).

We consider the semiprocess $\{U_\varepsilon(t,\tau)\} := \{U_\varepsilon(t,\tau), t \ge \tau \ge 0\}$ corresponding to problem (15) and (21) and acting in the space $\boldsymbol{H}$ (see [3]). Recall that the mapping $U_\varepsilon(t,\tau) : \boldsymbol{H} \to \boldsymbol{H},\ t \ge \tau \ge 0$, is defined by the formula

$$U_\varepsilon(t,\tau)u_\tau = u(t),\ t \ge \tau \ge 0,\ \forall u_\tau \in \boldsymbol{H},$$

where $u(t), t \ge \tau$, is a solution of equation (15) with initial data $u|_{t=\tau} = u_\tau$. It follows from estimates (27) and (29) that the semiprocess $\{U_\varepsilon(t,\tau)\}$ has the uniformly (with respect to $\tau \in \mathbb{R}_+$) absorbing set

$$B_{0,\varepsilon} = \{v \in \boldsymbol{H} \mid \|v\| \le 2(C_0 + C_1\varepsilon^{-\rho})\} \tag{31}$$

that is bounded in $\boldsymbol{H}$ for every fixed $\varepsilon > 0$.

We now demonstrate that the semiprocess $\{U_\varepsilon(t,\tau)\}$ has a compact in $\boldsymbol{H}$ uniformly (with respect to $\tau \in \mathbb{R}_+$) absorbing set

$$B_{1,\varepsilon} = \{v \in \boldsymbol{V} \mid \|v\|_{\boldsymbol{V}} \le C_0' + C_1'\varepsilon^{-\rho}\}. \tag{32}$$

To prove this fact we take the scalar product in $\boldsymbol{H}$ of equation (15) with the term $-t\Delta u$. After the standard transformations, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(t\|\nabla u\|^2\right) - \frac{1}{2}\|\nabla u\|^2 + t\|\Delta u\|^2 - R(t)t\|\nabla u\|^2 - -\left\langle(1+i\beta(t))|u|^2 u, t\Delta u\right\rangle = -\langle g_0, t\Delta u\rangle - \varepsilon^{-\rho}\langle g_1(x/\varepsilon), t\Delta u\rangle. \tag{33}$$

We denote

$$f(t,\boldsymbol{v}) = |\boldsymbol{v}|^2 \begin{pmatrix} 1 & -\beta(t) \\ \beta(t) & 1 \end{pmatrix} \boldsymbol{v},\ \ \boldsymbol{v} = (v_1, v_2).$$

We notice that since $|\beta(t)| \leq \sqrt{3}$ the matrix $f'_{\boldsymbol{v}}(t, \boldsymbol{v})$ is positive definite, that is,

$$(34) \qquad f'_{\boldsymbol{v}}(t, \boldsymbol{v})\boldsymbol{w} \cdot \boldsymbol{w} \geq 0, \ \forall \boldsymbol{v} = (v_1, v_2), \boldsymbol{w} = (w_1, w_2), \ \forall t \geq 0$$

(see [3, page 42]). Therefore, the term in (33) containing $\beta(t)$ is also positive. Indeed,

$$(35) \quad -\langle (1 + i\beta(t))|u|^2 u, t\Delta u \rangle = -f(t, \boldsymbol{u}) \cdot t\Delta \boldsymbol{u} = t\sum_{i=1}^{n} f'_{\boldsymbol{u}}(t, \boldsymbol{u})\partial_{x_i}\boldsymbol{u}\cdot\partial_{x_i}\boldsymbol{u} \geq 0, \ \forall t \geq 0.$$

Integrating both sides of (33) in t and taking into account (35), we have

$$(36) \qquad \frac{1}{2}t\|\nabla u(t)\|^2 - \frac{1}{2}\int_0^t \|\nabla u(s)\|^2 ds + \int_0^t s\|\Delta u(s)\|^2 ds - \int_0^t R(s)s\|\nabla u(s)\|^2 ds \leq$$

$$\leq -\int_0^t \langle g_0(s), s\Delta u(s)\rangle ds - \varepsilon^{-\rho}\int_0^t \langle g_1(x/\varepsilon, s), s\Delta u(s)\rangle ds.$$

Using (30), we obtain

$$(37) \quad \frac{1}{2}t\|\nabla u(t)\|^2 + C_5\int_0^t s\|\Delta u(s)\|^2 ds \leq \|R\|_{C_b}\int_0^t s\|\nabla u(s)\|^2 ds +$$

$$+ C_5\left(\int_0^t s\|g_0(s)\|^2 ds + \varepsilon^{-2\rho}\int_0^t s\|g_1(x/\varepsilon, s)\|^2 ds\right).$$

Applying in (37) an inequality similarly to (28), we find that

$$t\|\nabla u(t)\|^2 \leq C_7(t\|u(0)\|^2 + t + 1 + t\|g_0\|^2_{L_2^b(\mathbb{R}_+;\boldsymbol{H})} + t\varepsilon^{-2\rho}\|g_1\|^2_{L_2^b(\mathbb{R}_+;\boldsymbol{Z})}).$$

Assuming that $u(0) \in B_{0,\varepsilon}$ and setting $t = 1$, we obtain

$$(38) \qquad \|\nabla u(1)\| \leq C_8(1 + \|g_0\|_{L_2^b(\mathbb{R}_+;\boldsymbol{H})} + \varepsilon^{-\rho}\|g_1\|_{L_2^b(\mathbb{R}_+;\boldsymbol{Z})}).$$

Clearly, the same inequalities holds if we replace 0 and t with τ and $\tau + t$:

$$(39) \qquad t\|\nabla u(\tau + t)\|^2 \leq C_7(t\|u(\tau)\|^2 + t + 1 + t\|g_0\|^2_{L_2^b(\mathbb{R}_+;\boldsymbol{H})} + t\varepsilon^{-2\rho}\|g_1\|^2_{L_2^b(\mathbb{R}_+;\boldsymbol{Z})}),$$

$$\|\nabla u(\tau + 1)\| \leq C_8(1 + \|g_0\|_{L_2^b(\mathbb{R}_+;\boldsymbol{H})} + \varepsilon^{-\rho}\|g_1\|_{L_2^b(\mathbb{R}_+;\boldsymbol{Z})}), \ \forall \tau \geq 0.$$

It follows from (39) that the set

$$(40) \qquad B_{1,\varepsilon} = \{v \in \boldsymbol{V} \mid \|v\|_{\boldsymbol{V}} \leq C_8(1 + \|g_0\|_{L_2^b(\mathbb{R}_+;\boldsymbol{H})} + \varepsilon^{-\rho}\|g_1\|_{L_2^b})\}$$

is uniformly (w.r.t. $\tau \in \mathbb{R}_+$) absorbing for the semiprocess $\{U_\varepsilon(t, \tau)\}$ corresponding to the G.-L. equation (15). The set $B_{1,\varepsilon}$ is bounded in $\boldsymbol{V}$ and compact in $\boldsymbol{H}$ since the embedding $\boldsymbol{V} \Subset \boldsymbol{H}$ is compact. Recall that a semiprocess having a compact uniformly absorbing set is called *uniformly compact*. We have proved the following

Proposition 1.1: *For any fixed $\varepsilon > 0$, the semiprocess $\{U_\varepsilon(t, \tau)\}$ corresponding to equation* (15) *is uniformly (w.r.t. $\tau \in \mathbb{R}_+$) compact in the space $\boldsymbol{H}$. It has the compact uniformly absorbing set $B_{1,\varepsilon}$ defined in* (40).

Along with the G.–L. equation (15), we consider its "limit" equation

(41) $$\partial_t u^0 = (1 + i\alpha(t))\Delta u^0 + R(t)u^0 - (1 + i\beta(t))|u^0|^2 u^0 + g_0(x,t),\ u^0|_{\partial\Omega} = 0,$$

where the coefficients $\alpha(t), \beta(t), R(t)$ and the external force $g_0(x,t)$ are the same as in (15). In particular, conditions (16) and (18) hold. Therefore, the Cauchy problem for this equation with initial data

(42) $$u^0|_{t=0} = u_0(x),\ u_0(\,\cdot\,) \in \boldsymbol{H},$$

also has a unique weak solution $u^0(x,t)$ and there is the corresponding semiprocess $\{U_0(t,\tau)\}$ in $\boldsymbol{H}$: $U_0(t,\tau)u_\tau = u^0(t), t \geq \tau \geq 0$, where $u^0(t), t \geq \tau$ is a solution of equation (41) with initial data $u|_{t=\tau} = u_\tau$. Similar to (29), the main a priory estimate for equation (41) reads

(43) $$\|u^0(t)\|^2 \leq \|u^0(0)\|^2 e^{-2\lambda_1 t} + C_0^2.$$

Following the above reasoning, we prove that the semiprocess $\{U_0(t,\tau)\}$ has the uniformly (w.r.t. $\tau \in \mathbb{R}_+$) absorbing set

(44) $$B_{0,0} = \{v \in \boldsymbol{H} \mid \|v\| \leq 2C_0\}$$

(Comparing with (31), we observe that here the parameter ε is missing since the term $\dfrac{1}{\varepsilon^\rho} g_1\left(\dfrac{x}{\varepsilon}, t\right)$ is missing in equation (41).) Moreover, the semiprocess also has the uniformly absorbing set

(45) $$B_{1,0} = \{v \in \boldsymbol{V} \mid \|\nabla v\|_{\boldsymbol{V}} \leq C_8(1 + \|g_0\|_{L_2^b(\mathbb{R}_+;\boldsymbol{H})})\}$$

which is bounded in $\boldsymbol{V}$ and compact in $\boldsymbol{H}$. Consequently, the semiprocess $\{U_0(t,\tau)\}$ corresponding to the "limit" equation (41) is uniformly compact in $\boldsymbol{H}$ and Proposition 1.1 holds for the "limit" case $\varepsilon = 0$.

