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Linear Isometries of Vector-Valued Functions (**) 

SUMMARY. — Let $M$ be a compact Hausdorff space and let $C(M)$ be the Banach space of all complex-valued continuous functions on $M$. The classical Banach-Stone theorem, which associates to any surjective linear isometry $A : C(M) \to C(M)$ a homeomorphism of $M$, was generalized by W. Holsztynski to the case in which the linear isometry $A$ is not necessarily surjective. Holsztynski’s result — which was further extended by M. Cambern to Banach spaces of continuous vector-valued functions on $M$ — associates to $A$ a subset $K(A)$ of $M$ and a continuous surjective map $\psi : K(A) \to M$. In this paper, a maximal $\psi$-invariant subset of $M$ is constructed in terms of the iterates of $A$. Actually, the construction of the invariant subset is carried out replacing the discrete subgroup of the iterates of $A$ by a strongly continuous semigroup of linear isometries.

Isometrie lineari di funzioni a valori vettoriali

SUNTO. — Sia $M$ uno spazio compatto di Hausdorff, e sia $C(M)$ lo spazio di Banach delle funzioni continue a valori complessi su $M$. Il classico teorema di Banach-Stone, che associa ad ogni isometria lineare $A : C(M) \to C(M)$ un omeomorfismo di $M$, è stato generalizzato da W. Holsztynski al caso in cui l’isometria lineare $A$ non è necessariamente surgettiva. Il risultato di Holsztynski — esteso da M. Cambern a spazi di Banach di funzioni a valori vettoriali, continue su $M$ — associa a $A$ un sottoinsieme $K(A)$ di $M$ ed una applicazione continua $\psi$ di $K(A)$ su $M$. In questo lavoro, si costruisce un sottoinsieme $\psi$-invariante massimale di $M$ definito mediante le iterate di $A$. Di fatto, il sottoinsieme invariante viene costruito sostituendo al semigruppo discreto delle iterate di $A$ un sottoinsieme fortemente continuo di isometrie lineari.

In one of the final chapters of [2], S. Banach made the important observation that two compact metric spaces $M$ and $N$ are homeomorphic if, and only if, the uniform spaces of all continuous, real-valued functions on $M$ and $N$ are isometric. As a byproduct of his proof, if $A$ is such an isometry, there are a homeomorphism $\psi$ of $N$ onto $M$

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and a continuous function $\alpha$, with modulus one at all points of $N$, such that

$$
(Af)(y) = \alpha(y)f(\psi(y))
$$

at all $y \in N$ and for any real-valued, continuous function $f$ on $M$. This ground-breaking result was the starting point of a research field which is quite alive today. In [13] M. Stone extended Banach’s theorem to continuous, complex-valued functions on compact (not necessarily metric) Hausdorff spaces and set the stage, within the framework of Boolean algebras, of what would later be called the Banach-Stone problem (see [3] also for exhaustive historical references until 1979), involving continuous vector-valued functions.

In [9], W. Holsztynski considered the case in which the linear isometry $A$ is not surjective (1), and proved that (1) still holds, but gives only a partial description of $A$ in the sense that $\psi$ is then a continuous map of a closed subset $K(A)$ of $N$ onto $M$ and $y \in K(A)$. As was shown in [15], the case $K(A) = N$ can be characterized in terms of the behaviour of $A$ on the extreme points of the closed unit ball of the space of all continuous, complex-valued functions on $M$.

In [4] M. Cambern proved that Holsztynski’s result extends mutatis mutandis to Banach spaces of continuous vector-valued functions from $M$ to a complex Banach space $E$ and from $N$ to a strictly convex complex Banach space $\overline{E}$.

In the case in which $M = N$ the question arises, for both Holsztynski’s and Cambern’s theorems, whether there exists a subset $K(A) \subseteq M$ that is invariant under the action of $A$ and on which the action of $A$ is therefore completely described by (1) or by a generalization thereof. In this paper, a maximal invariant set will be constructed in terms of the iterates of $A$. However, instead of considering these iterates, a more general situation will be investigated, replacing $A$ by a strongly continuous semigroup of linear isometries.

After a first section devoted to the set of all extreme points of the closed unit ball of the Banach space of all continuous maps from $M$ to $E$, and of the closed unit ball of the dual space, n. 2 investigates the set $K(A) \subseteq N$, establishing a necessary and sufficient condition for $K(A)$ to coincide with $N$, and a sufficient condition for $K(A)$ to be closed, retrieving, as a consequence, a result of M. Cambern whereby $K(A)$ is closed when $\overline{E}$ has finite dimension.

In n. 3, $A$ is replaced — under the hypotheses $M = N$ and $\overline{E} = \overline{\overline{E}}$ — by a semigroup $T$ of linear isometries, which, in particular, may coincide with the family of all iterates of $A$. Under rather weak hypotheses on $T$ (that are fulfilled when $\overline{E}$ has finite dimension), a maximal «invariant» set $K_\omega(T) \subseteq M$ will be shown to exist, on which the action of $T$ is determined by a semiflow $\phi$ acting on $K_\omega(T)$ and by an operator-valued cocycle associated to $\phi$. If $K_\omega(T)$ is closed and the semigroup $T$ is assumed to be strongly continuous — as will be done in nn. 5 and 6 — the semiflow $\phi$ is continuous,

(1) According to the Mazur-Ulam theorem ([2], pp. 166-168) surjective isometries are linear over the reals. The case of non-linear isometries was briefly investigated in [15].
and the infinitesimal generator of the semigroup defined by \( T \) in \( K_\infty(T) \) is a bounded perturbation of the infinitesimal generator of the semigroup determined by \( \phi \).

Finally, in n. 7 the particular case of scalar-valued continuous functions will be considered, extending to semigroups of general linear isometries some results established in [17] under additional conditions.

1. Let \( \mathcal{E} \) be a complex Banach space with norm \( \| \cdot \| \). If \( M \) is a compact Hausdorff space, \( C(M, \mathcal{E}) \) will stand for the complex Banach space of all continuous functions \( f : M \to \mathcal{E} \), with the uniform norm \( \| f \|_{C(M, \mathcal{E})} = \sup \{ \| f(x) \| : x \in M \} \). For any complex Banach space \( \mathcal{E} \), \( \mathcal{E}' \) will stand for the strong dual of \( \mathcal{E} \); \( \mathcal{B}_\mathcal{E}, \mathcal{B}_\mathcal{E}' \) will indicate respectively the unit ball of \( \mathcal{E} \) and its closures.

**Proposition 1:** Let \( \mathfrak{A} \neq \{0\} \) be a closed linear subspace of \( C(M, \mathcal{E}) \). If \( f \in \mathfrak{A} \),

\[
\| f \|_{C(M, \mathcal{E})} = \sup \{ \langle f, A \rangle : A \text{ extreme point of } \mathcal{B}_\mathcal{E} \}.
\]

**Proof:** Obviously,

\[
\| f \|_{C(M, \mathcal{E})} \geq \sup \{ \langle f, A \rangle : A \text{ extreme point of } \mathcal{B}_\mathcal{E} \}.
\]

Let now \( \| f \|_{C(M, \mathcal{E})} = 1 \).

Since \( M \) is compact, there is some \( x_0 \in M \) such that \( 1 = \| f \|_{C(M, \mathcal{E})} = \| f(x_0) \|_\mathcal{E} \).

For any \( \lambda \in \partial \mathcal{B}_{\mathcal{E}'} \), with \( \| \lambda \|_{\mathcal{E}'} = 1 \), the continuous linear form on \( \mathfrak{A} \)

\[
\delta_{x_0} \hat{\otimes} \lambda : f \mapsto \langle f(x_0), \lambda \rangle
\]

has norm one, showing that the closed set

\[
S := \{ A \in \mathcal{B}_\mathcal{E}' : \langle f, A \rangle = 1 \} \subset \mathfrak{A}'
\]

is not empty. Since, for \( A_1, A_2 \in S \) and \( 0 < t < 1 \),

\[
\langle f, tA_1 + (1-t)A_2 \rangle = t + 1 - t = 1,
\]

\( S \) is also convex, and therefore is compact for the weak-star topology of \( \mathfrak{A}' \). By the Kreš-
Milman theorem, \( S \) has one extreme point at least.

