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Recent Trends on Nonlinear Elliptic Equations on $\mathbb{R}^n$

1. - Introduction

Many problems arising in Condensed Matter, Nonlinear Optics, Quantum Mechanics, etc. can be modeled by nonlinear elliptic equations inf $\mathbb{R}^n$, like

$$(1) \qquad \begin{cases} -\Delta u + V(x)u = K(x)u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0, \end{cases}$$

where Δ denotes the Laplace operator and $p > 1$. Above, the condition $u \in W^{1,2}(\mathbb{R}^n)$ is required to obtain a solution with physical interest. In the sequel we will always consider the case in which the space dimension n is greater or equal than 3 and that the exponent p is subcritical, namely that p satisfies

$$1 < p < \frac{n+2}{n-2}.$$

If $n = 1, 2$ this restriction would be unnecessary and any $p > 1$ could be allowed.

Equation (1) is the Euler-Lagrange equation of the functional defined (formally) on $W^{1,2}(\mathbb{R}^n)$ by setting

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} V(x)u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^n} K(x)|u|^{p+1} dx.$$

In order to find critical points of $I(u)$, the main difficulty is the lack of compactness which can bypassed assuming suitable conditions on V and/or K. For example, taking $K \equiv 1$, in [15] it has been shown, among other things, that I has a (Mountain-Pass) critical point

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provided $V > 0$ and $V(x) \to +\infty$ as $|x| \to +\infty$. On the other hand, using the *Concentration-Compactness* method of P.L. Lions, existence of solutions of (1) has been proved under the assumption that ($V \equiv 1$ and) $K > 0$, $\lim_{|x|\to\infty} K(x) = K_\infty > 0$ and $\exists\, R, C, \delta > 0$

$$(K_0) \qquad K(x) \geq K_\infty - C\exp(-\delta x), \qquad \text{for } |x| \geq R.$$

See [9, 10].

In Section 2 below, we will survey some recent results, dealing with the existence of solutions of (1), which do not make use of the Concentration-Compactness, see Theorems 1 and 3. In particular, the latter deals with the new case that V and K decay to zero at infinity.

Another interesting class of nonlinear equations on $\mathbb{R}^n$ is the following one

$$(2) \qquad -\varepsilon^2 \Delta u + V(x)u = K(x)u^p, \quad u \in W^{1,2}(\mathbb{R}^n), \quad u > 0,$$

arising in Quantum Mechanics (take $\varepsilon = \hbar$, the Planck constant). It is natural to consider (2) like a singular perturbation problem and to look for solutions of (2) when ε is sufficiently small. This problem, as well as the study of the behavior of solutions when ε tends to zero (semiclassical limits) is particularly important to explain the relationship between Quantum and Classical Mechanics.

There is a broad literature dealing with (2). Roughly, a typical result is that semiclassical states u_ε exist provided

(i) $0 < C_1 \leq V(x) \leq C_2$;
(ii) $0 < K(x) \leq C_3$;
(iii) the auxiliary potential

$$Q(x) = [V(x)]^\theta [K(x)]^{-2/(p-1)}, \quad \theta = \frac{p+1}{p-1} - \frac{n}{2},$$

has a *stable* stationary point x_0.

Stable critical points of Q include maxima, minima and, more in general, points x_0 such that the local degree of Q' at x_0 is different from zero.

Furthermore, one shows that the semiclassical state u_ε concentrates at x_0, namely it decays to zero away from x_0 in an uniform way, see e.g. [13, 16]. See also [1] and, for a more general result, [7] and references therein.

Recently, some new results dealing with the existence and concentration of semiclassical states has been obtained, see [6, 8]. The common feature of these results and the ones dealing with (1) outlined above (in particular those stated in Theorem 3), is that the potential V can decay to zero at infinity. These results will be discussed in Section 3.