Using this results, it follows easily that the semiprocess $\{U_\varepsilon(t,\tau)\}, \varepsilon > 0$, and $\{U_0(t,\tau)\}$ have the uniform (w.r.t. $\tau \in \mathbb{R}_+$) global attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$, respectively (see [3]). However, the formulated above conditions for the function $g_1(z,t)$ is not sufficient to establish that the global attractors $\mathcal{A}_\varepsilon$ are uniformly (with respect to $\varepsilon > 0$) bounded in $\boldsymbol{H}$. In the next sections, we shall present assumption that provide this uniform boundedness of global attractors $\mathcal{A}_\varepsilon$ for $0 < \varepsilon \leq 1$. Moreover, we are going to estimate the deviation of the attractors $\mathcal{A}_\varepsilon$ from $\mathcal{A}_0$ in terms of ε.

2. - Deviation estimate for solutions of the Ginzburg–Landau equation with oscillating external forces from solutions of the "limit" equation

We consider equation (15)

(46) $$\partial_t u = (1 + i\alpha(t))\Delta u + R(t)u - (1 + i\beta(t))|u|^2 u + g_0(x,t) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right),\ u|_{\partial\Omega} = 0.$$

We assume that the coefficients of this equation satisfy conditions (16)–(19) and

$0 < \rho \leq 1$. The corresponding "limit" equation is

$$\partial_t u^0 = (1 + i\alpha(t))\Delta u^0 + R(t)u^0 - (1 + i\beta(t))|u^0|^2 u^0 + g_0(x,t),\ u^0|_{\partial\Omega} = 0, \tag{47}$$

For $t = 0$, we consider the common initial data

$$u|_{t=0} = u_0(x),\ u^0|_{t=0} = u_0(x),\ u_0(\,\cdot\,) \in \boldsymbol{H}, \tag{48}$$

for equations (46) and (47). Let $u(x,t), t \geq 0$, and $u^0(x,t), t \geq 0$, be solutions of problems (46), (48) and (47), (48) respectively. We set $w(x,t) = u(x,t) - u^0(x,t)$. The function $w(t) := w(\cdot, t)$ satisfies the equation

$$\partial_t w = (1 + i\alpha(t))\Delta w + R(t)w - (1 + i\beta(t))\left(|u|^2 u - |u^0|^2 u^0\right) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right),\ w|_{\partial\Omega} = 0, \tag{49}$$

and has the initial data $w(0) = 0$.

We now assume that the function $g_1(z,t)$ satisfies the following additional condition.

Condition I. *There exist functions* $G_j(z,t) \in C_b(\mathbb{R}_+; \boldsymbol{Z})$ *with* $\dfrac{\partial G_j}{\partial z_j} \in L_2^{loc}(\mathbb{R}_+; \boldsymbol{Z})$ *for* $j = 1, 2, \ldots, n$ $(z = (z_1, \ldots, z_n))$, *such that*

$$\left\|G_j(\cdot,\cdot)\right\|^2_{C_b(\mathbb{R}_+;\, \boldsymbol{Z})} := \sup_{t\in\mathbb{R}_+} \sup_{z^0\in\mathbb{R}^n} \int_{z_1^0}^{z_1^0+1} \cdots \int_{z_n^0}^{z_n^0+1} \left|G_j(z,t)\right|^2 dz \leq M^2 \tag{50}$$

and

$$\sum_{j=1}^{n} \frac{\partial G_j}{\partial z_j}(z,t) = g_1(z,t),\ \forall z \in \mathbb{R}^n,\ t \in \mathbb{R}_+. \tag{51}$$

We set

$$\bar{R} = \sup_{t\in\mathbb{R}_+} R(t). \tag{52}$$

Theorem 2.1: *Under Condition I, the difference* $w(t) = u(\cdot,t) - u^0(\cdot,t)$ *of the solutions* $u(x,t)$ *and* $u^0(x,t)$ *of equations* (46) *and* (47) *respectively with common initial data* (48) *satisfies the following inequality:*

$$\|w(t)\| = \|u(\cdot,t) - u^0(\cdot,t)\| \leq C\varepsilon^{(1-\rho)} e^{rt}, \tag{53}$$

where

$$r = \begin{cases} 0, & \textit{for } \bar{R} < \lambda_1 \\ \bar{R} - \lambda_1 + \delta, & \textit{for } \bar{R} \geq \lambda_1 \end{cases}, \tag{54}$$

$\delta > 0$ *is arbitrary small, and* $C = C(\delta)$.

PROOF: Taking the scalar product in $\boldsymbol{H}$ of equation (49) and w, we have

$$(55)\quad \frac{1}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 - \langle R(t)w, w\rangle + \\ + \big\langle (1+i\beta(t))\big(|u|^2u - |u^0|^2u^0\big), u-u^0\big\rangle = \varepsilon^{-\rho}\Big\langle g_1\Big(\frac{x}{\varepsilon}, t\Big), w\Big\rangle.$$

Since $|\beta(t)| \le \sqrt{3}$, it follows from (34) that

$$(56)\qquad \big\langle (1+i\beta(t))\big(|u|^2u - |u^0|^2u^0\big), u-u^0\big\rangle \ge 0$$

(see, e.g., [3, page 42]). From (55) and (56), we obtain

$$(57)\qquad \frac{d}{dt}\|w\|^2 + 2\|\nabla w\|^2 \le 2\bar{R}\|w\|^2 + 2\varepsilon^{-\rho}\Big\langle g_1\Big(\frac{x}{\varepsilon}, t\Big), w\Big\rangle,$$

where $\bar{R} = \sup_{t\in\mathbb{R}_+} R(t)$. Applying (51), we find that

$$(58)\quad 2\varepsilon^{-\rho}\Big\langle g_1\Big(\frac{x}{\varepsilon}, t\Big), w\Big\rangle ds = 2\varepsilon^{-\rho}\sum_{j=1}^{n}\Big\langle \partial_{z_j} G_j\Big(\frac{x}{\varepsilon}, t\Big), w\Big\rangle = \\ = 2\varepsilon^{1-\rho}\sum_{j=1}^{n}\Big\langle \partial_{x_j} G_j\Big(\frac{x}{\varepsilon}, t\Big), w\Big\rangle = -2\varepsilon^{1-\rho}\sum_{j=1}^{n}\Big\langle G_j\Big(\frac{x}{\varepsilon}, t\Big), \partial_{x_j} w\Big\rangle \le \\ \le \frac{\lambda_1}{2\delta}\varepsilon^{2(1-\rho)}\sum_{j=1}^{n}\int_\Omega \Big|G_j\Big(\frac{x}{\varepsilon}, t\Big)\Big|^2 dx + \frac{2\delta}{\lambda_1}\int_\Omega |\nabla w(x,t)|^2 dx,\ \delta > 0.$$

Using (50), we have

$$(59)\qquad \int_\Omega \Big|G_j\Big(\frac{x}{\varepsilon}, t\Big)\Big|^2 dx = \varepsilon^n \int_{\varepsilon^{-1}\Omega} |G_j(z,t)|^2 dx \le C|G_j(\cdot,t)|^2_{\boldsymbol{Z}} \le CM^2.$$

It follows from (58) and (59) that

$$2\varepsilon^{-\rho}\Big\langle g_1\Big(\frac{x}{\varepsilon}, t\Big), w\Big\rangle \le \frac{\lambda_1}{2\delta} nCM^2\varepsilon^{2(1-\rho)} + \frac{2\delta}{\lambda_1}\|\nabla w\|^2,\ \delta > 0.$$

Consequently from (57), we have

$$(60)\qquad \frac{d}{dt}\|w\|^2 + (2 - 2\delta\lambda_1^{-1})\|\nabla w\|^2 \le 2\bar{R}\|w\|^2 + \frac{\lambda_1}{2\delta}C_1M^2\varepsilon^{2(1-\rho)}.$$

We assume that $\delta < \lambda_1$. From the Poincaré inequality, we conclude that

$$(61)\qquad \frac{d}{dt}\|w\|^2 \le 2\big(\bar{R} - \lambda_1 + \delta)\big)\|w\|^2 + \frac{\lambda_1}{2\delta}C_1M^2\varepsilon^{2(1-\rho)}.$$

If now $\bar{R} \ge \lambda_1$, then $r = \bar{R} - \lambda_1 + \delta > 0$ and hence

$$\frac{d}{dt}\|w\|^2 \le r\|w\|^2 + \frac{\lambda_1}{2\delta}C_1M^2\varepsilon^{2(1-\rho)}.$$

Integrating this differential inequality, we have

$$(62)\qquad \|w(t)\|^2 \le \left(\|w(0)\|^2 + \delta^{-1}\frac{\lambda_1 C_1 M^2}{2r}\varepsilon^{2(1-\rho)}\right)e^{rt},\ \forall t \ge 0.$$

Recall that $w(0) = 0$ and we obtain

$$\|w(t)\| \leq C(\delta)\varepsilon^{(1-\rho)}e^{rt},$$

where $r = \bar{R} - \lambda_1 + \delta$ and $C(\delta) = \left(\delta^{-1}\dfrac{\lambda_1 C_1 M^2}{2r}\right)^{1/2}$.

If $\bar{R} < \lambda_1$, then $-r_1 = \bar{R} - \lambda_1 + \delta < 0$ for a sufficiently small $\delta > 0$ and we have

$$\frac{d}{dt}\|w\|^2 \leq -r_1\|w\|^2 + \frac{\lambda_1}{2\delta}C_1M^2\varepsilon^{2(1-\rho)}. \tag{63}$$

Integrating this inequality we have

$$\|w(t)\|^2 \leq \|w(0)\|^2 e^{-r_1 t} + \delta^{-1}\frac{\lambda_1 C_1 M^2}{2r_1}\varepsilon^{2(1-\rho)}, \ \forall t \geq 0,$$

and since $w(0) = 0$

$$\|w(t)\| \leq C(\delta)\varepsilon^{(1-\rho)},$$

where $C(\delta) = \left(\delta^{-1}\dfrac{\lambda_1 C_1 M^2}{2r_1}\right)^{1/2}$, $r_1 = \lambda_1 - \bar{R} - \delta > 0$.