Let \( A_0 \) be one of these extreme points, and let \( A_1, A_2 \in \mathcal{B}_\mathcal{E}' \), \( 0 < t < 1 \) be such that

\[
A_0 = tA_1 + (1-t)A_2.
\]

Since \( A_0 \in S \),

\[
\ell(f, A_1) + (1-\ell)(f, A_2) = 1,
\]

where \( \ell(f, A) := \langle f, A \rangle \).
whence
\[ 1 \leq t|\langle f, A_1 \rangle| + (1 - t)|\langle f, A_2 \rangle| \]
\[ \leq t\|f\|_\alpha \|A_1\|_\alpha + (1 - t)\|f\|_\alpha \|A_2\|_\alpha \]
\[ \leq t + (1 - t) = 1 , \]
and therefore
\[ |\langle f, A_1 \rangle| = |\langle f, A_2 \rangle| = 1 ; \]
(3) yields then
\[ \langle f, A_1 \rangle = \langle f, A_2 \rangle = 1 , \]
\text{i.e.} \( A_1, A_2 \in S. \)

Hence
\[ 1 = \|f\|_{CM, \alpha} = \langle f, A_0 \rangle , \]
and this fact, together with (2) completes the proof of the proposition (2) \( \Box \)

**Lemma 1:** Let the closed linear subspace \( \mathcal{A} \) of \( CM, \{0\} \) be such that, for every \( x \in M \) and every open neighbourhood \( U \) of \( x \) in \( M \) there is \( g \in \mathcal{A} \setminus \{0\} \) with \( \text{Supp} \, g \subset U \). If \( f \in \mathcal{A} \) is a complex extreme point of \( B_{\mathcal{A}} \), then \( \|f(x)\|_\varepsilon = 1 \) for all \( x \in M \).

**Proof:** If \( \|f(x_0)\|_\varepsilon < 1 \) for some \( x_0 \in M \), there exist an open neighbourhood \( U \) of \( x_0 \) and some \( \varepsilon > 0 \) for which
\[ \|f(x)\|_\varepsilon < 1 - \varepsilon \quad \forall \, x \in U . \]
Let \( g \in \mathcal{A} \setminus \{0\} \) be such that \( \text{Supp} \, g \subset U \) and \( \|g\|_{CM, \alpha} \leq \varepsilon \). Given any \( \zeta \in \mathcal{A} = \{ r \in \mathbb{C} : |r| < 1 \} \),
\[ \|f(x) + \zeta g(x)\|_\varepsilon \leq \|f(x)\|_\varepsilon + \|\zeta\| \|g(x)\|_\varepsilon \]
\[ \leq \|f(x)\|_\varepsilon + \|g(x)\|_\varepsilon \]
\[ < 1 - \varepsilon + \varepsilon = 1 \]
if \( x \in U \), and
\[ \|f(x) + \zeta g(x)\|_\varepsilon = \|f(x)\|_\varepsilon \]
if \( x \in M \setminus U \). Thus,
\[ \|f + \zeta g\|_{CM, \alpha} \leq 1 \]
(2) The proof follows the ideas in [7], pp. 145-146.
for all $\zeta \in \mathcal{A}$, contradicting the hypothesis whereby $f$ is a complex extreme point of $\overline{B_{\mathcal{E}}}$.

Lemma 1 and the following lemma characterize all extreme points of $\overline{B_{\mathcal{E}}}$, where $\mathcal{E}$ is strictly convex.

**Lemma 2:** Let $\mathcal{E}$ be strictly convex. If, and only if, 
$$
\|f(x)\|_{\mathcal{E}} = 1 \quad \forall x \in M,
$$
$f \in C(M, \mathcal{E})$ is an extreme point of $\overline{B_{\mathcal{E}}}$.

**Proof:** Let $g \in C(M, \mathcal{E})$ and let $t \in (0, 1) \setminus \{0\}$ be such that 
$$
\|f + tg\|_{C(M, \mathcal{E})} \leq 1.
$$
Then
$$
\|f(x) + tg(x)\|_{\mathcal{E}} \leq 1 \quad \forall x \in M.
$$
Since $f(x) \in \partial B_{\mathcal{E}}$ is an extreme point of $\overline{B_{\mathcal{E}}}$, then $g(x) = 0$ for all $x \in M$.

Let
$$
\Theta(\mathcal{E}) = \{g \in \overline{B_{\mathcal{E}}}: g \text{ extreme point of } \overline{B_{\mathcal{E}}}\}.
$$
Lemma 1 and Lemma 2 yield

**Theorem 1:** If $\mathcal{E}$ is strictly convex and $\mathcal{A} \neq \{0\}$ is a closed linear subspace of $C(M, \mathcal{E})$ such that, for every $x \in M$ and every open neighbourhood of $x$ in $M$ there is $g \in \mathcal{A} \setminus \{0\}$ with $\text{Supp } g \subset U$, then 
$$
\Theta(\mathcal{E}) = \{g \in \mathcal{A}: \|g(x)\|_{\mathcal{E}} = 1 \quad \forall x \in M\}.
$$
In particular, if $\mathcal{E}$ is strictly convex, then

$$
\Theta(C(M, \mathcal{E}')) = \{f \in C(M, \mathcal{E}): \|f(x)\|_{\mathcal{E}} = 1 \quad \forall x \in M\}.
$$
We will now describe $\Theta(C(M, \mathcal{E}'))$.

Let 
$$
C := \{\delta_x \otimes \lambda: x \in M, \lambda \in \overline{B_{\mathcal{E}'}}\} \subset \overline{B_{C(M, \mathcal{E})}}.
$$

**Lemma 3:** The set $C$ is weak-star closed in $C(M, \mathcal{E}')$.

**Proof:** If $\Omega$ is contained in the weak-star closure of $C$, there is a generalized sequence $\{\delta_{x_j} \otimes \lambda_j\}$, with $x_j \in M$ and $\lambda_j \in \overline{B_{\mathcal{E}'}}$, converging to $\Omega$, i.e., such that

$$
\langle f, \Omega \rangle = \lim \langle f(x_j), \lambda_j \rangle \quad \forall f \in C(M, \mathcal{E}).
$$
Up to replacing this generalized sequence by a generalized subsequence, there is
no restriction in assuming that \( \{ x_i \} \) converges to a point \( x_0 \in M \), and that \( \{ \lambda_i \} \) converges to \( \lambda_0 \in B_E^\prime \) for the weak-star topology. Hence, (5) yields
\[
\langle f, \Omega \rangle = \langle f(x_0), \lambda_0 \rangle \quad \forall f \in C(M, \sigma)^
\]

**Lemma 4:** If \( \Omega \in C(M, \delta)' \) is an extreme point of \( \overline{B_{C(M, \delta')}} \), there exist \( x_0 \in M \) and \( \lambda_0 \) an extreme point of \( B_E^\prime \) such that \( \Omega = \delta_{x_0} \otimes \lambda_0 \).

**Proof:** The closure \( \overline{\text{co}}(C) \) of the convex hull \( \text{co}(C) \) of \( C \) coincides with the closed convex hull \( \overline{\text{co}}(C) \), which is closed in \( B_E^\prime \).

If \( \Omega \notin \overline{\text{co}}(C) \), there exist, ([6], p. 417), \( f \in C(M, \delta) \), \( c \in \mathbb{R} \) and \( \epsilon > 0 \) such that
\[
\Re(f, \Omega) \geq c
\]
and
\[
\Re(f, A) \leq c - \epsilon \quad \forall A \in C ,
\]
**i.e.,**
\[
\Re(f(x), \lambda) \leq c - \epsilon \quad \forall x \in M, \lambda \in B_E^\prime.
\]
Since
\[
\|f(x)\|_\infty = \sup \{ \|f(x), \lambda\| : \lambda \in B_E^\prime \},
\]
then
\[
\|f(x)\|_\infty \leq c - \epsilon \quad \forall x \in M ,
\]
and therefore
\[
\|f\|_{C(M, \delta)} \leq c - \epsilon .
\]
If \( \|\Omega\| \leq 1 \), then
\[
\epsilon \leq \Re(f, \Omega) \leq \|f, \Omega\| \leq \|f\|_{C(M, \delta)} \|\Omega\| \leq \|f\|_{C(M, \delta)} \leq c - \epsilon .
\]
This contradiction shows that
\[
\Omega \notin \overline{\text{co}}(C) \Rightarrow \Omega \notin \overline{B_{C(M, \delta')}} ,
\]
**i.e.,**
\[
\overline{B_{C(M, \delta')}} \subset \overline{\text{co}}(C) \subset \overline{B_{C(M, \delta')}} ,
\]
and therefore
\[
\overline{\text{co}}(C) = \overline{B_{C(M, \delta')}} .
\]
Since the extreme points of $\mathfrak{C}(C)$ are contained in $C$ (see, e.g., [6], pp. 440-441), there are $x_0 \in M$ and $\lambda_0 \in \overline{B}_C$ such that $\Omega = \delta_{x_0} \otimes \lambda_0$.