2. - Existence of bound states of (1)

The first result we discuss is taken from [3] and deals with the case in which $V \equiv 1$ and

$$K(x) = K_\infty + \varepsilon k(x),$$

so that (1) becomes

$$(3)\qquad \begin{cases} -\Delta u + u = K_\infty u^p + \varepsilon k(x) u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0, \end{cases}$$

THEOREM 1: *Assume that k satisfies one of the following two conditions:*

(k_1) *there is $r \in [1,2]$ such that $k \in L^s(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ with $s = \dfrac{2^*}{2^* - (p+1)}$;*

(k_2) *$k \in L^\infty(\mathbb{R}^n)$ and $\lim\limits_{|x|\to\infty} k(x) = 0$.*

Then for all $|\varepsilon|$ small, problem (3) *has a solution.*

With respect to the results cited above, the new feature of Theorem 1 is that no assumption like (K_0) is required. The proof relies on some abstract critical point results, perturbative in nature, that we are going to outline.

Let E be a Hilbert space and suppose that we are looking for the critical points of a functional $I_\varepsilon \in C^2(E, \mathbb{R})$ of the form

$$I_\varepsilon(u) = I_0(u) + \varepsilon G(u),$$

where the unperturbed functional I_0 satisfies the following assumptions:

(a) there exists a finite dimensional C^2 manifold Z such that every $z \in Z$ is a critical point of I_0. Such a Z is called *critical manifold*.

(b) for all $z \in Z$, $I_0''(z)$ is an index 0 Fredholm map.

(c) $T_z Z = Ker[I_0''(z)], \quad \forall\, z \in Z$ (1)

A critical manifold Z is called *non-degenerate* if (b) – (c) hold.

Roughly , if the critical manifold Z is unbounded, there is a lack of compactness which can be recovered under suitable assumptions on the perturbation G. This idea can be carried out when Z is non-degenerate. Actually, by means of a finite dimensional reduction, one can prove various results on the existence of critical points of I_ε in cases in which the compactness Palais-Smale (PS, in short) condition fails. Referring to [4] for a broad exposition, here we limit ourselves to state the following result.

THEOREM 2: [4, Thm. 2.17] *Let $I_0, G \in C^2(E, \mathbb{R})$ and suppose that I_0 has a non-degenerate critical manifold Z. Moreover, let us assume that G has on Z a constrained stable critical point $\bar{z}$. Then, for $|\varepsilon|$ sufficiently small, I_ε has a critical point u_ε such that $u_\varepsilon \to \bar{z}$ as $\varepsilon \to 0$.*

In order to apply Theorem 2 to (3), we take $E = W^{1,2}(\mathbb{R}^n)$,

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} u^2 dx - K_\infty \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx,$$

(1) $T_z Z$ denotes the tangent space to Z at z.

and

$$G(u) = -\frac{1}{p+1}\int_{\mathbb{R}^n} k(x)|u|^{p+1}dx.$$

It is known that I_0 has a critical manifold Z. Precisely, if U denotes the unique positive radial function satisfying $-\Delta U + K_\infty U = U^p$, then $Z = \{z_\xi(x) = U(x-\xi) : \xi \in \mathbb{R}^n\}$. It is also known that such a Z is non-degenerate. One can prove that $\lim_{|\xi|\to\infty} G(z_\xi) = 0$. Moreover, by some Fourier analysis one shows that G is not identically zero on Z, provided (k_1) holds. Thus G has a strict maximum or minimum $\bar{z} \in Z$ and the existence of solutions to (3) follows from Theorem 2. When k satisfies (k_2) the proof requires a slightly different abstract theorem, see [4, Thm. 2.12].

Our next existence result deals with (1) in the case that V and K decay to zero at infinity. Precisely, we will assume that $V, K : \mathbb{R}^n \to \mathbb{R}$ are smooth and satisfy

(V_1) $$\exists\, a, A > 0, \text{ and } \alpha \in [0,2[\text{ such that } \quad \frac{a}{1+|x|^\alpha} \le V(x) \le A,$$

(K_1) $$\exists\, b, \beta > 0, \text{ such that } \quad 0 < K(x) \le \frac{b}{1+|x|^\beta} \; .$$

It is convenient to introduce the following weighted Sobolev space

$$E_V = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left[|\nabla u(x)|^2 + V(x)u^2(x)\right] dx < \infty\},$$

endowed with scalar product and norm given, respectively, by

$$(u|v)_V = \int_{\mathbb{R}^n} [\nabla u(x)\cdot\nabla v(x) + V(x)u(x)v(x)]dx, \quad \|u\|_V^2 = (u|u)_V.$$