Inequality (53) is proved. ■

We note that estimate (53) is a generalization of the classical estimate of N. N. Bogolubov for the deviation of solutions of equations with oscillating term from the solution of the corresponding averaged equations (see [25]).

REMARK 2.1: *It is clear, that an estimate of the form* (53) *holds for the difference of solutions* $u(t), t \geq \tau$, *and* $u^0(t), t \geq \tau$, *of equations* (46) *and* (47), *respectively, with the common initial data at* $t = \tau$ *for every* $\tau \geq 0$.

3. - A.p. FUNCTIONS WITH VALUES IN THE SPACES $\boldsymbol{H}$ AND $\boldsymbol{Z}$

A function $\varphi(z,t)$ belongs to the space $C_b(\mathbb{R}; \boldsymbol{H})$ if it is continuous and bounded in $t \in \mathbb{R}$ with values in $\boldsymbol{H}$. A function $\varphi(z,t) \in C_b(\mathbb{R}; \boldsymbol{H})$ is called *almost periodic* (a.p.) with values in $\boldsymbol{H}$ if the family of its translations $\{\varphi(z, t+h),\ h \in \mathbb{R}\}$ is precompact in $C_b(\mathbb{R}; \boldsymbol{H})$. Notice that this definition of an almost periodic function with values in a Banach space E was formulated by Bochner and Amerio (see, e.g., [26, 30]). Recall that the set

$$\mathcal{H}(\varphi) := [\{\varphi(t+h),\ h \in \mathbb{R}\}]_{C_b(\mathbb{R}; \boldsymbol{H})}$$

is called the *hull* of the function φ in the space $C_b(\mathbb{R}; \boldsymbol{H})$. Here, $[X]_\Xi$ denotes the closure of a set X in a topological space Ξ. By definition, a function $\varphi(t) \in C_b(\mathbb{R}; \boldsymbol{H})$ is a.p. if and only if the hull $\mathcal{H}(\varphi)$ is compact in $C_b(\mathbb{R}; \boldsymbol{Z})$.

Similarly, one defines a.p. functions with values in the space $\boldsymbol{Z} = L_2^b(\mathbb{R}^n; \mathbb{C})$. By definition, a function $\psi(t) \in C_b(\mathbb{R}; \boldsymbol{Z})$ is a.p. if and only if the hull

$$\mathcal{H}(\psi) := [\{\psi(t+h),\ h \in \mathbb{R}\}]_{C_b(\mathbb{R}; \boldsymbol{Z})}$$

is compact in $C_b(\mathbb{R}; \boldsymbol{Z})$.

Proposition 3.1: *If a function $g_1(z,t)$ is a.p. with values in $\boldsymbol{Z}$, then for every fixed ε, $0 < \varepsilon \leq 1$, the function $g_1\left(\frac{x}{\varepsilon}, t\right)$ is a.p. with values in $L_2(\Omega; \mathbb{C}) = \boldsymbol{H}$.*

Proof: We have to show that the set $\left\{g_1\left(\frac{x}{\varepsilon}, t+h\right),\ h \in \mathbb{R}\right\}$ is precompact in $C_b(\mathbb{R}; \boldsymbol{H})$. Let $\{h_n, n = 1, 2, \ldots\}$ be an arbitrary sequence of real numbers. Since the function $g_1(z,t)$ is a.p. there exists a subsequence of indices $\{n'\} \subset \{n\}$ such that the sequence $g_1(z, t + h_{n'})$ converges to a function $\hat{g}_1(z,t) \in C_b(\mathbb{R}; \boldsymbol{Z})$ as $n' \to \infty$, i.e.,

$$\|g_1(\cdot, t + h_{n'}) - \hat{g}_1(\cdot, t)\|_{C_b(\mathbb{R}; \boldsymbol{Z})} \to 0\ (n' \to \infty). \tag{64}$$

Recall that the norm in the space $C_b(\mathbb{R}; \boldsymbol{Z})$ is defined by the following formula:

$$\|g_1\|^2_{C_b(\mathbb{R}; \boldsymbol{Z})} = \sup_{\tau \in \mathbb{R}} \left(\sup_{z \in \mathbb{R}^n} \int_{z_1}^{z_1+1} \cdots \int_{z_n}^{z_n+1} |g_1(\zeta_1, \ldots, \zeta_n, \tau)|^2 d\zeta_1 \cdots d\zeta_n \right).$$

We set $\bar{g}_{n'}(z,t) := g_1(z, t + h_{n'}) - \hat{g}_1(z,t)$. Then we have

$$\left\|\bar{g}_{n'}\left(\frac{\cdot}{\varepsilon}, t\right)\right\|^2_{\boldsymbol{H}} = \int_{\Omega} \left|\bar{g}_{n'}\left(\frac{x}{\varepsilon}, t\right)\right|^2 dx = \varepsilon^n \int_{\varepsilon^{-1}\Omega} |\bar{g}_{n'}(z,t)|^2 dz \leq \tag{65}$$

$$\leq C\varepsilon^{-n} \sup_{z_0 \in \mathbb{R}^n} \varepsilon^n \int_{z_0}^{z_0+1} \cdots \int_{z_n}^{z_n+1} |\bar{g}_{n'}(z,t)|^2 dz = C\|g_1(\cdot, t)\|^2_{\boldsymbol{Z}}.$$

Here, in the second equality, we have changed the variable $x/\varepsilon = z,\ dx = \varepsilon^n dz$, and then, in the last inequality, we have used the fact that the domain $\varepsilon^{-1}\Omega$ can be covered by unit boxes the number of which does not exceed $C\varepsilon^{-n}$ for some positive $C = C(\Omega)$.

Thus, from (64) and (65), we observe that

$$\sup_{t \in \mathbb{R}} \left\|\bar{g}_{n'}\left(\frac{\cdot}{\varepsilon}, t\right)\right\|^2_{\boldsymbol{H}} \leq C \sup_{t \in \mathbb{R}} \|\bar{g}_{n'}(\cdot, t)\|^2_{\boldsymbol{Z}} \to 0\ (n' \to \infty). \tag{66}$$

Therefore, the set $\left\{g_1\left(\frac{x}{\varepsilon}, t+h\right),\ h \in \mathbb{R}\right\}$ is precompact in the space $C_b(\mathbb{R}_+; \boldsymbol{H})$. ■

Similarly to Proposition 3.1, we prove

Proposition 3.2: *If a function $g_1(z,t)$ belong to the space $C_b(\mathbb{R}; \boldsymbol{Z})$, then, for every $\varepsilon, 0 < \varepsilon \leq 1$, the function $g_1\left(\frac{x}{\varepsilon}, t\right)$ belongs to $C_b(\mathbb{R}; \boldsymbol{H})$ and moreover*

$$\left\|g_1\left(\frac{\cdot}{\varepsilon}, t\right)\right\|^2_{\boldsymbol{H}} \leq C\|g_1(\cdot, t)\|^2_{\boldsymbol{Z}},\ \forall t \in \mathbb{R}, \tag{67}$$

where C is independent of t and $\varepsilon, 0 < \varepsilon \leq 1$.

PROPOSITION 3.3: *Let a function $g_0(x,t)$ be a.p. with values in $\boldsymbol{H}$ and let a function $g_1(z,t)$ be a.p. with values in $\boldsymbol{Z}$. For every fixed ε, $0<\varepsilon\le 1$, we consider the function $g^\varepsilon(x,t) = = g_0(x,t)+\dfrac{1}{\varepsilon^\rho}g_1\left(\dfrac{x}{\varepsilon},t\right)$ as an element of the space $C_b(\mathbb{R};\boldsymbol{H})$ (here, $\rho>0$) and its hull $\mathcal{H}(g^\varepsilon)$ is taken in the space $C_b(\mathbb{R};\boldsymbol{H})$. Then the hull $\mathcal{H}(g^\varepsilon)=\mathcal{H}\left(g_0(x,t)+\dfrac{1}{\varepsilon^\rho}g_1\left(\dfrac{x}{\varepsilon},t\right)\right)$ (in the space $C_b(\mathbb{R};\boldsymbol{H})$) consists of (a.p. in $C_b(\mathbb{R};\boldsymbol{H})$) functions $\hat{g}^\varepsilon(x,t)$ of the form $\hat{g}^\varepsilon(x,t) = = \hat{g}_0(x,t)+\dfrac{1}{\varepsilon^\rho}\hat{g}_1\left(\dfrac{x}{\varepsilon},t\right)$ for some $\hat{g}_0(x,t)\in\mathcal{H}(g_0(x,t))$ and $\hat{g}_1(z,t)\in\mathcal{H}(g_1(z,t))$, where $\mathcal{H}(g_0)$ and $\mathcal{H}(g_1)$ are the hulls of functions $g_0(x,t)$ and $g_1(z,t)$ in the spaces $C_b(\mathbb{R};\boldsymbol{H})$ and $C_b(\mathbb{R};\boldsymbol{Z})$, respectively.*

PROOF: Let $\{h_n\}$ be an arbitrary sequence of real numbers. Consider the sequence of functions $\{g_0(x,t+h_n)\}$ in $C_b(\mathbb{R};\boldsymbol{H})$. Since the function $g_0(x,t)$ is a.p. with values in $\boldsymbol{H}$ there is a subsequence $\{h_{n'}\}\subset\{h_n\}$ such that

$$g_0(x,t+h_{n'})\to\hat{g}_0(x,t) \text{ as } n'\to\infty \text{ in } C_b(\mathbb{R};\boldsymbol{H}) \tag{68}$$

for some $\hat{g}_0(x,t)\in C_b(\mathbb{R};\boldsymbol{H})$. Consider now the sequence of the functions $\{g_1(z,t+h_{n'})\}$ in $C_b(\mathbb{R};\boldsymbol{Z})$. The function $g_1(z,t)$ is a.p. with values in $\boldsymbol{Z}$. Therefore, passing to a subsequence $\{h_{n''}\}\subset\{h_{n'}\}$, we may assume that

$$g_1(z,t+h_{n''})\to\hat{g}_1(z,t) \text{ as } n''\to\infty \text{ in } C_b(\mathbb{R};\boldsymbol{Z}), \tag{69}$$

where $\hat{g}_1(z,t)\in C_b(\mathbb{R};\boldsymbol{H})$. It follows from (67) that

$$\left\|g_1\left(\frac{\cdot}{\varepsilon},t+h_{n''}\right)-\hat{g}_1\left(\frac{\cdot}{\varepsilon},t\right)\right\|_{\boldsymbol{H}}^2\le C\|g_1(\cdot,t+h_{n''})-\hat{g}_1(\cdot,t)\|_{\boldsymbol{Z}}^2,\ \forall t\in\mathbb{R}.$$