If $\lambda_0$ is not an extreme point of $\overline{B}_C$, there are $\lambda_1, \lambda_2 \in \overline{B}_C$ and $t \in (0, 1)$ such that

$$\Omega = \delta_{x_0} \otimes \lambda_0 = t\delta_{x_0} \otimes \lambda_1 + (1-t)\delta_{x_0} \otimes \lambda_2.$$

In conclusion, the following theorem holds.

**Theorem 2:** A linear form $A \in C(M, \mathfrak{C})'$ is an extreme point of $\overline{B}_{C(M, \mathfrak{C})}$ if, and only if, there exist $x \in M$ and an extreme point $\lambda$ of $\overline{B}_C$ such that $A = \delta_x \otimes \lambda$.

2. Let $M$ and $N$ be compact Hausdorff spaces and let $\mathfrak{E}$ and $\mathfrak{F}$ be complex Banach spaces, with $\mathfrak{F}$ strictly convex. In [4], M. Cambern has characterized all linear isometries of $C(M, \mathfrak{E})$ into $C(N, \mathfrak{F})$, proving the following theorem, which extends previous results established by W. Holsztyński in [9] for the case $\mathfrak{E} = \mathfrak{F} = C$.

**Theorem 3:** Let $A \in \mathfrak{L}(C(M, \mathfrak{E}), C(N, \mathfrak{F}))$ be a linear isometry. If $\mathfrak{F}$ is strictly convex, there exist:
- a set $K(A) \subset N$;
- a continuous, surjective map $\psi : K(A) \rightarrow M$;
- a map $\gamma : N \ni y \mapsto C_y \in \mathfrak{L}(\mathfrak{E}, \mathfrak{F})$, which is continuous for the strong operator topology in $\mathfrak{L}(\mathfrak{E}, \mathfrak{F})$, such that

$$\gamma((Af)(y)) = C_y(f \circ \psi(y))$$

for all $y \in K(A)$ and all $f \in C(M, \mathfrak{E})$.

The set $K(A)$ and the map $\psi$ are described as follows. For $x \in M$, $\xi \in \partial B(M, \mathfrak{E})$, let

$$F(\xi, x) = \{ f \in C(M, \mathfrak{E}) : f(x) = \| f \|_{C(M, \mathfrak{E})} \xi \},$$

$$K_A(\xi, x) = \{ y \in N : \| (Af)(y) \|_\mathfrak{E} = \| f \|_{C(M, \mathfrak{E})} \forall f \in F(\xi, x) \},$$

$$K_A(x) = \bigcup \{ K(\xi, x) : \xi \in \partial B(M, \mathfrak{E}) \},$$

$$K(A) = \bigcup \{ K_A(x) : x \in M \}.$$

In [4], Cambern shows that $K_A(\xi, x) \neq \emptyset$ for all $x \in M$, and

$$x_1 \neq x_2 \Rightarrow K_A(x_1) \cap K_A(x_2) = \emptyset.$$

Hence, for every $y \in K(A)$ there is a unique $x \in M$ such that $y \in K_A(x)$. The map $\psi : K(A) \rightarrow M$ is defined by setting $x = \psi(y)$.
Any $\xi \in \mathcal{E}$ defines a function $\xi \in C(M, \mathcal{E})$ as follows:

$$\xi(x) = \xi \quad \forall x \in M.$$ For $y \in N$, the operator $C_y \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is given by

$$C_y(\xi) = A(\xi).$$

Since, for any $y \in N$,

$$\|C_y\xi\|_\sigma = \|(A(\xi))(y)\|_\sigma \leq \|A\| \|\xi\|_{C(M, \mathcal{E})}$$

$$= \|\xi\|_{C(M, \mathcal{E})} = \|\xi\|_\sigma,$$

then

$$\|C_y\| \leq 1 \quad \forall y \in N.$$ Being $\xi \in F(\xi, x)$ for all $x \in M$, then

$$\|C_y\xi\|_\tau = \|\xi\|_\tau \quad \forall \xi \in \mathcal{E}, \quad \forall y \in K(A).$$

Since $y \mapsto C_y \xi$ is continuous for all $\xi \in \mathcal{E}$, that proves

**Lemma 5:** For any $y \in K(A)$, $C_y$ is a linear isometry of $\mathcal{E}$ into $\mathcal{F}$.

In [4] M. Cambern shows that, if $y \in K(A)(x)$, then

$$(Af)(y) = C_y(f(x)) \quad \forall f \in C(M, \mathcal{E}).$$

By the construction of $\psi$, that yields (6).

**Proposition 2:** If the map $C : y \mapsto C_y$ of $N$ into $\mathcal{L}(\mathcal{E}, \mathcal{F})$ is continuous for the uniform operator topology of $\mathcal{L}(\mathcal{E}, \mathcal{F})$, the set $K(A)$ is closed.

**Proof:** Let $y_0 \in K(A)$.

For any $f \in B_{C(M, \mathcal{E})}$ and for $n = 1, 2, \ldots$ there is some $y_n \in K(A)$ such that

$$\|(Af)(y_0) - (Af)(y_n)\|_\sigma < \frac{1}{n},$$

i.e.,

$$\|(Af)(y_0) - C_{y_n}(f(\psi(y_n)))\|_\sigma < \frac{1}{n},$$

and moreover

$$\|C_{y_0} - C_{y_n}\| < \frac{1}{n}.$$
Suppose that the set \( \{ \psi(y_n) \} \) is infinite. Because \( M \) is compact, the set \( \{ \psi(y_n) \} \) has at least one cluster point \( x_0 \). For any \( \varepsilon > 0 \) there is an open neighbourhood \( U \) of \( x_0 \) in \( M \) such that

\[
\| f(x) - f(x_0) \|_\varepsilon < \varepsilon \quad \forall x \in U.
\]

Let \( n_0 > 0 \) be so large that \( \frac{1}{n_0} < \varepsilon \), and let \( n > n_0 \) be such that \( x_n \in U \). Then

\[
\|(Af)(y_0) - C_{y_0}(f(x_n))\|_\varepsilon \leq \|(Af)(y_0) - C_{y_0}(f(x_0))\|_\varepsilon +
\]

\[
+ \|(C_{y_0} - C_{y_0})(f(x_n))\|_\sigma +
\]

\[
+ \|C_{y_0}(f(x_n) - f(x_0))\|_\sigma
\]

\[
\leq \|(Af)(y_0) - C_{y_0}(f(x_0))\|_\varepsilon +
\]

\[
+ \|C_{y_0} - C_{y_0}\|_\sigma \|f(x_n)\|_\sigma +
\]

\[
+ \|C_{y_0}\|_\sigma \|f(x_n) - f(x_0)\|_\sigma
\]

\[
< \frac{1}{n} + \frac{1}{n} + \varepsilon < 3 \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, that shows that

\[(Af)(y_0) = C_{y_0}(f(x_0)).\]

Obviously, the same conclusion holds when the set \( \{ \psi(y_n) \} \) is finite; in which case \( x_0 \in \{ \psi(y_n) \} \) can be chosen such that \( \psi(y_n) = x_0 \) for \( n_1 < n_2 < \ldots \).

Let now \( u_0 \) be another cluster point of the set \( \{ \psi(y_n) \} \) when this latter set is infinite, or such that \( \psi(y_m) = u_0 \) for \( m_1 < m_2 < \ldots \). By the same argument as before, one shows that

\[(Af)(y_0) = C_{y_0}(f(u_0)).\]

Hence,

\[C_{y_0}(f(x_0) - f(u_0)) = 0,
\]

and therefore

\[f(x_0) = f(u_0) \quad \forall f \in C(M, \mathcal{E})
\]

because \( C_{y_0} \) is injective. If \( x_0 \neq u_0 \), given any two vectors \( \xi_1 \) and \( \xi_2 \) in \( \mathcal{E} \), there is a function \( f \in C(M, \mathcal{E}) \) such that

\[f(x_0) = \xi_1, \quad f(u_0) = \xi_2.
\]

Thus \( x_0 = u_0 \), and \( y_0 \in \psi_A(x_0). \)
In view of the definition of $C_{\epsilon}$, the hypothesis of Proposition 2 can be rephrased by requiring that the restriction of $A$ to the closed subspace of $C(M, \mathcal{E})$ consisting of all $\mathcal{E}$-valued constant functions on $M$ be continuous for the uniform operator topology.