Let us also consider the weighted Lebesgue space $L^q_K(\mathbb{R}^n)$ of measurable $u : \mathbb{R}^n \mapsto \mathbb{R}$ such that $\int_{\mathbb{R}^n} K(x)|u(x)|^q dx < +\infty$. It is known, see e.g. [14], that if (V_1) and (K_1) hold then E_V is compactly embedded into $L^{p+1}_K(\mathbb{R}^n)$ provided

(4) $$\sigma < p < \frac{n+2}{n-2}, \qquad \sigma = \begin{cases} \dfrac{n+2}{n-2} - \dfrac{4\beta}{\alpha(n-2)}, & \text{if } 0 < \beta < \alpha \\ 1 \quad \text{otherwise.} \end{cases}$$

It is worth pointing out that to get the embedding of E_V into L^{p+1}_K, it suffices to assume that $0 < \alpha \le 2$ as well as $\sigma \le p \le \frac{n+2}{n-2}$. In particular, we can consider the functional $I \in C^2(E_V, \mathbb{R})$ defined by setting

$$I(u) = \frac{1}{2}\|u\|_V^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} K(x)|u(x)|^{p+1}dx.$$

Moreover, when (4) holds, using this compactness result stated above, a straight application of the Mountain-Pass Theorem yields a critical point $\overline{u} \in E_V$ of $I \in C^2(E_V, \mathbb{R})$, which is a positive solution of $-\Delta u + Vu = Ku^p$. It remains to show that $\overline{u}$ belongs to $L^2(\mathbb{R}^n)$. This is accomplished by proving that, if $0 < \alpha < 2$ then there exists $C > 0$ such that $\forall R \gg 1$ one has

$$\int\limits_{|x|>R} \left[|\nabla \overline{u}|^2 + V(x)\overline{u}^2\right] dx \leq C \exp\left\{-CR^{\frac{2-\alpha}{2}}\right\},$$

and this readily implies that $\overline{u} \in L^2(\mathbb{R}^n)$. In conclusion we can state:

THEOREM 3: [2, Thm. 1] *Suppose that* (V_1) *and* (K_1) *hold and let* p *satisfy* (4). *Then* (1) *has a solution* $\overline{u} \in C^2(\mathbb{R}^n)$ *such that* $\lim\limits_{|x|\to\infty} \overline{u}(x) = 0$. *Moreover,* $\overline{u}$ *is a ground state, in the sense that* $I_V(\overline{u}) = \inf\{I_V(u); u \neq 0,\ I'_V(u) = 0\}$.

REMARK 4: (*a*) Of course, the same arguments show that (2) has a solution u_ε for every $\varepsilon > 0$.

(*b*) If $1 < p < \sigma$, or if K is bounded away from zero and $V \sim |x|^{-\alpha}$ with $0 < \alpha < 2$, there are no ground states of (1), see [2, Rem. 14 and Prop. 15]. ∎

We conclude this section by pointing out that the solutions u_ε of (2) found in Theorem 3, see also Remark 4-(*a*), concentrate as $\varepsilon \to 0$. First, let us remark that the potential Q introduced before, has a global minimum when (V_1), (K_1) and (4) hold.

THEOREM 5: [2, Thm. 3] *Under the same assumptions made in Theorem 3, the solution* u_ε *of (2) concentrates at the global minimum of* Q.

The proof relies on two facts: first of all, one shows the following estimate on the norm of the ground states

$$\exists\, \gamma > 0 \quad : \quad \|u_\varepsilon\|^2_{E_{\varepsilon,V}} \leq \gamma \varepsilon^n,$$

where $E_{\varepsilon,V} = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^n) : \int\limits_{\mathbb{R}^n} \left[\varepsilon^2|\nabla u(x)|^2 + V(x)u^2(x)\right] dx < \infty\}$. Moreover, the following uniform pointwise estimates hold: $\exists\, C,\ d,\ R > 0$, such that

$$|u_\varepsilon(x)| \leq C\left|\frac{x}{\varepsilon}\right|^d \exp\left\{-\frac{C}{\varepsilon}\left(|x|^{\frac{2-\alpha}{2}} - R^{\frac{2-\alpha}{2}}\right)\right\}; \qquad \text{for } |x| > 2R.$$