Then owing to (69) we conclude that

$$g_1\left(\frac{x}{\varepsilon},t+h_{n''}\right)\to\hat{g}_1\left(\frac{x}{\varepsilon},t\right) \text{ as } n''\to\infty \text{ in } C_b(\mathbb{R};\boldsymbol{H}),$$

and, combining this relation with (68) we obtain

$$g_0(x,t+h_{n''})+\varepsilon^{-\rho}g_1\left(\frac{x}{\varepsilon},t+h_{n''}\right)\to\hat{g}_0(x,t)+\varepsilon^{-\rho}\hat{g}_1\left(\frac{x}{\varepsilon},t\right) \text{ as } n''\to\infty \text{ in } C_b(\mathbb{R};\boldsymbol{H}). \tag{70}$$

Consequently, the hull $\mathcal{H}\left(g_0(x,t)+\dfrac{1}{\varepsilon^\rho}g_1\left(\dfrac{x}{\varepsilon},t\right)\right)$ consists of a.p. functions with values in $\boldsymbol{H}$ and each function $\hat{g}^\varepsilon(x,t)\in\mathcal{H}\left(g_0(x,t)+\dfrac{1}{\varepsilon^\rho}g_1\left(\dfrac{x}{\varepsilon},t\right)\right)$ has the form $\hat{g}^\varepsilon(x,t)= \hat{g}_0(x,t)+\dfrac{1}{\varepsilon^\rho}\hat{g}_1\left(\dfrac{x}{\varepsilon},t\right)$, where $\hat{g}_0(x,t)\in\mathcal{H}(g_0(x,t))$ and $\hat{g}_1(z,t)\in\mathcal{H}(g_1(z,t))$. ■

4. - UNIFORM BOUNDEDNESS OF THE GLOBAL ATTRACTORS $\mathcal{A}^\varepsilon$

We assume that the coefficients $\alpha(t)$, $\beta(t)$, and $R(t)$ in equation (15) are defined for all $t\in\mathbb{R}$ and they are a.p. functions with values in $\mathbb{R}$. In addition, we assume that the functions $g_0(x,t)$ and $g_1(z,t)$ are also known for all $t\in\mathbb{R}$ and they are a.p. with values in $\boldsymbol{H}=L_2(\Omega;\mathbb{C})$ and in $\boldsymbol{Z}=L_2^b(\mathbb{R}^n;\mathbb{C})$, respectively.

In this section, we consider the equation (15) on the entire time axis:

$$(71)\qquad \partial_t u = (1 + i\alpha(t))\Delta u + R(t)u - (1 + i\beta(t))|u|^2 u + g_0(x,t) + \frac{1}{\varepsilon^\rho} g_1\left(\frac{x}{\varepsilon}, t\right), t \in \mathbb{R}.$$

It follows from Proposition 3.1 that the external force $g^\varepsilon(t) = g_0(x,t) + \varepsilon^{-\rho} g_1\left(\frac{x}{\varepsilon}, t\right), t \in \mathbb{R}$, is an a.p. function with values in $\boldsymbol{H}$.

The function $\sigma^\varepsilon(t) = (\alpha(t), \beta(t), R(t), g^\varepsilon(t)), t \in \mathbb{R}$, is called the *symbol* of equation (71). Here the parameter ε is fixed. We note that the function $\sigma^\varepsilon(t), t \in \mathbb{R}$, is a.p. with values in $\mathbb{R}^3 \times \boldsymbol{H}$. By definition, the *hull* $\mathcal{H}(\sigma^\varepsilon)$ of the function σ^ε in $C_b(\mathbb{R}; \mathbb{R}^3 \times \boldsymbol{H})$ is the following family of functions:

$$(72)\qquad \mathcal{H}(\sigma^\varepsilon) = [\{\sigma^\varepsilon(t+h),\ h \in \mathbb{R}\}]_{C_b(\mathbb{R};\mathbb{R}^3 \times \boldsymbol{H})} =: \Sigma^\varepsilon,$$

Recall that, for a fixed ε, the hull $\mathcal{H}(\sigma^\varepsilon)$ is compact in $C_b(\mathbb{R}; \mathbb{R}^3 \times \boldsymbol{H})$ since the function $\sigma^\varepsilon(t)$ is a.p. in this space. Elements of the hull $\mathcal{H}(\sigma^\varepsilon)$ will be denoted by $\hat{\sigma}^\varepsilon(t)$, $\hat{\sigma}^\varepsilon \in \mathcal{H}(\sigma^\varepsilon)$.

It follows from Proposition 3.3 that, for any $\hat{\sigma}^\varepsilon(t) = (\hat{\alpha}(t), \hat{\beta}(t), \hat{R}(t), \hat{g}^\varepsilon(t)) \in \mathcal{H}(\sigma^\varepsilon)$, the last component $\hat{g}^\varepsilon(t)$ can be written in the form $\hat{g}^\varepsilon(t) = \hat{g}_0(x,t) + \varepsilon^{-\rho}\hat{g}_1\left(\frac{x}{\varepsilon}, t\right)$ for some $\hat{g}_0(x,t) \in \mathcal{H}(g_0)$ and $\hat{g}_1(z,t) \in \mathcal{H}(g_1)$, where the hulls $\mathcal{H}(g_0)$ and $\mathcal{H}(g_1)$ of the functions $g_0(x,t)$ and $g_1(z,t)$ are taken in the space $C_b(\mathbb{R}; \boldsymbol{H})$ and $C_b(\mathbb{R}; \boldsymbol{Z})$, respectively.

For every fixed $\tau \in \mathbb{R}$ similar to Section 1, we consider weak solutions $u(t), t \geq \tau$, of equation (71) belonging to the space $C_\mathrm{b}(\mathbb{R}_\tau; \boldsymbol{H}) \cap L_2^\mathrm{b}(\mathbb{R}_\tau; \boldsymbol{V}) \cap L_4^\mathrm{b}(\mathbb{R}_\tau; \boldsymbol{L}_4)$ that satisfy equation (71) in the sense of distributions of the space $\mathcal{D}'(\mathbb{R}_\tau; \boldsymbol{H}^{-r})$. Here, we denote $\mathbb{R}_\tau = [\tau, +\infty)$. Recall that any weak solution $u(t)$, $t \geq \tau$, is uniquely determined by its initial value $u(\tau)$. Therefore, to the equation (71), there corresponds a process $\{U_{\sigma^\varepsilon}(t,\tau)\} = \{U_{\sigma^\varepsilon}(t,\tau), t \geq \tau, \tau \in \mathbb{R}\}$ acting in the space $\boldsymbol{H}$ by the formula

$$U_{\sigma^\varepsilon}(t,\tau)u_\tau = u(t),\ t \geq \tau,\ \tau \in \mathbb{R},\ u_\tau \in \boldsymbol{H},$$

where $u(t)$ is a solution of (71) with initial data $u(\tau) = u_\tau$.

Along with equation (71), we consider the family of equations

$$(73)\qquad \partial_t \hat{u}^\varepsilon = (1 + i\hat{\alpha}(t))\Delta\hat{u}^\varepsilon + \hat{R}(t)\hat{u}^\varepsilon - (1 + i\hat{\beta}(t))|\hat{u}^\varepsilon|^2\hat{u}^\varepsilon + \hat{g}_0(x,t) + \frac{1}{\varepsilon^\rho}\hat{g}_1\left(\frac{x}{\varepsilon}, t\right), \hat{u}^\varepsilon|_{\partial\Omega} = 0,$$

with symbols $\hat{\sigma}^\varepsilon(t) = (\hat{\alpha}(t), \hat{\beta}(t), \hat{R}(t), \hat{g}^\varepsilon(t)) \in \mathcal{H}(\sigma^\varepsilon) = \Sigma^\varepsilon$, where the function $\hat{g}^\varepsilon(t) = \hat{g}_0(x,t) + \varepsilon^{-\rho}\hat{g}_1\left(\frac{x}{\varepsilon}, t\right)$, $\hat{g}_0(x,t) \in \mathcal{H}(g_0)$, $\hat{g}_1(z,t) \in \mathcal{H}(g_1)$. We note that the processes $\{U_{\hat{\sigma}^\varepsilon}(t,\tau), t \geq \tau \in \mathbb{R}\}$ corresponding to equations (73) satisfy the same properties as the semiprocess $\{U_{\sigma^\varepsilon}(t,\tau), t \geq \tau \geq 0\} = \{U_\varepsilon(t,\tau), t \geq \tau \geq 0\}$ considered in Section 1. In particular, for every fixed ε, $0 < \varepsilon \leq 1$, the family of processes $\{U_{\hat{\sigma}^\varepsilon}(t,\tau)\}$, $\hat{\sigma}^\varepsilon \in \Sigma^\varepsilon$, has the compact absorbing set $\hat{B}_{1,\varepsilon}$ defined in (40). It is necessarily to replace $\mathbb{R}_+$ with $\mathbb{R}$ in definitions (18) and (19). These norms are also bounded since we assume in this section that the functions $g_0(x,t)$ and $g_1(z,t)$ are a.p. with values in $\boldsymbol{H}$ and in $\boldsymbol{Z} = L_2^b(\mathbb{R}^n; \mathbb{C})$, respectively. For the absorbing set $\hat{B}_{1,\varepsilon}$, we clearly obtain the estimate

$$\hat{B}_{1,\varepsilon} = \{v \in \boldsymbol{V} \mid \|v\|_{\boldsymbol{H}} \leq C_8'(1 + \|\hat{g}_0\|_{C_b(\mathbb{R};\boldsymbol{H})} + \varepsilon^{-\rho}\|\hat{g}_1\|_{C_b(\mathbb{R};\boldsymbol{Z})})\}.$$