**Corollary 1:** [4] If $\dim \mathcal{E} < \infty$, $K(A)$ is closed in $N$.

**Lemma 6:** Let $\mathcal{F}$ be strictly convex and $\mathcal{E}$ reflexive. If $y \in N$ and there is $\mu \in \partial B_{\mathcal{F}}$ such that
\[
A'(\delta_\lambda \otimes \mu) = \delta_\lambda \otimes \lambda
\]
for some $x \in M$ and $\lambda \in \partial B_{\mathcal{F}}$, then $y \in K(A)$.

**Proof:** Since $\mathcal{E}$ is reflexive, there exists $\xi \in \mathcal{E}$ such that $\langle \xi, \lambda \rangle = 1$. If $f \in C(M, \mathcal{E})$ is such that $f(x) = \|f\|_{C(M, \mathcal{E})} \xi$, then
\[
\langle (Af)(y), \mu \rangle = \langle Af, \delta_\lambda \otimes \mu \rangle = \langle f, A'(\delta_\lambda \otimes \mu) \rangle
= \langle f, \delta_\lambda \otimes \lambda \rangle = \langle f(x), \lambda \rangle
= \|f\|_{C(M, \mathcal{E})} \langle \xi, \lambda \rangle = \|f\|_{C(M, \mathcal{E})}.
\]

Since
\[
\|f\|_{C(M, \mathcal{E})} = \langle (Af)(y), \mu \rangle \leq \|Af\|_{\mathcal{E}} \|\mu\|_{\mathcal{F}}
= \langle (Af)(y) \rangle_{\mathcal{F}} \leq \|Af\|_{C(M, \mathcal{E})} = \|f\|_{C(M, \mathcal{E})},
\]
then
\[
\|Af\|_{\mathcal{F}} = \|f\|_{C(M, \mathcal{E})},
\]
and therefore $f \in K(A)$. 

On the other hand, if $y \in K(A)$, for any $\mu \in \partial B_{\mathcal{F}}$ and all $f \in C(M, \mathcal{E})$
\[
\langle f, A'(\delta_\lambda \otimes \mu) \rangle = \langle Af, \delta_\lambda \otimes \mu \rangle
= \langle (Af)(y), \mu \rangle = \langle C_\lambda(f(\psi(y))), \mu \rangle
= \langle f(\psi(y)), C_\lambda'(\mu) \rangle = \langle f, \delta_{\psi(y)} \otimes C_\lambda'(\mu) \rangle.
\]

In conclusion, in view of Theorem 2, the following theorem holds

**Theorem 4:** If $\mathcal{F}$ is strictly convex, and $\mathcal{E}$ is uniformly convex, then $K(A) = N$ if, and only if,
\[
A'(\Theta(C(N, \mathcal{F}'))) \subseteq \Theta(C(M, \mathcal{E}')).
\]
3. Let $M$ be, as before, a compact Hausdorff space, let $\mathcal{E}$ be a strictly convex complex Banach space, and let $T : R_+ \to \mathcal{E}(C(M, \mathcal{E}))$ be a semigroup of linear isometries $T(t) : C(M, \mathcal{E}) \to C(M, \mathcal{E})$.

According to Theorem 3, for every $t \geq 0$ there exist:

a subset $K(T(t))$ of $M$;

a continuous surjective map $\phi_t : K(T(t)) \to M$;

a map $x \mapsto C_{t, x}$ of $M$ into $\mathcal{E}(\mathcal{E})$, continuous for the strong operator topology in $\mathcal{E}(\mathcal{E})$, such that

\begin{equation}
(T(t) f)(x) = C_{t, x}(f(\phi_t(x))) \quad \forall f \in C(M, \mathcal{E}), \forall x \in K(T(t)).
\end{equation}

If $t = 0$, then $K(I) = M$, $\phi_0 = I$ and $C_{0, x} = I$ for all $x \in M$.

If $t > 0$, for all $x \in M \|C_{t, x}\| \leq 1$, and, if $x \in K(T(t))$, $C_{t, x}$ is a linear isometry of $\mathcal{E}$.

**Lemma 7:** Let $t, s \geq 0$ and $x \in M$. If $x \in K(T(t))$ and $\phi_t(x) \in K(T(s))$, then $x \in K(t + s)$. If $x \in K(T(t)) \cap K(T(t + s))$, then $\phi_t(x) \in K(T(s))$.

**Proof:** If $\phi_t(x) \in K(T(s))$, then $x \in K(T(t)) \cap K(T(t + s))$ and, for all $f \in C(M, \mathcal{E})$,

\begin{equation}
(T(t + s) f)(x) = (T(t) \circ T(s) f)(x) = C_{t, x}((T(s) f)(\phi_t(x))) = C_{t, x} \circ C_{t, \phi_t(x)}(f(\phi_t \circ \phi_t(x)))
\end{equation}

\begin{equation}
= C_{t, x} \circ C_{t, \phi_t(x)}(f(z)) = C_{t, x} \circ C_{t, \phi_t(x)}(f(z)),
\end{equation}

where $z = (\phi_t \circ \phi_t)(x)$. If $f(z) = \|f\|_{C(M, \mathcal{E})} \xi$, with $\|\xi\|_{\mathcal{E}} = 1$, then

\begin{equation}
\left\|T(t + s) f(x)\right\|_{\mathcal{E}} = \|f(z)\|_{\mathcal{E}} = \|f\|_{C(M, \mathcal{E})} = \left\|T(t + s) f\right\|_{C(M, \mathcal{E})}.
\end{equation}

Therefore $x \in K(T(t + s))$ and

\begin{equation}
T(t + s) f(x) = C_{t + s, x}(f(\phi_{t + s}(x))).
\end{equation}

Choosing $f = \xi$, for any $\xi \in \mathcal{E}$, (8) and (9) yield

\begin{equation}
C_{t + s, x}(\xi) = T(t + s) \xi(x) = C_{t, x} \circ C_{t, \phi_t(x)}(\xi),
\end{equation}

whence

\begin{equation}
C_{t + s, x} = C_{t, x} \circ C_{t, \phi_t(x)} \quad \forall t, s \in R_+,
\end{equation}

and therefore

\begin{equation}
f(\phi_{t + s}(x)) = f(\phi_t \circ \phi_t(x)) \quad \forall f \in C(M, \mathcal{E}).
\end{equation}
If $x \in K(T(t)) \cap K(T(t + s))$, then
\[ C_{t+s}(f(x)) = (T(t + s) f)(x) = (T(t) \circ T(s) f)(x) = C_t(f(x)). \]
Letting $z = \phi_{t+s}(x)$, if $f(z) = \|f\|_{C_{t+s}(D)} \xi$, with $\|\xi\|_\infty = 1$, then
\[ \|f(T(t) f)(\phi_{t+s}(x))\|_\infty = \|C_{t+s}(f(x))\|_\infty = \|f(T(t + s) f)(x)\|_\infty \]
and therefore $\phi_{t+s}(x) \in K(T(s))$.  

**Corollary 2:** If $t, s \geq 0$,
\[ K(T(t)) \cap K(T(t + s)) = \phi_t^{-1}(K(T(s))), \]
and $\phi_{t+s} = \phi_t \circ \phi_s$ on $\phi_t^{-1}(K(T(s)))$.

In general, the family $\{K(T(t)) : t > 0\}$ is not increasing, as the following lemma shows.

**Lemma 8:** If
\[ (11) \]
for some $t \geq 0$ and some $s > 0$, then $K(T(t)) = M$ for all $r \geq 0$.

**Proof:** If (11) holds for some $t \geq 0$ and some $s > 0$, then
\[ K(T(t)) = K(T(t)) \cap K(T(t + s)) = \phi_t^{-1}(K(T(s))), \]
and therefore
\[ M = \phi_t(K(T(t))) = K(T(s)). \]
Hence, if $0 < l < s$ and $r = s - l$, then
\[ K(T(r)) = K(T(r)) \cap K(T(s)) = K(T(r)) \cap K(T(r + l)) = \phi_{s}^{-1}(K(T(l))), \]
and therefore
\[ M = \phi_{s}(K(T(r))) = K(T(l)), \]
showing that, if $K(T(s)) = M$ for some $s > 0$, then $K(T(r)) = M$ for all $r \in [0, s]$.