3. - EXISTENCE OF SEMICLASSICAL STATES

The existence of solutions of (2) for $\varepsilon \ll 1$, can be proved under assumptions much weaker than those made for (1). Roughly, the presence of a small parameter ε allows us to use perturbation methods to bypass the problem concerning the lack of compactness. As a byproduct, this approach will provide solutions that concentrate. This procedure is illustrated below discussing some recent results of [6], where the equation (2) is considered, under the following assumption on V and K:

$$(V_1') \qquad \exists\, a, A > 0, \text{ such that } \quad \frac{a}{1+|x|^2} \le V(x) \le A,$$

$$(V_2) \qquad \exists\, A_1 > 0 \;:\; |V'(x)| \le A_1, \quad \forall\, x \in \mathbb{R}^n.$$

$$(K_2) \qquad \exists\, \kappa > 0 \;:\; 0 < K(x) \le \kappa, \quad |K'(x)| \le \kappa, \quad \forall\, x \in \mathbb{R}^n.$$

Let us point out that, with respect to (V_1), the new feature of (V_1') is that the latter the exponent $\alpha = 2$ is allowed.

THEOREM 6: [6, Thm. 1] *Let* $1 < p < \dfrac{n+2}{n-2}$ *and suppose that V and K are smooth and satisfy* (V_1'), (V_2) *and* (K_2). *Moreover, let* x_0 *be an isolated stable stationary point of* Q. *Then for* $\varepsilon \ll 1$, *equation* (2) *has a solution which concentrates at* x_0.

REMARK 7: (*a*) The new feature of the preceding Theorem is that we neither need to assume the growth restriction (4), nor to impose that K decays to zero at infinity. Moreover, $\alpha = 2$ is allowed. On the other hand, the solutions found above exist for ε small and might not be ground states. Let us also point out that, in general, these solutions do not concentrate at the global minima of Q, like the ground states found in Theorem 3. Actually Q might possess no global minimum (see (*b*)).

(*b*) If $0 < \beta < \alpha < 2$ and $p < \sigma$, then $Q(x) \to 0$ as $|x| \to \infty$ and it has a global maximum. The same happens if $\beta = 0$. In both the cases, Theorem 6 applies while Theorem 3 does not. ■

Let us outline the proof of Theorem 6. Performing the change of variable $x \mapsto \varepsilon x$, equation (2) becomes

$$(5) \qquad -\Delta u + V(\varepsilon x) u = K(\varepsilon x) u^p, \quad u \in W^{1,2}(\mathbb{R}^n), \quad u > 0,$$

and this makes clear that the problem is perturbative in nature. To take advantage of this fact, it is convenient to use an approach closely related to the abstract results such as Theorem 2 above.

More precisely, let us consider the space H_ε,

$$H_\varepsilon = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(\varepsilon x) u^2(x) dx < \infty\},$$

endowed with the norm

$$(6) \qquad \|u\|_\varepsilon^2 = \int_{\mathbb{R}^n} \Big[|\nabla u(x)|^2 + V(\varepsilon x) u^2(x) \Big] dx,$$

and let us suppose for the moment that the functional

$$(7) \qquad I_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 - \int_{\mathbb{R}^n} K(\varepsilon x) |u|^{p+1} dx, \qquad u \in H_\varepsilon,$$

is well defined. However, the critical points of I_ε such that

$$\int_{\mathbb{R}^n} K(\varepsilon x)|u|^{p+1}dx < +\infty, \tag{8}$$

give rise to solutions of (5).

Consider the autonomous equation

$$-\Delta u + V(\varepsilon\xi)u = K(\varepsilon\xi)u^p, \quad u \in W^{1,2}(\mathbb{R}^n), \quad u > 0, \tag{9}$$

obtained from (5) by "freezing" the potentials V and K at $x = \varepsilon\xi$. The positive solutions of (9) are given by

$$z_{\varepsilon,\xi}(x) = \sigma U(\lambda(x-\xi)), \qquad \sigma = \left[\frac{V(\varepsilon\xi)}{K(\varepsilon\xi)}\right]^{\frac{1}{p-1}}, \quad \lambda = [V(\varepsilon\xi)]^{\frac{1}{2}},$$

where $U > 0$ is radial and satisfies $-\Delta U + U = U^p$. We expect to find solutions concentrating at a stable stationary point x_0 of the auxiliary potential Q defined in the introduction. To simplify the notation we take $x_0 = 0$. Then we will look for critical points of I_ε in the form $u = z_{\varepsilon,\xi} + w$, with $\varepsilon\xi \sim 0$. More precisely, let us set