We also have that

$$\|\hat{g}_0\|_{C_b(\mathbb{R};\boldsymbol{H})} \le \|g_0\|_{C_b(\mathbb{R};\boldsymbol{H})},$$
$$\|\hat{g}_1\|_{C_b(\mathbb{R};\boldsymbol{Z})} \le \|g_1\|_{C_b(\mathbb{R};\boldsymbol{Z})},$$

since the functions $g_0(x,t)$ and $g_1(z,t)$ are a.p. in the corresponding spaces. Thus, repeating the reasonings from Section 1, we prove

PROPOSITION 4.1: *For any fixed* $\varepsilon, 0 < \varepsilon \le 1$, *the family of processes* $\{U_{\hat{\sigma}^\varepsilon}(t,\tau)\}$, $\hat{\sigma}^\varepsilon \in \mathcal{H}(\sigma^\varepsilon) = \Sigma^\varepsilon$, *corresponding to equation* (73) *has the compact (in* $\boldsymbol{H}$*) uniformly (w.r.t.* $\tau \in \mathbb{R}$ *and* $\hat{\sigma}^\varepsilon \in \mathcal{H}(\sigma^\varepsilon)$*) absorbing set* $B_{1,\varepsilon}$. *The set* $B_{1,\varepsilon}$ *satisfies the inequality*

$$\|B_{1,\varepsilon}\|_{\boldsymbol{V}} \le C\varepsilon^{-\rho}, \tag{74}$$

where C *is independent of* $\hat{\sigma}^\varepsilon(t) \in \mathcal{H}(\sigma^\varepsilon)$ *and* ε.

Since the Cauchy problem for equations (73) is uniquely solvable, the family of processes $\{U_{\hat{\sigma}^\varepsilon}(t,\tau)\}$, $\hat{\sigma}^\varepsilon \in \Sigma^\varepsilon$, satisfies the following translation identity:

$$U_{\hat{\sigma}^\varepsilon}(t+h,\tau+h) = U_{T(h)\hat{\sigma}^\varepsilon}(t,\tau),\ \forall t \ge \tau,\ \ \tau \in \mathbb{R},\ \ \forall h \ge 0, \tag{75}$$

where $T(h)$ is the time translation operator acting on a symbol $\hat{\sigma}^\varepsilon$ by the formula $T(h)\hat{\sigma}^\varepsilon(t) = \hat{\sigma}^\varepsilon(t+h)$, $\forall t \in \mathbb{R}, h \ge 0$.

For every fixed ε, the family $\{U_{\hat{\sigma}^\varepsilon}(t,\tau)\}$, $\hat{\sigma}^\varepsilon \in \Sigma^\varepsilon$, is $(\boldsymbol{H} \times \Sigma^\varepsilon, \boldsymbol{H})$-continuous in the following sense: for every fixed t and τ, the mapping $(u_\tau, \hat{\sigma}^\varepsilon) \to U_{\hat{\sigma}^\varepsilon}(t,\tau)u_\tau$ acting from $\boldsymbol{H} \times \Sigma^\varepsilon$ into $\boldsymbol{H}$ is continuous in the norm of these spaces:

$$U_{\hat{\sigma}^\varepsilon_n}(t,\tau)u^n_\tau \to U_{\hat{\sigma}^\varepsilon}(t,\tau)u_\tau \text{ (in } \boldsymbol{H}\text{) as } \hat{\sigma}^\varepsilon_n \to \hat{\sigma}^\varepsilon \text{ (in } \Sigma^\varepsilon\text{) and } u^n_\tau \to u_\tau \text{ (in } \boldsymbol{H}\text{)}. \tag{76}$$

The proof of this property is standard (see, e.g., [3, Proposition 2.3, page 116], where a more general reaction-diffusion systems were considered). (We note that the complex Ginzburg–Landau equation (73) is a particular case of a reaction-diffusion system). The proof is based on the inequality $|\hat{\beta}(t)| \le \sqrt{3}, \forall t \in \mathbb{R}$, which is true since $|\hat{\beta}(t)| \le$ $\le \sup_{t\in\mathbb{R}} |\beta(t)| \le \sqrt{3}, \forall t \in \mathbb{R}$. Then one uses the key inequality

$$\left\langle (1 + i\hat{\beta}_n(t))\left(|\hat{u}^\varepsilon|^2\hat{u}^\varepsilon - |\hat{u}^\varepsilon_n|^2\hat{u}^\varepsilon_n\right), \hat{u}^\varepsilon - \hat{u}^\varepsilon_n \right\rangle \ge 0$$

similar to (56).

Proposition 4.1 and properties (75) and (76) imply that the family of processes $\{U_{\hat{\sigma}^\varepsilon}(t,\tau)\}$, $\hat{\sigma}^\varepsilon \in \Sigma^\varepsilon$, has a uniform (w.r.t. $\tau \in \mathbb{R}$ and $\hat{\sigma}^\varepsilon \in \Sigma^\varepsilon$) global attractor $\mathcal{A}^\varepsilon \Subset \boldsymbol{H}$ such that $\mathcal{A}^\varepsilon \subset B_{1,\varepsilon}$, where $B_{1,\varepsilon}$ is the uniformly (w.r.t. $\tau \in \mathbb{R}$ and $\hat{\sigma}^\varepsilon \in \mathcal{H}(\sigma^\varepsilon)$) absorbing set (see (74)). Recall that $\mathcal{A}^\varepsilon$ has the following properties:

(i) for any bounded (in $\boldsymbol{H}$) set B,

$$\sup_{\hat{\sigma}^\varepsilon\in\Sigma^\varepsilon} \operatorname{dist}_{\boldsymbol{H}}(U_{\hat{\sigma}^\varepsilon}(t,\tau)B, \mathcal{A}^\varepsilon) \to 0 \ (t-\tau \to +\infty); \tag{77}$$

(ii) $\mathcal{A}^\varepsilon$ is the minimal closed set that satisfies (77).

We now define the *kernel* $\mathcal{K}_{\hat{\sigma}^\varepsilon}$ for equation (73) (and for the corresponding process

$\{U_{\hat{\sigma}^\varepsilon}(t,\tau)\}$). The kernel $\mathcal{K}_{\hat{\sigma}^\varepsilon}$ is the family of all complete weak solutions $\hat{u}^\varepsilon(t), t\in\mathbb{R}$, of equation (73) which are bounded in the norm of $\boldsymbol{H}$:

$$\|\hat{u}^\varepsilon(t)\| \le M_{\hat{u}},\ \forall t\in\mathbb{R}. \tag{78}$$

The set

$$\mathcal{K}_{\hat{\sigma}^\varepsilon}(s)=\{\hat{u}^\varepsilon(s)\mid \hat{u}^\varepsilon\in\mathcal{K}_{\hat{\sigma}^\varepsilon}\},\ s\in\mathbb{R},$$

is called the kernel section at time $t=s$.

We have the following

THEOREM 4.1: *Under the above assumptions, the uniform global attractor $\mathcal{A}^\varepsilon$ of equations* (73) *can be represented in the form*

$$\mathcal{A}^\varepsilon=\bigcup_{\hat{\sigma}^\varepsilon\in\mathcal{H}(\sigma^\varepsilon)}\mathcal{K}_{\hat{\sigma}^\varepsilon}(0). \tag{79}$$

Moreover, the kernel $\mathcal{K}_{\hat{\sigma}^\varepsilon}$ is non-empty for every $\hat{\sigma}^\varepsilon\in\mathcal{H}(\sigma^\varepsilon)$.

(The set $\mathcal{H}(\sigma^\varepsilon)$ is translation invariant. Therefore, the number 0 in (79) can be replaced by an arbitrary fixed $s\in\mathbb{R}$). The proof of this theorem is given in [3] (see Theorem 5.1 at page 89). Since the absorbing set $B_{1,\varepsilon}$ satisfies (74) and $\mathcal{A}^\varepsilon\Subset\boldsymbol{H}$ we conclude that

$$\|\mathcal{A}^\varepsilon\|_{\boldsymbol{V}}\le C\varepsilon^{-\rho},\ 0<\varepsilon\le 1. \tag{80}$$

We now formulate the condition that provides the uniform boundedness (w.r.t. $\varepsilon\in(0,1]$) of the global attractors $\mathcal{A}^\varepsilon$ in the space $\boldsymbol{H}$.

CONDITION $\hat{\mathrm{I}}$. *For every $\hat{g}_1(z,t)\in\mathcal{H}(g_1)$, there exist functions $\hat{G}_j(z,t)\in C_b(\mathbb{R};\boldsymbol{Z})$ with $\partial_{z_j}\hat{G}_j\in L_2^{loc}(\mathbb{R};\boldsymbol{Z})$ for $j=1,2,\dots,n$ ($z=(z_1,\dots,z_n)$), such that*

$$\sup_{t\in\mathbb{R}}\sup_{z^0\in\mathbb{R}^n}\int_{z_1^0}^{z_1^0+1}\cdots\int_{z_n^0}^{z_n^0+1}\left|\hat{G}_j(z_1,\dots,z_n,t)\right|^2dz_1\cdots dz_n=\left\|\hat{G}_j(\cdot,\cdot)\right\|^2_{C_b(\mathbb{R};\boldsymbol{Z})}\le M^2 \tag{81}$$

and

$$\sum_{j=1}^n\frac{\partial\hat{G}_j}{\partial z_j}(z,t)=\hat{g}_1(z,t),\ \forall z\in\mathbb{R}^n,\ t\in\mathbb{R}, \tag{82}$$

where the constant M is independent of $\hat{g}_1(z,t)$ (recall that the functions $g_1(z,t)$ and $\hat{g}_1(z,t)$ are a.p. with values in $\boldsymbol{Z}$).