Let
\[ s_0 = \sup \{ s \geq 0 : K(T(s)) = M \}. \]
If $0 < s_0 < \infty$, there are $t, s$, with $0 < t < s_0$ and $0 < s < s_0$, such that $t + s > s_0$. 

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Then $K(T(t)) = M = K(T(s))$, and therefore
$$K(T(t + s)) = K(T(t)) \cap K(T(t + s)) = \phi_t^{-1}(K(T(s)))$$
$$= \phi_t^{-1}(M) = K(T(t)) = M.$$  

This contradiction shows that either $s_0 = 0$ or $s_0 = +\infty$, and completes the proof of the lemma.  

If (11) holds for some $t \geq 0$ and some $s > 0$, (7) holds for all $t \geq 0, f \in C(M), x \in M$.

Let $n > 1$ and let $t_j > 0$ for $j = 1, 2, \ldots, n$. Then

(12) 
$$K(T(t_1)) \cap K(T(t_1 + t_2)) \cap \ldots \cap K(T(t_1 + t_2 + \ldots + t_n)) =$$
$$= (K(T(t_1)) \cap K(T(t_1 + t_2))) \cap (K(T(t_1 + t_2))) \cap \ldots \cap$$
$$\cap (K(T(t_1 + t_2 + \ldots + t_n))) = \phi_1^{-1}(K(T(t_2))) \cap \ldots \cap$$
$$\cap \phi_{t_1}^{-1}(K(T(t_2))) \cap \ldots \cap \phi_{t_1}^{-1}(K(T(t_2 + \ldots + t_n))) =$$
$$= \phi_{t_1}^{-1}(K(T(t_2))) \cap \ldots \cap K(T(t_2 + \ldots + t_n)) =$$
$$= \phi_{t_1}^{-1} \circ \phi_{t_2}^{-1} \circ \ldots \circ \phi_{t_{n-1}}^{-1}(K(T(t_n))) = \ldots =$$
$$= \phi_{t_1}^{-1} \circ \phi_{t_2}^{-1} \circ \ldots \circ \phi_{t_{n-1}}^{-1}(K(T(t_n))) = \ldots =$$

**Lemma 9:** The set
$$\bigcap \{K(T(t)) : t \geq 0\}$$
is compact and non-empty.

**Proof:** By the chain of equalities above, the family $\{K(T(t)) : t \geq 0\}$ of closed sub-
sets of the compact space $M$ has the finite intersection property.  

**Corollary 3:** If $K(T(t))$ is closed for all $t \in R_+$, the set

(13) 
$$K_\infty(T) = \bigcap \{K(T(t)) : t \geq 0\}$$
is compact and non-empty.

The fact that the set $K_\infty(T)$ is non-empty follows from weaker conditions.

**Theorem 5:** If there is some $s > 0$ such that $K(T(t))$ is closed whenever $0 \leq t \leq s$, 
the set $K_\infty(T)$ defined by (13) is non-empty.
PROOF: Consider the set (12), where \( t_p > 0 \) for \( p = 1, 2, \ldots, n \). Letting \( t_p = q_p s + r_p \), with \( q_p \in \mathbb{Z}_+ \) and \( 0 \leq r_p < s \) for \( p = 1, 2, \ldots, n \), the set (12) contains the set

\[
G(t_1, \ldots, t_n) := K(T(t_1)) \bigcap \left( \bigcap_{p=1}^{n} \left( K(T(t_1 + \cdots + t_{p-1} + j_s)) \right) \right),
\]

which — as was noticed before — is not empty. Since \( K(T(t_1 + \cdots + t_{p-1} + q_p s)) \bigcap K(T(t_1 + \cdots + t_{p-1} + q_p s + r_p)) = \phi_{t_1 + \cdots + t_{p-1} + q_p s}^{-1}(K(T)) \), the set \( G(t_1, \ldots, t_n) \) is closed. By the finite intersection property, the intersection of all sets \( G(t_1, \ldots, t_n) \) is not empty. Hence \( K_\infty(T) \) is not empty. \( \blacksquare \)

As a consequence of Proposition 2, the following lemma holds.

**Lemma 10:** If there is some \( t_0 > 0 \) such that the map \( x \mapsto C_{t, s} \) of \( M \) into \( \mathcal{E}(\mathcal{E}) \) is continuous for the uniform operator topology whenever \( t \in [0, t_0] \), then \( K_\infty(T) \neq \emptyset \). If the hypothesis holds for all \( t \in \mathbb{R}_+ \), \( K_\infty(T) \) is also closed.

Corollary 4 yields

**Corollary 4:** If \( \dim_{\mathcal{E}} C_{t, s} \) \( \leq \infty \), \( K_\infty(T) \) is closed and non-empty.

Let \( K_\infty(T) \) be non-empty.

Since \( K(T(s)) = \phi_s^{-1}(M) \), for all \( s \geq 0 \)

\[
\phi_s^{-1}(K_\infty(T)) = \phi_s^{-1}(\bigcap \{ K(T(s)) : s \geq 0 \}) = \bigcap \{ \phi_s^{-1}(K(T(s))) : s \geq 0 \}
\]

\[
= \bigcap \{ K(T(t + s)) : s \geq 0 \} = \bigcap \{ K(T(s)) : s \geq t \}
\]

\[
\supset \bigcap \{ K(T(s)) : s \geq 0 \} = K_\infty(T),
\]
and therefore

\[ \phi_t(K_\infty(T)) \subset K_\infty(T) \quad \forall t \geq 0. \]  

Remark: The set \( K_\infty(T) \) — if non-empty — is the largest subset of \( M \) which is \( \phi_t \)-invariant for all \( t \geq 0 \). Let \( x \in M \). Then \( x \in \phi_t^{-1}(K_\infty(T)) \setminus K_\infty(T) \) for some \( t > 0 \) if, and only if,

\[ x \in K(T(t)) \cap K(T(t+s)) \quad \forall s \geq 0, \]

i.e.,

\[ x \in K(T(s)) \quad \forall s \geq t, \]

and moreover

\[ x \notin K(T(r)) \quad \text{for some } r \in (0, t). \]

Hence

\[ \phi_t^{-1}(K_\infty(T)) \setminus K_\infty(T) \subset \bigcap \{ K(T(s)) : s \geq t \} \setminus K(T(r)) \]

for some \( r \in (0, t) \).

If

\[ K(T(t)) \subset K_\infty(T) \]

for some \( t > 0 \), then \( K(T(s)) \supset K(T(t)) \) for all \( s > 0 \), and Lemma 8 yields

**Theorem 6:** If, and only if, (16) holds for some \( t > 0 \), then \( K_\infty(T) = M \), and (7) holds for all \( t \geq 0 \).

Let \( K_\infty(T) \) be closed and non-empty. In view of the \( \phi_t \)-invariance of \( K_\infty(T) \), one defines a semigroup \( \bar{T} : \mathbb{R}_+ \to \mathbb{L}(C(K_\infty(T), \mathcal{E})) \) of linear contractions of \( C(K_\infty(T), \mathcal{E}) \), by

\[ (\bar{T}(t)g)(x) = C_{t,s}(g(\phi_x(t, x))) \]

for all \( t \geq 0, g \in C(K_\infty(T), \mathcal{E}), x \in C(K_\infty(T)). \)

4. Let \( M, N, P \) be compact Hausdorff spaces, \( \mathcal{E}, \mathcal{F}, \mathcal{G} \) be complex Banach spaces, with \( \mathcal{F}, \mathcal{G} \) strictly convex, and let

\[ A \in \mathbb{L}(C(M, \mathcal{E}), C(N, \mathcal{F})), \quad B \in \mathbb{L}(C(N, \mathcal{F}), C(P, \mathcal{G})) \]

be linear isometries. Then \( B \circ A \) is a linear isometry of \( C(M, \mathcal{E}) \) into \( C(P, \mathcal{G}) \).

Arguing as in the proof of Lemma 7, one shows that

\[ K(B) \cap K(B \circ A) = \psi^{-1}(K(A)) \]
and

\[ \psi_B \circ \psi_A = \psi_B \circ \psi_A \] on \( \psi_B^{-1}(K(A)) \).