$$Z_\varepsilon = \{z_{\varepsilon,\xi}(x) : \ \xi \in \mathbb{R}, \ |\varepsilon\xi| < 1\},$$

and let try to solve the equation $I'_\varepsilon(z_{\varepsilon,\xi} + w) = 0$, with $z_{\varepsilon,\xi} \in Z_\varepsilon$. Let $P = P_{\varepsilon,\xi}$ denote the orthogonal projection onto $(T_{z_{\varepsilon,\xi}}Z)^\perp$ (orthogonality is meant with respect to the scalar product in (6)) and consider the auxiliary equation

$$PI'_\varepsilon(z_{\varepsilon,\xi} + w) = 0. \tag{10}$$

It is possible to show that for ε is small enough, $PI''_\varepsilon(z_{\varepsilon,\xi})$ is uniformly invertible for every $\xi \in \mathbb{R}^n$, with $|\varepsilon\xi| \leq 1$. This allows us to re-write (10) in the more convenient form $S_\varepsilon(w) = w$, where

$$S_\varepsilon(w) = w - \left[PI''_\varepsilon(z_{\varepsilon,\xi})\right]^{-1}\left(PI'_\varepsilon(z_{\varepsilon,\xi} + w)\right).$$

Fixed points of the map S_ε are found in a subset Γ_ε of H_ε of functions satisfying appropriate estimates and suitable decays. To carry out this argument in a more precise way, some preliminaries are in order.

From (V'_1) it follows that, for any $m > 0$ there exist $R > 0$ such that

$$V(\varepsilon x + y) \geq \frac{m}{|x|^2}, \qquad \forall\, |x| \geq R, \ |y| \leq 1,$$

provided ε is sufficiently small. To use this estimate, it is convenient to consider the problem

$$-u''(r) - (n-1)\frac{u'(r)}{r} + m\,\frac{u(r)}{r^2} = 0, \tag{11}$$

which has two linearly independent solutions given by

$$\phi_1(r) = r^{\frac{2-n-\sqrt{(n-2)^2+4m}}{2}}, \qquad \phi_2(r) = r^{\frac{2-n+\sqrt{(n-2)^2+4m}}{2}}. \tag{12}$$

Given any positive $h : (R, +\infty) \mapsto \mathbb{R}$, let $\psi = \psi_h$ be the radial solution

$$(L_R) \qquad \begin{cases} -\Delta u + \dfrac{m}{|x|^2}\, u = h(|x|), & |x| > R, \\ u(x) = 1 & |x| = R, \\ u(x) \to 0 & |x| \to \infty, \end{cases}$$

It is possible to show that, if

$$(13) \qquad \int_R^{+\infty} r^{n-1} h(r) \phi_2(r)\, dr < +\infty,$$

then there exists $\gamma_R > 0$, depending upon R (and h), such that

$$(14) \qquad |\psi(|x|)| \le \gamma_R \phi_1(|x|), \qquad \forall\, |x| \ge R.$$

Next, let $\mathcal{W}_\varepsilon$ denote the set of the functions $w \in H_\varepsilon$ such that

$$(15) \qquad |w(x+\xi)| \le \begin{cases} \gamma_R \sqrt{\varepsilon}\, \phi_1(|x|), & \text{if } |x| \ge R, \\ \sqrt{\varepsilon}, & \text{if } |x| \le R, \end{cases}$$

and set

$$\Gamma_\varepsilon = \{ w \in E : \|w\|_\varepsilon \le c\,\varepsilon, \quad w \in \mathcal{W}_\varepsilon \cap (T_{z_{\varepsilon,\xi}})^\perp,\ |\varepsilon\xi| \le 1 \}.$$

The key lemma is the following one:

LEMMA 8: *There exist $R > m$ and $c > 0$, such that S_ε maps Γ_ε into itself and is a contraction, provided ε is sufficiently small.*