THEOREM 4.2: *Let Condition $\hat{I}$ hold and let the exponent ρ (see the initial equation* (15)*) satisfies the following inequality:*

$$0\le\rho<\rho_0\equiv\begin{cases}1, & \text{for } \bar{R}<\lambda_1\\ \lambda_1/\bar{R}, & \text{for } \bar{R}\ge\lambda_1\end{cases}.$$

Then the uniform global attractors $\mathcal{A}^{\varepsilon}$ *of equations* (73) *are uniformly* (*w.r.t.* $\varepsilon \in (0,1]$) *bounded in the norm* $\boldsymbol{H}$, *i.e.,*

$$\|\mathcal{A}^{\varepsilon}\|_{\boldsymbol{H}} \leq C,\ \forall \varepsilon \in (0,1]\ (C = C(\rho)).$$

Proof: From formula (79), we conclude that the global attractor $\mathcal{A}^{\varepsilon}$ is the union of all the values of all the bounded complete trajectories $\hat{u}^{\varepsilon}(t), t \in \mathbb{R}$, of equation (73):

(83) $$\mathcal{A}^{\varepsilon} = \bigcup_{\hat{\sigma}^{\varepsilon} \in \mathcal{H}(\sigma^{\varepsilon})} \{\hat{u}^{\varepsilon}(s) \mid \hat{u}^{\varepsilon} \in \mathcal{K}_{\hat{\sigma}^{\varepsilon}}\},\ \forall s \in \mathbb{R}.$$

Along with equations (73), we consider the family of their "limit" equations:

(84) $$\partial_t \hat{u}^0 = (1 + i\hat{\alpha}(t))\Delta \hat{u}^0 + \hat{R}(t)\hat{u}^0 - (1 + i\hat{\beta}(t))|\hat{u}^0|^2 \hat{u}^0 + \hat{g}_0(x,t), \hat{u}^0|_{\partial\Omega} = 0,$$

with symbols $\hat{\sigma}^0(t) = (\hat{\alpha}(t), \hat{\beta}(t), \hat{R}(t), \hat{g}^0(t)) \in \mathcal{H}(\sigma^0) = \Sigma^0$, where $\hat{g}^0(t) = \hat{g}_0(x,t)$, $\hat{g}_0(x,t) \in \mathcal{H}(g_0)$. Similar to Section 2, we consider the difference

$$\hat{w}(x,t) = \hat{u}^{\varepsilon}(x,t) - \hat{u}^0(x,t),\ t \geq \tau,$$

of solutions of equations (73) and (84) having the common terms $\hat{\alpha}(t), \hat{\beta}(t), \hat{R}(t)$, and $\hat{g}_0(t)$, while the term $\hat{g}_1(z,t)$ in (73) is taken quite arbitrary, $\hat{g}_1(z,t) \in \mathcal{H}(g_1)$. Moreover, we assume that initial data at $t = \tau$ are identical, i.e., $\hat{u}^{\varepsilon}(\cdot,\tau) = \hat{u}^0(\cdot,\tau)$. Since, by assumption, every function $\hat{g}_1(z,t) \in \mathcal{H}(g_1)$ satisfies conditions (81) and (82), we can apply Theorem 2.1 (replacing the initial time $t = 0$ with an arbitrary $t = \tau$, see Remark 2.1). Then we have the following estimate

(85) $$\|\hat{w}(\tau + t)\| = \|\hat{u}^{\varepsilon}(\tau + t) - \hat{u}^0(\tau + t)\| \leq C(\delta)\varepsilon^{(1-\rho)}e^{rt},\ \forall t \geq 0.$$

where $\rho > 0$ is taken from (73) and r is defined in (54). Here δ is small, $\delta < \lambda_1$.

Now let u^{ε} be an arbitrary point of the global attractor $\mathcal{A}^{\varepsilon}$. It follows from (83) that there exists a symbol $\hat{\sigma}^{\varepsilon} = \left(\hat{\alpha}(t), \hat{\beta}(t), \hat{R}(t), \hat{g}_0(x,t) + \varepsilon^{-\rho}\hat{g}_1\left(\frac{x}{\varepsilon}, t\right)\right) \in \mathcal{H}(\sigma^{\varepsilon})$ and a bounded complete solution $\hat{u}^{\varepsilon}(t), t \in \mathbb{R}$, of equation (73) such that

(86) $$\hat{u}^{\varepsilon}(0) = u^{\varepsilon}.$$

We consider the point $\hat{u}^{\varepsilon}(-T)$, where the time T will be specified below. Since $\hat{u}^{\varepsilon}(-T) \in \mathcal{A}^{\varepsilon}$ from (80), we observe that

(87) $$\|\hat{u}^{\varepsilon}(-T)\| \leq \|\mathcal{A}^{\varepsilon}\| \leq C'\varepsilon^{-\rho}, 0 < \varepsilon \leq 1,\quad (\text{recall that } \|\cdot\| = \|\cdot\|_{\boldsymbol{H}}).$$

Let $\hat{u}^0(t), t \geq -T$, be a solution of the "limit" equation (84) with symbol $\hat{\sigma}^0 = (\hat{\alpha}(t), \hat{\beta}(t), \hat{R}(t), \hat{g}_0(x,t))$ and with initial data

(88) $$\hat{u}^0(-T) = \hat{u}^{\varepsilon}(-T).$$

According to estimate (43), we have

(89) $$\|\hat{u}^0(-T + t)\| \leq \|\hat{u}^0(-T)\|e^{-\lambda_1 t} + C_0.$$

Using in (89) identity (88) and inequality (87), we find that

(90) $$\|\hat{u}^0(-T + t)\| \leq C'\varepsilon^{-\rho}e^{-\lambda_1 t} + C_0,\ \forall t \geq 0.$$

We set $\tau = -T$ in inequality (85)

(91) $$\|\hat{w}(-T + t)\| = \|\hat{u}^{\varepsilon}(-T + t) - \hat{u}^0(-T + t)\| \leq C(\delta)\varepsilon^{(1-\rho)}e^{rt}.$$

Using (90) and (91), we observe that

$$\text{(92)} \qquad \|\hat{u}^\varepsilon(-T+t)\| \le \|\hat{u}^\varepsilon(-T+t) - \hat{u}^0(-T+t)\| + \|\hat{u}^0(-T+t)\| \le C(\delta)\varepsilon^{(1-\rho)}e^{rt} + C'\varepsilon^{-\rho}e^{-\lambda_1 t} + C_0.$$

We now specify the number T from the equation

$$\varepsilon^{(1-\rho)}e^{rT} = \varepsilon^{-\rho}e^{-\lambda_1 T},$$

that is

$$\text{(93)} \qquad T = \frac{1}{r+\lambda_1}\log\left(\frac{1}{\varepsilon}\right).$$

In (92), we set $t = T$, where T is defined above. Then we obtain

$$\text{(94)} \qquad \|\hat{u}^\varepsilon(0)\| \le (C(\delta)+C')\varepsilon^{(1-\rho)}\varepsilon^{-\frac{r}{r+\lambda_1}} + C_0 = (C(\delta)+C')\varepsilon^{\frac{\lambda_1}{r+\lambda_1}-\rho} + C_0.$$

Now, if $\bar{R} < \lambda_1$, then $r = 0$ (see (54)), i.e., $\dfrac{\lambda_1}{r+\lambda_1} = 1$, and owing to (94)

$$\text{(95)} \qquad \|\hat{u}^\varepsilon(0)\| \le (C(\delta)+C')\varepsilon^{1-\rho} + C_0.$$

In this case, if $\rho \le \rho_0 := 1$, then

$$\text{(96)} \qquad \|u^\varepsilon\| = \|\hat{u}^\varepsilon(0)\| \le C(\delta) + C' + C_0 = C, \ \forall\varepsilon, 0 < \varepsilon \le 1.$$

If $\bar{R} \ge \lambda_1$, then $r = \bar{R} - \lambda_1 + \delta$, (see (54)) and, therefore, $\dfrac{\lambda_1}{r+\lambda_1} = \dfrac{\lambda_1}{\bar{R}+\delta}$. From (94) we obtain

$$\|\hat{u}^\varepsilon(0)\| \le (C(\delta)+C')\varepsilon^{\frac{\lambda_1}{\bar{R}+\delta}-\rho} + C_0.$$

Hence, if $\rho < \rho_0 := \dfrac{\lambda_1}{\bar{R}}$, then $\rho \le \dfrac{\lambda_1}{\bar{R}+\delta}$ for some small $\delta = \delta(\rho)$ and

$$\|u^\varepsilon\| = \|\hat{u}^\varepsilon(0)\| \le C, \ \forall\varepsilon, 0 < \varepsilon \le 1.$$

where $C = (C(\delta) + C') + C_0$.

In both cases, since u^ε is an arbitrary point of $\mathcal{A}^\varepsilon$, we conclude that

$$\|\mathcal{A}^\varepsilon\|_{\boldsymbol{H}} \le C, \ \forall\varepsilon \in (0,1]$$

if $\rho < \rho_0$, where $C = C(\rho)$. The proof is complete. ■

REMARK 4.0: *If $\bar{R} < \lambda_1$, then the statement of Theorem remains true for the limit case $\rho = \rho_0 = 1$ (see (95) and (96)).*

5. - Global attractor with exponential attraction rate for the "limit" equations

In this section, we study the family of the "limit" equations

$$\text{(97)} \qquad \partial_t\hat{u} = (1+i\hat{\alpha}(t))\Delta\hat{u} + \hat{R}(t)\hat{u} - (1+i\hat{\beta}(t))|\hat{u}|^2\hat{u} + \hat{g}_0(x,t), \hat{u}|_{\partial\Omega} = 0,$$

with symbols $\hat{\sigma}(t) = (\hat{\alpha}(t), \hat{\beta}(t), \hat{R}(t), \hat{g}_0(t)) \in \mathcal{H}(\sigma)$, where $\mathcal{H}(\sigma)$ is the hull of the symbol

$\sigma(t) = (\alpha(t), \beta(t), R(t), g_0(t)), t \in \mathbb{R}$, taken from equation (41). We assume, that the function $\sigma(t), t \in \mathbb{R}$, is a.p. with values in $\mathbb{R}^3 \times \boldsymbol{H}$. Then, clearly, $\sigma \in C_b(\mathbb{R}; \mathbb{R}^3 \times \boldsymbol{H})$ and every function $\hat{\sigma} \in \mathcal{H}(\sigma)$ is a.p. as well.