If \( M = P \) and \( \delta = \gamma \), and if \( B \circ A \) is the identity on \( M \), then \( K(B \circ A) = P \), and (17) becomes

\[ \psi_B(K(B)) = K(A) \]

whence \( K(A) = N \). That implies M. Jerison's extension, [10], of the classical Banach-Stone theorem to vector-valued, continuous functions.

Let now \( M = N \) and \( \delta = \delta' \). By similar arguments to those developed in n. 3, one can handle the discrete case, in which the semigroup \( T \) is replaced by the iterates \( \{ A^n \colon n \in \mathbb{N} \} \) of an isometry \( A \in \mathcal{E}(C(M, \delta)) \), and the Banach space \( \delta \) is strictly convex. Assuming in Theorem 3 \( N = M \), \( \delta = \delta' \), and replacing \( A \) by \( A^n \), \( K(A) \) by \( K(A^n) \), \( C \) by \( C_{A^n} \), \( \psi \) by \( \psi_{A^n} \), one shows, as in n. 3, that

\[ K(A^n) \cap K(A^{n+\delta}) = \psi_{A^n}^{-\delta}(K(A^n)) \]

Let \( n_1, n_2, \ldots, n_p \) be positive integers. As in n. 3 one proves that

\[ K(A^{n_1}) \cap K(A^{n_1+\delta_2}) \cap \cdots \cap K(A^{n_1+\cdots+\delta_p}) = \psi_{A_1}^{-1} \circ \cdots \circ \psi_{A_p}^{-1}(K(A^{n_1})) \neq \emptyset \]

and this shows that

\[ \bigcap \{ K(A^n) \colon n \in \mathbb{Z}_+ \} \neq \emptyset \]

Since the left-hand side of (18) contains the set

\[ \bigcap_{n=1}^{n_1+\cdots+n_p} K(A^n) = \psi_{A_1}^{-1} \circ \psi_{A_2}^{-1} \circ \cdots \circ \psi_{A_p}^{-1}(K(A)) \]

which is (non-empty and) closed when \( K(A) \) is closed, the following proposition holds.

**Proposition 3:** If \( K(A) \) is closed, the set

\[ K_\infty(A) := \bigcap \{ K(A^n) \colon n \in \mathbb{Z}_+ \} \]

is non-empty.

Similar arguments as those developed in the proof of Lemma 8 lead to
Lemma 11: If

\[ K(A^p) \subset K(A^{p+q}) \]

for two positive integers \( p \) and \( q \), then \( K(A) = M \).

Arguing as in Theorem 6 one proves

Theorem 7: If, and only if,

\[ K(A^p) \subset K_\infty(A) \]

for some \( p \geq 0 \), then \( K(A) = M \).

If \( \bar{A} \in \mathscr{L}(C(K_\infty(A), \mathcal{E})) \) is defined by

\[ (\bar{A} \sigma)(x) = C_{A, \sigma}(\sigma(x)) \]

for all \( x \in K_\infty(A) \) and all \( \sigma \in C(K_\infty(A), \mathcal{E}) \), then \( \bar{A} \) is a contraction of \( C(K_\infty(A), \mathcal{E}) \).

If \( A_{\xi} = \xi_{\xi} \) for some \( \xi \in \mathcal{C} \) and \( \xi \in \mathcal{E} \setminus \{0\} \), then \( |\xi| = 1 \) and \( A_{\xi} = \xi_{\xi} \), i.e.,

\[ C_{A, \sigma}(\xi) = \xi_{\xi} \quad \forall x \in K_\infty(A), \]

and vice versa. That proves

Lemma 12: Let \( K_\infty(A) \neq \emptyset \). If, and only if, \( \xi \) is an eigenvalue of \( C_{A, \sigma} \) with an eigenvector \( \xi \in \mathcal{E} \setminus \{0\} \) for all \( x \in K_\infty(A) \), then \( |\xi| = 1 \) and \( \xi \) is an eigenvalue of \( \bar{A} \) with an eigenvector \( \xi_{\xi} \).

Let now

\[ (Af)(y) = \xi f(y) \quad \forall f \in C(M, \mathcal{E}) \]

and for some \( y \in M \) and \( \xi \in \mathcal{C} \). Then \( |\xi| \leq 1 \). If \( f \in C(\mathcal{E}, y) \) for some \( \xi \in \mathcal{E} \) with \( \|\xi\|_{\mathcal{E}} = 1 \), then

\[ \|(Af)(y)\|_{\mathcal{E}} = |\xi| \|f\|_{C(M, \mathcal{E})} = |\xi| \|Af\|_{C(M, \mathcal{E})}. \]

Thus

\[ \xi \in \partial \mathcal{A} \Rightarrow y \in K(A), \]

and therefore

\[ C_{A, \sigma}(f(\psi_A(y))) = (Af)(y) = \xi f(y) \quad \forall f \in C(M, \mathcal{E}). \]
Because $C_{\gamma}$ is an isometry, that implies that

$$\|f(\psi_{\gamma}(y))\| = \|f(y)\|$$

for all $f \in C(M, \mathcal{B})$, and therefore $\psi_{\gamma}(y) = y$, proving thereby

**Proposition 4:** If $y \in M$ and $\zeta \in \mathcal{A}$ satisfy (19), then $y \in K(A)$, $\psi_{\gamma}(y) = y$ and $C_{\gamma, \gamma} = \zeta I$.

We shall conclude this section with a result on the compression spectrum of $A$ in the case in which $M = N$, $\tilde{\sigma} = \sigma = \sigma$ and $A$ is a linear isometry of $C(M)$ onto $C(N)$. Now $K(A) = M$, and $A$ is expressed by (1) for all $y \in M$ and all $f \in C(M)$, with $a \in \Theta(C(M))$ and $\psi$ a homeomorphism of $M$ onto itself.

The compression spectrum of $A$ is, by definition, the point spectrum $p\sigma(A')$ of the dual operator $A'$ of $A$. If $\zeta \in p\sigma(A')$, there is some $\lambda \in C(M) \setminus \{0\}$ such that

$$\langle Af, \lambda \rangle = \zeta(f, \lambda) \quad \forall f \in C(M),$$

i.e.,

$$\int a(x) f(\psi(x)) \, d\lambda(x) = \zeta \int f(x) \, d\lambda(x)$$

for all $f \in C(M)$, where $\lambda$ has been identified with its representative Borel measure.

This implies, first of all, that $\zeta \neq 0$.

Let $x_0 \in \text{Supp} \lambda$ be such that $\psi(x_0) \notin \text{Supp} \lambda$. Let $U$ be an open neighbourhood of $x_0$ in $M$, disjoint from $\text{Supp} \lambda$, and let $V = \psi^{-1}(U)$.

For any $f \in C(M)$ such that $\text{Supp} f \subset U$,

$$\int f(x) \, d\lambda(x) = 0,$$

and therefore

$$\int a(x) f(\psi(x)) \, d\lambda(x) = 0. \quad (21)$$

If $g \in C(M)$ is such that $\text{Supp} g \subset V$, then, setting $f = g \circ \psi^{-1}$, $\text{Supp} f \subset U$, and (21) yields

$$\int a(x) g(x) \, d\lambda(x) = 0,$$

showing that $x_0 \notin \text{Supp} \lambda$: which is a contradiction.

Hence, $\psi(\text{Supp} \lambda) \subset \text{Supp} \lambda$, and therefore $\psi(\text{Supp} \lambda) = \text{Supp} \lambda$ because $\psi$ is a homeomorphism. That proves

**Theorem 8:** If $A \in \mathcal{L}(C(M))$ is a bijective isometry and if $\zeta \in p\sigma(A')$, then
$\zeta \not= 0$. Furthermore, the support of any $\lambda \in C(M) \setminus \{0\}$ satisfying (20), is $\psi$-invariant.

As a consequence, if $\text{Supp} \lambda = \{x_0\}$, then $x_0$ is fixed by $\psi$. In that case, $\zeta = f(x_0)$.

5. – Applying some of the results of n. 4 to $T(t)$, for any $t > 0$, we see that, if $K(T(t))$ is closed, the set

$$K_0(T(t)) := \bigcap \{K(T(nt)) : n \in \mathbb{N}\}$$

is non-empty and $\bar{T}(t)$ is a contraction of $C(K_0(T(t)), \delta)$.

**Lemma 13:** If $(T(t)f)(x) = \zeta f(x)$ for some $\tau > 0$, $x \in M$ and $\zeta \in \partial A$, and for all $f \in C(M, \delta)$, then $x \in K(T(\tau))$, $\phi_j(x) = x$ and $C_{\tau, x} = \zeta I$.