The proof of this Lemma cannot be reported here in details. To have a flavor of the arguments, let us give an idea how one shows that $\widetilde{w} := S_\varepsilon(w)$ satisfies the first inequality in (15). From the definition of S_ε it follows that the function $\widetilde{w}$ satisfies

$$\begin{cases} -\Delta \widetilde{w} + V(\varepsilon x + \varepsilon\xi)\widetilde{w} - pK(\varepsilon\xi + \varepsilon x) z_{\varepsilon,\xi}^{p-1}(x+\xi)\widetilde{w} = h^*(|x|), & |x| > R, \\ \widetilde{w}(x) = 1 & |x| = R, \\ \widetilde{w}(x) \to 0 & |x| \to \infty, \end{cases}$$

for a suitable h^* depending on w, such that $0 < h^*(r) \le \sqrt{\varepsilon}\, \phi_1^{2\wedge p}(r)$ for $r \ge R$, provided $R \gg 1$. Taking m sufficiently large, one has that $\phi_1^{2\wedge p}(r)$ verifies (13) and hence any solution ψ of (L_R), with $h = \phi_1^{2\wedge p}(r)$, satisfies (14). Since

$$V(\varepsilon x + \varepsilon\xi) - pK(\varepsilon x + \varepsilon\xi) z_0^{p-1}(x) \ge \frac{m}{|x|^\alpha}, \qquad (|x| > R),$$

a straight comparison argument implies that $|\widetilde{w}(x+\xi)| \le \sqrt{\varepsilon}\psi(|x|) \le \gamma_R \sqrt{\varepsilon}\phi_1(|x|)$, as required.

From Lemma 8 we deduce that S_ε has a fixed point in Γ_ε and thus equation (10) has a unique solution $w = w_{\varepsilon,\xi} \in \Gamma_\varepsilon$, provided $\varepsilon \ll 1$. At this point, in order to find a solution of $I'_\varepsilon(z_{\varepsilon,\xi}+w)=0$ it remains to solve the equation $(I-P)I'_\varepsilon(z_{\varepsilon,\xi}+w)=0$. One shows (see [4, Chapter 2] for details) that a solution of this latter equation can be found by looking for critical points of the *reduced* (finite dimensional) functional $\Phi_\varepsilon(\xi)=I_\varepsilon(z_{\varepsilon,\xi}+w_{\varepsilon,\xi})$. The final step consists in proving that

$$\Phi_\varepsilon(\xi)=C_0Q(\varepsilon\xi)+o(\varepsilon),\qquad \Phi'_\varepsilon(\xi)=C_1\,\varepsilon\,Q'(\varepsilon\xi)+o(\varepsilon),$$

where C_0, C_1 are positive constants. Since $x_0=0$ is a maximum or minimum of Q, or, more in general, a stable critical point of Q, we infer that Φ_ε possesses a critical point $\xi_\varepsilon/\varepsilon$ of , with $\xi_\varepsilon\sim x_0=0$, which corresponds to a solution u_ε of (5). Scaling back, $\widetilde{u}_\varepsilon(x):=u_\varepsilon(x/\varepsilon)$ is a solution of (2). Since $\widetilde{u}_\varepsilon(x)\sim z_{\varepsilon,\xi}\left(\dfrac{x-\xi_\varepsilon}{\varepsilon}\right)$, it follows that $\widetilde{u}_\varepsilon$ belongs to $W^{1,2}(\mathbb{R}^n)$ and concentrates at $x_0=0$.

Finally, we have to justify that we can deal with the functional I_ε given by (7). Actually, we can choose $c>0$ and $\vartheta>0$ such that for all $u=z_{\varepsilon,\xi}+w$, $w\in\Gamma_\varepsilon$, there holds

$$\vartheta(p-1)>n,\qquad |u(x)|<c(1+|\varepsilon x|)^{-\vartheta},\qquad (|\varepsilon\xi|\le 1).$$

This implies that

$$\int_{\mathbb{R}^n}K(\varepsilon x)|u(x)|^{p+1}dx<c\int_{\mathbb{R}^n}K(\varepsilon x)(1+|\varepsilon x|)^{\vartheta(p+1)}dx<+\infty,$$

for all such u. In particular, the solution u_ε found above satisfies (8). Now we can define $F_\varepsilon(x,\cdot)\in C^2(\mathbb{R})$ satisfying

$$F_\varepsilon(x,u)=\begin{cases}\dfrac{1}{p+1}|u|^{p+1} & \text{if}\quad |u|<(1+|\varepsilon x|)^{-\vartheta}\\[2ex] c\,(1+|\varepsilon x|)^{-\vartheta(p+1)} & \text{if}\quad |u|>2(1+|\varepsilon x|)^{-\vartheta}.\end{cases}$$