We assume that

$$|\beta(t)| \leq \sqrt{3}, \tag{98}$$

$$R(t) \leq \lambda_1 - \kappa, \ \forall t \in \mathbb{R}, \tag{99}$$

where the number $\kappa > 0$ and λ_1 is the first eigenvalue of the operator $\{-\Delta, \ u|_{\partial\Omega} = 0\}$. We note that these inequalities also hold for any functions $\hat{\beta}(t)$ and $\hat{R}(t)$, such that $(\hat{\alpha}(t), \hat{\beta}(t), \hat{R}(t), \hat{g}_0(t)) \in \mathcal{H}(\sigma)$.

THEOREM 5.1: *Let the above assumptions hold. Then*

(i) *the family of processes* $\{U_{\hat{\sigma}}(t,\tau)\}, \hat{\sigma} \in \mathcal{H}(\sigma)$, *corresponding to equations* (97) *has the uniform global attractor* $\mathcal{A}$;

(ii) *the set* $\mathcal{A}$ *attracts bounded (in* $\boldsymbol{H}$*) set of solutions of* (97) *with exponential rate, that is,*

$$\sup_{\hat{\sigma} \in \mathcal{H}(\sigma)} \operatorname{dist}_{\boldsymbol{H}}(U_{\hat{\sigma}}(t,\tau)B, \mathcal{A}) \leq Ce^{-\kappa(t-\tau)}, \ C = C(\|B\|_{\boldsymbol{H}}), \tag{100}$$

where κ *is taken from* (99)*;*

(iii) *for every* $\hat{\sigma} \in \mathcal{H}(\sigma)$, *there exists a unique bounded (in* $\boldsymbol{H}$*) complete solution* $z_{\hat{\sigma}}(t), t \in \mathbb{R}$, *of equation* (97) *with symbol* $\hat{\sigma} \in \mathcal{H}(\sigma)$, *i.e., the kernel* $\mathcal{K}_{\hat{\sigma}}$ *consists of the unique element* $z_{\hat{\sigma}}$, *and, in this case, the formula* (79) *for the global attractor* $\mathcal{A}$ *has the form*

$$\mathcal{A} = \bigcup_{\hat{\sigma} \in \mathcal{H}(\sigma)} \{z_{\hat{\sigma}}(s)\}, \ \forall s \in \mathbb{R}; \tag{101}$$

(iv) *the complete solution* $z_{\hat{\sigma}}(t), t \in \mathbb{R}$, *attracts any solution* $\hat{u}_{\hat{\sigma}}(t) = U_{\hat{\sigma}}(t,\tau)u_\tau, t \geq \tau$, *with exponential rate:*

$$\|\hat{u}_{\hat{\sigma}}(t) - z_{\hat{\sigma}}(t)\| \leq \|\hat{u}_{\hat{\sigma}}(\tau) - z_{\hat{\sigma}}(\tau)\| e^{-\kappa(t-\tau)}, \ \forall t \geq \tau, \ \tau \in R. \tag{102}$$

PROOF: The existence of the uniform global attractor $\mathcal{A}$ follows from the reasoning given in the beginning of Section 4 and from Theorem 4.1 since we can set $g_1(x,z) \equiv 0$ in equation (71). Then, in this case, the family (73) is independent of ε and coincides with (97) (see also [3] for more details). Here, we only use inequality (98).

We now prove that, under the assumption (99), the uniform global attractor $\mathcal{A}$ is exponential, more exactly, i.e. it attracts bounded sets of initial data $B = \{u_\tau\}$ with exponential rate. We consider an arbitrary symbol $\hat{\sigma} \in \mathcal{H}(\sigma)$. Then, by Theorem 71, the kernel $\mathcal{K}_{\hat{\sigma}}$ is non-empty and, therefore, there is at least one bounded complete trajectory $z_{\hat{\sigma}} \in \mathcal{K}_{\hat{\sigma}}$, i.e. $z_{\hat{\sigma}}(t), t \in \mathbb{R}$, is a solution of (97) and $\|z_{\hat{\sigma}}(t)\| \leq M$, for all $t \in \mathbb{R}$. Let us prove inequality (102). We set $w(t) = \hat{u}_{\hat{\sigma}}(t) - z_{\hat{\sigma}}(t), t \geq \tau$. The function $w(t)$ satisfies the equation

$$\partial_t w = (1 + i\alpha(t))\Delta w + R(t)w - (1 + i\beta(t))\left(|\hat{u}_{\hat{\sigma}}|^2 \hat{u}_{\hat{\sigma}} - |z_{\hat{\sigma}}|^2 z_{\hat{\sigma}}\right), \ w|_{\partial\Omega} = 0. \tag{103}$$

Taking the scalar product in $\boldsymbol{H}$ of (103) with w, we obtain

$$(104)\quad \frac{1}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 - R(t)\|w\|^2 + \big\langle (1+i\beta(t))\big(|\hat{u}_{\hat{\sigma}}|^2\hat{u}_{\hat{\sigma}} - |z_{\hat{\sigma}}|^2 z_{\hat{\sigma}}\big), \hat{u}_{\hat{\sigma}} - z_{\hat{\sigma}}\big\rangle = 0.$$

Since the last term in (104) is non-negative (recall that $|\beta(t)| \le \sqrt{3}$, see (56)) we have that

$$(105)\qquad \frac{1}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 - R(t)\|w\|^2 \le 0.$$

Using the Poincaré inequality $\|\nabla w\|^2 \ge \lambda_1\|w\|^2$ and condition (99) we find that

$$\frac{d}{dt}\|w(t)\|^2 + 2\kappa\|w(t)\|^2 \le 0,\ \forall t \ge \tau.$$

Then

$$(106)\qquad \|w(t)\| \le \|w(\tau)\|e^{-\kappa(t-\tau)},\ \forall t \ge \tau,$$

and (102) is proved.

It follows from (102) that the function $z_{\hat{\sigma}}(t), t \in \mathbb{R}$, is unique. Indeed, let $z'_{\hat{\sigma}}(t), t \in \mathbb{R}$, be any bounded complete solution of (97) with symbol $\hat{\sigma}$. Then the difference $w(t) = z'_{\hat{\sigma}}(t) - z_{\hat{\sigma}}(t)$ satisfies inequality (106), i.e.

$$(107)\qquad \|z'_{\hat{\sigma}}(t) - z_{\hat{\sigma}}(t)\| \le \|z'_{\hat{\sigma}}(\tau) - z_{\hat{\sigma}}(\tau)\|e^{-\kappa(t-\tau)},\ \forall t \ge \tau,\ \tau \in \mathbb{R}.$$

By assumption, the function $\|z'_{\hat{\sigma}}(\tau) - z_{\hat{\sigma}}(\tau)\|$ is bounded for all $\tau \in \mathbb{R}$, thus, letting $\tau \to -\infty$ and fixing t, we deduce from (107) that

$$\|z'_{\hat{\sigma}}(t) - z_{\hat{\sigma}}(t)\| = 0,\ \forall t \in \mathbb{R}.$$

Consequently, a bounded complete solution is unique. Thus, we have proved point (iii) of the theorem. The formula (101) is also established.

Finally, we notice that the set $\{z_{\hat{\sigma}}(t), t \in \mathbb{R}\} \subset \mathcal{A}$. Moreover, $\|\mathcal{A}\| \le C'$. Combining (101) and (102), we obtain that

$$\begin{aligned}
\operatorname{dist}_{\boldsymbol{H}}(U_{\hat{\sigma}}(t,\tau)B, \mathcal{A}) &= \sup_{u_\tau \in B} \operatorname{dist}_{\boldsymbol{H}}(U_{\hat{\sigma}}(t,\tau)u_\tau, \mathcal{A}) \\
&\le \sup_{u_\tau \in B}\|U_{\hat{\sigma}}(t,\tau)u_\tau - z_{\hat{\sigma}}(t)\| \le \sup_{u_\tau \in B}\|\hat{u}_{\hat{\sigma}}(\tau) - z_{\hat{\sigma}}(\tau)\|e^{-\kappa(t-\tau)} \\
&\le (\|B\| + C')e^{-\kappa(t-\tau)} = C(B)e^{-\kappa(t-\tau)},\ \forall \hat{\sigma} \in \mathcal{H}(\sigma),
\end{aligned}$$

and point (ii) of Theorem 5.1 is also proved.

6. - Estimate for the deviation of the attractor $\mathcal{A}^\varepsilon$ from the attractor $\mathcal{A}^0$

In this section, we consider the family of the G.–L. equations (73) with singularly oscillating external force

$$(108)\quad \partial_t\hat{u}^\varepsilon = (1+i\hat{\alpha}(t))\Delta\hat{u}^\varepsilon + \hat{R}(t)\hat{u}^\varepsilon - (1+i\hat{\beta}(t))|\hat{u}^\varepsilon|^2\hat{u}^\varepsilon + \hat{g}_0(x,t) + \frac{1}{\varepsilon^\rho}\hat{g}_1\Big(\frac{x}{\varepsilon}, t\Big),\ \hat{u}^\varepsilon|_{\partial\Omega} = 0,$$

having symbols $\hat{\sigma}^\varepsilon(t) = (\hat{a}(t), \hat{\beta}(t), \hat{R}(t), \hat{g}^\varepsilon(t)) \in \mathcal{H}(\sigma^\varepsilon)$, where $\hat{g}^\varepsilon(t) = \hat{g}_0(x,t) + \varepsilon^{-\rho}\hat{g}_1\left(\frac{x}{\varepsilon}, t\right)$ for some $\hat{g}_0(x,t) \in \mathcal{H}(g_0)$ and $\hat{g}_1(z,t) \in \mathcal{H}(g_1)$. Recall that the symbol $\sigma^\varepsilon(t) = (a(t), \beta(t), R(t), g^\varepsilon(t))$ of the original equation (15) is a.p. We assume that the assumptions of Theorems 4.1 and 5.1 hold. Thus properties (81), (82) and inequalities (98), (99) are valid.