**Corollary 5:** Let $K(T(\tau))$ be closed. If $x \in K_0(T)$ and $\tau > 0$ are such that

$$(\bar{T}(t)g)(x) = g(x) \quad \forall g \in C(K_0(T), \delta)$$

and if, for every $t \in (0, \tau)$ there is some $k \in C(K_0(T), \delta)$ for which

$$(\bar{T}(t)k)(x) \not= k(x),$$

then $C_{\tau, x} = I$ and the semiflow $\phi$ is periodic with period $\tau$ at the point $x$.

So far, no hypothesis on the topological structure of the semigroups $T$ and $\bar{T}$ has been introduced.

Throughout this and the following sections, $K_0(T)$ will be assumed to be closed and non-empty.

For any $t \geq 0$ and any $x \in K_0(T)$,

$$(T(t)f)(x) = C_{\tau, x}(f(\phi_j(x))) = (\bar{T} f_{|K_0(T)})(x)$$

for all $f \in C(K_0(T), \delta)$.

Let the semigroup $\bar{T}$ be strongly continuous.

Since, for any $\xi \in \delta$,

$$C_{\tau, x}(\xi) = (\bar{T}(t)\xi)(x),$$

the map $(t, x) \mapsto C_{\tau, x}$ of $\mathbb{R}_+ \times K_0(T)$ into $\mathcal{E}(\delta)$ is continuous for the strong operator topology in $\mathcal{E}(\delta)$.

We will show now that $\phi : t \mapsto \phi_j$ is a continuous semiflow in $K_0(T)$, i.e.,

$(t, x) \mapsto \phi_j(x)$ is a continuous map of $\mathbb{R}_+ \times K_0(T)$ into $K_0(T)$.

If that is not the case, there exist $t_0 \geq 0$, $x_0 \in K_0(T)$ and an open neighbourhood $U$
of \( \phi_t(x_0) \) such that, for every \( \delta > 0 \) and for every open neighbourhood \( V \) of \( x_0 \) there are \( t \in \mathbb{R}_+ \cap (t_0 - \delta, t_0 + \delta) \) and \( x \in V \) for which \( \phi_t(x) \notin U \). In view of the compactness of \( K_u(T) \), there are generalised sequences \( \{t_j\} \) in \( \mathbb{R}_+ \) and \( \{x_j\} \) in \( K_u(T) \) converging to \( t_0 \) and to \( x_0 \), such that \( \phi_{t_j}(x_j) \notin U \) and that \( \{\phi_{t_j}(x_j)\} \) converges to some \( y_0 \in K_u(T) \setminus U \).

Hence, for any \( f \in C(K_u(T), \mathbb{R}) \),
\[
C_{t_j, y_0}(f(\phi_{t_j}(x_0))) = C_{t_j, y_0}(f(y_0)).
\]
The injectivity of \( C_{t_j, y_0} \) implies then that \( f(\phi_{t_j}(x_0)) = f(y_0) \) for all \( f \in C(K_u(T), \mathbb{R}) \), and therefore \( \phi_{t_j}(x_0) = y_0 \), contradicting (22) and proving thereby that the semiflow \( \phi \) is continuous.

If \( L : \mathbb{R}_+ \rightarrow \mathcal{L}(C(K_u(T), \mathbb{R})) \) is the semigroup defined by the continuous semiflow \( t \rightarrow \phi_t \) on \( K_u(T) \); i.e.
\[
L(t) g = g \circ \phi_t,
\]
for all \( t \geq 0 \) and all \( g \in C(K_u(T), \mathbb{R}) \), then
\[
(\overline{T}(t) g)(x) = C_{t, x}((L(t)g)(x)) \quad \forall t \geq 0, g \in C(K_u(T), \mathbb{R}), x \in K_u(T).
\]

The map \( \overline{T}(t) \) is a linear isometry if, and only if, \( \phi_t \) is surjective. It is easily seen, [18], that the set of all \( t > 0 \) for which \( \overline{T}(t) \) is an isometry is either \( \mathbb{R}_+^* \) or the empty set.

If the semigroup \( T \) is strongly continuous, Corollary 5 may yield more information on the global behaviour of \( \phi_t \) and \( C_{t, x} \). As an example, assume now that \( M \) is the unit circle: \( M = \mathbb{S} \). According to Proposition 3 of [19], if the continuous semiflow \( \phi \) has a periodic point with period \( \tau > 0 \), then \( \phi \) is periodic with period \( \tau \). Hence, the following theorem holds.

**Theorem 9:** Let the semigroup \( T \) be strongly continuous. If \( M \) is the unit circle and \( x \) and \( \tau \) satisfy the hypotheses of Corollary 5, then \( \phi \) is the restriction to \( \mathbb{R}_+ \) of a continuous periodic flow, and \( T \) is the restriction to \( \mathbb{R}_+ \) of a strongly continuous periodic group \( \mathbb{R} \times C(\mathbb{S}, \mathbb{R}) \rightarrow C(\mathbb{S}, \mathbb{R}) \) of surjective linear isometries of \( C(\mathbb{S}, \mathbb{R}) \).

For any \( t \in \mathbb{R} \) and \( g \in C(\mathbb{S}, \mathbb{R}) \), \( x \in \mathbb{S} \), \( T(t) g \) is expressed by
\[
(T(t) g)(x) = C_{t, x}(g(\phi_t(x))),
\]
where, \( C_{t, x} \) is invertible in \( \mathcal{L}(C(M, \mathbb{R})) \) for all \( t \in \mathbb{R} \), and, if \( t \leq 0 \), \( C_{t, x} \) is expressed by
\[
C_{t, x} = C_{-t, x}^{-1}.
\]

Going back to the general case of \( C(M, \mathbb{R}) \), since \( K_u(T) \) is closed and non-empty, the contraction semigroup \( \overline{T} \) acting on the Banach space \( C(K_u(T), \mathbb{R}) \) is strongly con-
continuous, its infinitesimal generator \( \tilde{X} : \mathcal{O}(\tilde{X}) \subset C(K_* (T), \mathcal{E}) \to C(K_* (T), \mathcal{E}) \) is \( m \)-dissipative.

If the semigroup \( T \) is strongly continuous — in which case its infinitesimal generator \( X : \mathcal{O}(X) \subset C(M, \mathcal{E}) \to C(M, \mathcal{E}) \) is conservative and \( m \)-dissipative, \[16\] — also \( \tilde{T} \) is strongly continuous.

The space \( \mathcal{A} \) consisting of the restrictions to \( K_* (T) \) of the elements of \( \mathcal{O}(X) \) is contained in \( \mathcal{O}(\tilde{X}) \). Hence, if \( Y \) is the linear operator with domain \( \mathcal{O}(Y) = \mathcal{A} \) defined on the restriction to \( K_* (T) \) of any \( f \in \mathcal{O}(X) \) by

\[
(Yf|_{K_* (T)})(x) = (Xf)(x) \quad \forall x \in K_* (T),
\]

then \( Y \subset \tilde{X} \).

Because \( T(t) \mathcal{O}(X) \subset \mathcal{O}(X) \), then

\[
\tilde{T}(t) \mathcal{O}(Y) \subset \mathcal{O}(Y).
\]

Since \( \mathcal{O}(X) \) is dense in \( C(M, \mathcal{E}) \), if the space \( C(M, \mathcal{E})|_{K_* (T)} \) of the restrictions to \( K_* (T) \) of all \( f \in C(M, \mathcal{E}) \) is dense in \( C(K_* (T), \mathcal{E}) \), then \( \mathcal{A} \) is dense in \( C(K_* (T), \mathcal{E}) \). Thus \( \mathcal{A} = \mathcal{O}(Y) \) is a core of \( \tilde{X} \), and the following lemma holds.

**Lemma 14**: If \( C(M, \mathcal{E})|_{K_* (T)} \) is dense in \( C(K_* (T), \mathcal{E}) \), the operator \( \tilde{X} \) is the closure of \( Y \).

If \( \tilde{T} \) is strongly continuous, also the semigroup \( L \) is strongly continuous. Denoting by \( D : \mathcal{O}(D) \subset C(K_* (T), \mathcal{E}) \to C(K_* (T), \mathcal{E}) \) the infinitesimal generator of \( L \), then, for any \( \xi \in \mathcal{E} \), \( \xi \in \mathcal{O}(D) \) and \( D\xi = 0 \).