and such that

$$\widetilde{I}_\varepsilon(u):=\frac{1}{2}\|u\|^2_\varepsilon-\int_{\mathbb{R}^n}K(\varepsilon x)F_\varepsilon(x,u)dx,$$

is well defined and of class C^2 on H_ε. Clearly, $\widetilde{I}_\varepsilon(u)=I_\varepsilon(u)$ for all $u=z_{\varepsilon,\xi}+w$, with $w\in\Gamma_\varepsilon$, and hence all the preceding arguments can repeated with I_ε substituted by $\widetilde{I}_\varepsilon$.

REMARK 9: If $V(x)\sim|x|^{-a}$ as $|x|\to\infty$, and $0<a<2$, we could consider, instead of (11), the Bessel equation $-u''(r)-(n-1)\dfrac{u'(r)}{r}+m\,\dfrac{u(r)}{r^a}=0$. In such a case, the functions ϕ_i considered in (12) can be substituted by Bessel functions, giving rise to solutions with a decay rate more than polinomial. ■

Our last result deals with radial nonlinear Schrödinger equations like

$$-\varepsilon^2\Delta u+V(|x|)u=u^p,\quad u\in W^{1,2}(\mathbb{R}^n),\quad u>0. \tag{16}$$

In [5] it has been shown that (16) has a radial solution concentrating at the sphere of radius $\bar{r} > 0$ provided $\varepsilon \ll 1$, $p > 1$, V is smooth with $0 < C_1 \leq V(|x|) \leq C_2$ and $\bar{r}$ is a strict maximum or minimum of the auxiliary *weighted* potential

$$M(r) = r^{n-1} V^{\tau}(r), \qquad \text{where} \quad \tau = \frac{p+1}{p-1} - \frac{1}{2}.$$

This result can be extended to cover the case in which V is smooth, radial and satisfies (V_2) and

$$(V_3) \qquad \exists\, a, A, R > 0, \ : \ \frac{a}{1+r^2} \leq V(r) \leq A, \quad \forall\, r \geq R.$$

By modifying in an appropriate way the definition of Γ_ε and carrying out arguments similar to those used to prove Theorem 6, one can show

THEOREM 10: [8] *Let $p > 1$ and let V be smooth, radial and satisfy (V_2) and (V_3). Moreover, suppose that the auxiliary weighted potential M has a strict maximum or minimum of M at some $r^* \geq R$. Then for $\varepsilon \ll 1$, equation* (16) *has a radial solution concentrating at the sphere of radius r^*.*

REMARK 11: (a) The following one is an example in which Theorem 10 applies. Let $V(r) > 0$ for all $r > 0$, $V(r) \sim r^{-a}$ as $r \to +\infty$ and let (V_2) hold. Using the definition of M, namely $M(r) = r^{n-1} V^{\tau}(r)$, one deduces that
(i) $M(0) = 0$ and $M(r) > 0$ for all $r > 0$;
(ii) $M(r) \sim r^{n-1-a\tau}$ as $r \to +\infty$.

Therefore, if $a\,\tau > (n-1)$ then $M(r) \to 0$ as $r \to +\infty$ and hence M has a maximum at some $r^* > 0$.

(b) One should be able to prove that a radial solution concentrating at the sphere of radius r^* exists, provided $V(r) \geq 0$, $M'(r^*) = 0$ and $V(r^*) > 0$. Another interesting result to pursue would be to find a solution concentrating at a sphere of radius $r_0 > 0$, where r_0 is such that $V(r_0) = 0$. Remark that such a r_0 is a minimum of M.

(c) The case in which V can be zero at some $\bar{x} \in \mathbb{R}^n$, but V is bounded away from zero at infinity, has been handled in [12]. Let us point out that in [12] V does not need to be radial and solutions concentrating at $\bar{x}$ are found. It is worth pointing out that the solutions found in [12] are different in nature from those discussed above, because their peaks tend to zero as $\varepsilon \to 0$.

(d) It is an open problem to extend [12] to the case in which $V(x) \sim |x|^{-a}$ as $|x| \to +\infty$, with $0 < a \leq 2$. ■

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