We consider the difference $w(t) = \hat{u}^\varepsilon(\tau + t) - \hat{u}^0(\tau + t)$, where $\hat{u}^\varepsilon(t)$ and $\hat{u}^0(t)$ satisfies, respectively, equations (108) and (97) with common terms $\hat{a}(t), \hat{\beta}(t), \hat{R}(t), \hat{g}_0(t)$ and with common initial data at $t = \tau$, $\hat{u}^\varepsilon|_{t=\tau} = \hat{u}^0|_{t=\tau} = u_\tau \in \boldsymbol{H}$. The function $\hat{g}_1(z,t) \in \mathcal{H}(g_1)$ is arbitrary. In a standard way, we obtain the identity

$$(109)\quad \frac{1}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 - \langle R(t)w, w\rangle + \\ + \left\langle (1 + i\beta(t))\left(|\hat{u}^\varepsilon|^2\hat{u}^\varepsilon - |\hat{u}^0|^2\hat{u}^0\right), \hat{u}^\varepsilon - \hat{u}^0\right\rangle = \varepsilon^{-\rho}\left\langle \hat{g}_1\left(\frac{x}{\varepsilon}, t\right), w\right\rangle.$$

Since Condition $\hat{\mathrm{I}}$ holds we repeat the reasonings from the proof of Theorem 4.2 from (55) to (57) and we obtain that

$$(110)\qquad \frac{d}{dt}\|w\|^2 + 2\|\nabla w\|^2 - 2\bar{R}\|w\|^2 \le 2\varepsilon^{-\rho}\left\langle \hat{g}_1\left(\frac{x}{\varepsilon}, t\right), w\right\rangle,$$

where $\bar{R} = \sup_{t\in\mathbb{R}} R(t)$. We estimate the right-hand side of (110) similar to (58). We set $\delta = \kappa/2$ and we obtain similar to (63)

$$\frac{d}{dt}\|w\|^2 \le -r_1\|w\|^2 + \frac{\lambda_1}{\kappa}C_1M^2\varepsilon^{2(1-\rho)},$$

where $r_1 = \lambda_1 - \bar{R} - \kappa/2 \ge \kappa/2 > 0$ (see (99)). Then

$$(111)\qquad \|w(t)\|^2 \le \kappa^{-1}\frac{\lambda_1 C_1 M^2}{r_1}\varepsilon^{2(1-\rho)},\ \forall t \ge 0,$$

and we find from (111) that

$$\|w(t)\| \le C\varepsilon^{(1-\rho)},\ \forall t \ge 0,$$

where $C = \left(\frac{\lambda_1 C_1 M^2}{\kappa r_1}\right)^{1/2}$. Consequently,

$$(112)\qquad \|\hat{u}^\varepsilon(\tau + t) - \hat{u}^0(\tau + t)\| \le C\varepsilon^{(1-\rho)},\ \forall t \ge 0.$$

It was proved in Section 4 that the family of processes $\{U_{\hat{\sigma}^\varepsilon}(t,\tau)\}$, $\hat{\sigma}^\varepsilon \in \mathcal{H}(\sigma^\varepsilon)$, corresponding to equations (108) has the uniform (w.r.t. $\tau \in \mathbb{R}$ and $\hat{\sigma}^\varepsilon \in \mathcal{H}(\sigma^\varepsilon)$) global attractor $\mathcal{A}^\varepsilon \Subset \boldsymbol{H}$ such that

$$(113)\qquad \mathcal{A}^\varepsilon = \bigcup_{\hat{\sigma}^\varepsilon \in \mathcal{H}(\sigma^\varepsilon)} \mathcal{K}_{\hat{\sigma}^\varepsilon}(s)$$

where $s \in \mathbb{R}$ is arbitrary and fixed (see Theorem 4.1). It follows from Theorem 4.2 that if $\rho < \rho_0 = 1$ (we note that $\bar{R} < \lambda_1$, see (99)), then the global attractors $\mathcal{A}^\varepsilon$ are uniformly bounded (in $\boldsymbol{H}$) with respect to ε :

$$(114)\qquad \|\mathcal{A}^\varepsilon\| \le C(\rho),\ \forall \varepsilon,\ 0 < \varepsilon \le 1.$$

In Section 5, it was established that under the conditions (98) and (99), the global attractor $\mathcal{A}^0$ of the "limit" equations (97) is exponential. We use this result to prove the main theorem of this section.

THEOREM 6.1: *Let* $0 < \rho < 1$. *Then, under the assumptions of Theorems* 4.1 *and* 5.1, *the Hausdorff distance (in* $\boldsymbol{H}$*) from the global attractor* $\mathcal{A}^\varepsilon$ *to the "limit" global attractor* $\mathcal{A}^0$ *satisfies the inequality*

$$\text{dist}_{\boldsymbol{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leq C(\rho)\varepsilon^{1-\rho}, \ \forall\varepsilon, \ 0 < \varepsilon \leq 1. \tag{115}$$

PROOF: We fix ε. Let u^ε be an arbitrary element of $\mathcal{A}^\varepsilon$. By (113), there exists a bounded complete solution $\hat{u}^\varepsilon(t), t \in \mathbb{R}$, of equation (108) with some symbol $\hat{\sigma}^\varepsilon = \left(\hat{a}(t), \hat{\beta}(t), \hat{R}(t), \hat{g}_0(x,t) + \varepsilon^{-\rho}\hat{g}_1\left(\frac{x}{\varepsilon}, t\right)\right) \in \mathcal{H}(\sigma^\varepsilon)$, such that

$$u^\varepsilon = \hat{u}^\varepsilon(0). \tag{116}$$

We consider the point $\hat{u}^\varepsilon(-T)$ which clearly belongs to $\mathcal{A}^\varepsilon$ and hence

$$\|\hat{u}^\varepsilon(-T)\| \leq C(\rho), \tag{117}$$

where $C(\rho)$ is independent of ε and T.

Consider the "limit" symbol $\hat{\sigma}^0 = (\hat{a}(t), \hat{\beta}(t), \hat{R}(t), \hat{g}_0(x,t)) \in \mathcal{H}(\sigma^0)$ and the corresponding "limit" equation (97). We set $\tau = -T$. Let $\hat{u}^0(t), t \geq -T$, be a solution of this equation with initial data

$$\hat{u}^0|_{t=-T} = \hat{u}^\varepsilon(-T). \tag{118}$$

It follows from Points (iii) an (iv) of Theorem (5.1) that there is a unique bounded complete solution $z^0(t), t \in \mathbb{R}$, of equation (97) with symbol $\hat{\sigma}^0$ such that

$$\|\hat{u}^0(-T+t) - z^0(-T+t)\| \leq \|\hat{u}^0(-T) - z^0(-T)\|e^{-\kappa t}, \ \forall t \geq 0. \tag{119}$$

Recall that $z^0(t) \in \mathcal{A}^0$ for all $t \in \mathbb{R}$ and in particular

$$\|z^0(-T)\| \leq C', \tag{120}$$

where C' is independent of T and by (118) and (114)

$$\|\hat{u}^0(-T)\| = \|\hat{u}^\varepsilon(-T)\| \leq C(\rho). \tag{121}$$

From (119) and (121) we find that

$$\|\hat{u}^0(-T+t) - z^0(-T+t)\| \leq C''e^{-\kappa t}, \ \forall t \geq 0, \tag{122}$$

where $C'' = C' + C(\rho)$.

We set in (112) $\tau = -T$ and have

$$\|\hat{u}^\varepsilon(-T+t) - \hat{u}^0(-T+t)\| \leq C\varepsilon^{(1-\rho)}, \ \forall t \geq 0. \tag{123}$$

Using (122) and (123), we see that

$$\begin{aligned} &\|\hat{u}^\varepsilon(-T+t) - z^0(-T+t)\| \leq \\ &\leq \|\hat{u}^\varepsilon(-T+t) - \hat{u}^0(-T+t)\| + \|\hat{u}^0(-T+t) - z^0(-T+t)\| \leq \\ &\leq C\varepsilon^{(1-\rho)} + C''e^{-\kappa t}. \end{aligned} \tag{124}$$

We now choose T from the equation

$$\varepsilon^{(1-\rho)} = e^{-\kappa T}, \text{ that is, } T = \frac{1-\rho}{\kappa}\log\left(\frac{1}{\varepsilon}\right)$$

and we set in (124) $t = T$. We obtain that

$$\|\hat{u}^\varepsilon(0) - z^0(0)\| \le (C + C'')\varepsilon^{(1-\rho)} = C(\rho)\varepsilon^{(1-\rho)}$$

and hence

$$\text{dist}_{\boldsymbol{H}}(u^\varepsilon, \mathcal{A}^0) \le \|u^\varepsilon - z^0(0)\| = \|\hat{u}^\varepsilon(0) - z^0(0)\| \le C(\rho)\varepsilon^{(1-\rho)}.$$

Since u^ε is an arbitrary point of $\mathcal{A}^\varepsilon$ we find that

$$\text{dist}_{\boldsymbol{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \le C(\rho)\varepsilon^{(1-\rho)}.$$

The theorem is proved. ■

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Direttore responsabile: Prof. A. Ballio - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5 - Bologna