The space \( C(K_* (T), \mathcal{E}) \) is a module over the ring \( C(K_* (T)) \) of all complex-valued continuous functions on \( K_* (T) \). The infinitesimal generator \( D_0 \) of the Markov lattice semigroup \( L_0 \) defined in \( C(K_* (T)) \) by the semiflow \( \phi \) is a derivation \( D_0 : \mathcal{O}(D_0) \subset C(K_* (T)) \to C(K_* (T)) \). If \( q \in \mathcal{O}(D_0) \) and \( f \in \mathcal{O}(D) \), then \( qf \in \mathcal{O}(D) \) and

\[
D(qf) = D_0 q \cdot f + q \cdot Df.
\]

Hence, if \( \xi \in \mathcal{E} \),

\[
D(q \cdot \xi) = D_0 q \cdot \xi.
\]

Since all non-trivial derivations in \( C(K_* (T)) \) are unbounded \[3\], and since \( D \) is closed, the following lemma holds.

**Lemma 15**: If \( \mathcal{O}(D) = C(K_* (T), \mathcal{E}) \), then \( D = 0 \).

\[3\] See \[12\], or also \[17\] for a direct proof.
For all \( t > 0 \) and all \( g \in C(K_\alpha(T), \mathcal{H}) \),
\[
\frac{1}{t} (\tilde{T}(t) g - g)(x) = \frac{1}{t} (C_{t, \alpha} - I)((L(t) g)(x))
+ \frac{1}{t} (\tilde{L}(t) - I)(g(x)) .
\]

Hence, if \( g \in \mathcal{O}(\tilde{X}) \cap \mathcal{O}(D) \), the limit
\[
\lim_{t \downarrow 0} \frac{1}{t} (C_{t, \alpha} - I)(g(x)) = \lim_{t \downarrow 0} \frac{1}{t} (C_{t, \alpha} - I)(g(x)) ,
\]
exists for all \( x \in K_\alpha(T) \), and
\[
(\tilde{X}g)(x) = \lim_{t \downarrow 0} \frac{1}{t} (C_{t, \alpha} - I)(g(x)) + (Dg)(x) .
\]

In particular, letting
\[
\mathcal{K} = \{ \xi \in \mathcal{E} : \xi \in \mathcal{O}(\tilde{X}) \} ,
\]
then
\[
(\tilde{X}\xi)(x) = \lim_{t \downarrow 0} \frac{1}{t} (\tilde{T}(t) \xi - \xi)(x)
= \lim_{t \downarrow 0} \frac{1}{t} (C_{t, \alpha} - I)(\xi)
\]
for all \( \xi \in \mathcal{K} \) and all \( x \in K_\alpha(T) \).

Since \( \tilde{X} \) is closed and also the image \( \mathcal{K} \) of \( \mathcal{K} \) in \( C(K_\alpha(T), \mathcal{H}) \) by the map \( \xi \mapsto \tilde{\xi} \) is a closed subspace of \( \mathcal{O}(\tilde{X}) \), the operator \( \tilde{X}|_{\mathcal{K}} \) is closed. As a consequence:

**Lemma 16:** If \( \tilde{T} \) is strongly continuous, for every \( x \in K_\alpha(T) \) the linear operator \( Z_\alpha : \mathcal{O}(Z_\alpha) = \mathcal{K} \subset \mathcal{E} \rightarrow \mathcal{E} \)
defined by
\[
Z_\alpha \xi = (\tilde{X}\xi)(x)
\]
is closed\(^4\).

\(^4\) Here is a direct proof. Let \( \xi \in \mathcal{O}(Z_\alpha) \) and let \( \{ \xi_n \} \) be a sequence in \( \mathcal{O}(Z_\alpha) \), converging to \( \xi \) and such that \( \{ Z_\alpha \xi_n \} \) converges to some \( \eta \in \mathcal{E} \). Since the sequences \( \{ \xi_n \} \) and \( \{ Z_\alpha \xi_n \} = \{ \tilde{X}\xi_n \} \) in \( C(M, \mathcal{E}) \) converge respectively to \( \xi \) and to \( \eta \), then \( \tilde{\xi} \in \mathcal{O}(\tilde{X}) \) and \( \tilde{\eta} = \tilde{X}\tilde{\xi} \), i.e., \( \tilde{\xi} \in \mathcal{O}(Z_\alpha) \) and \( \eta = Z_\alpha \xi \).
Let $g \in \mathcal{O}(\mathcal{X}) \cap \mathcal{O}(D)$. Since $g(x) \in \mathcal{X}$, (25) yields
\[(Xg)(x) = Z_x(g(x)) + (Dg)(x)\]
for all $x \in K_x(T)$.

If $\mathcal{X} = \mathcal{E}$, that is, if $\xi \in \mathcal{O}(\mathcal{X})$ for all $\xi \in \mathcal{E}$, then $g(x) \in \mathcal{O}(\mathcal{X})$, and the following lemma holds.

**Lemma 17:** If $\mathcal{X} = \mathcal{E}$, then $Z_x \in \mathcal{L}(\mathcal{E})$, $\mathcal{O}(D) = \mathcal{O}(\mathcal{X})$ and (27) holds for all $g \in \mathcal{O}(D)$ and all $x \in K_x(T)$.

Since the closed operator $X$ is densely defined, conservative and $m$-dissipative, its spectrum $\sigma(X)$ is non-empty, \([16]\). Either $\sigma(X)$ is the closed left half-plane \(\{ \xi \in \mathbb{C} : \Re \xi \leq 0 \} \), or $\sigma(X)$ is contained in the imaginary axis; in which case $T$ is the restriction to $\mathbb{R}_+$ of a strongly continuous group of surjective linear isometries of $C(M, \mathcal{E})$ (and $K_x(T) = M$).

If $T$ is an eventually differentiable semigroup, according to a theorem of A. Pazy (see [11], Theorem 4.7, pp. 54-57), there are $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$ such that the resolvent set of $X$ contains the set
\[\{ \xi \in \mathbb{C} : \Re \xi \geq a - b \log |\Im \xi| \} .\]

As a consequence, the first of the two possibilities listed above is ruled out, and $\sigma(X)$ turns out to be a compact subset of the imaginary axis. But then (see [5], Corollary 8.20), $X \in \mathcal{L}(C(M, \mathcal{E}))$. Hence $\mathcal{O}(X) = C(M, \mathcal{E})$, and (25) — which holds (with $X$ replaced by $\mathcal{X}$) for all $g \in C(M, \mathcal{E})$ and at all $x \in M$ — yields: $\mathcal{O}(D) = C(M, \mathcal{E})$. Thus, by Lemma 15 the following proposition holds.

**Proposition 5:** If $T$ is an eventually differentiable semigroup, there is a conservative operator $X \in \mathcal{L}(C(M, \mathcal{E}))$ such that $T$ is the restriction to $\mathbb{R}_+$ of the group $G : \mathbb{R} \to \mathcal{L}(C(M, \mathcal{E}))$ of surjective linear isometries defined by
\[(G(t)f)(x) = ((\exp tX)f))(x)\]
for all $f \in C(M, \mathcal{E})$, $t \in \mathbb{R}$ and $x \in M$.

**Remark:** The same argument as before shows, more in general, that any strongly continuous, eventually differentiable semigroup of linear isometries of a complex Banach space $\mathcal{F}$ is the restriction to $\mathbb{R}_+$ of a strongly continuous group of surjective linear isometries of $\mathcal{F}$.

\(^{(5)}\) We correct a misprint in [16], where the inclusion $r(X) \subset \mathcal{M}$, displayed at p. 309, shall be replaced by $r(X) \supset \mathcal{M}$.\)
6. Since, for $t > 0$ and $\beta > 0$,
\[ C_{t + h, x} = C_{t, x} \circ C_{h, \phi_x(s)} , \]
then, for any $\xi \in \mathcal{K}$, (25) yields
\[ \lim_{h \to 0} \frac{1}{h} (C_{t + h, x} - C_{t, x})(\xi) = C_{t, x} \circ \lim_{h \to 0} \frac{1}{h} (C_{h, \phi_x(s)} - I)(\xi) \]
\[ = C_{t, x} ((\bar{X}(\xi))(\phi_x(s))) = C_{t, x} (Z_{\phi_x(s)}(\xi)). \]
Hence, the map $t \mapsto C_{t, x}(\xi)$ of $\mathcal{R}_+$ into $\mathcal{E}$ is of class $C^1$ on $\mathcal{R}_+$, and
\[ (28) \quad \frac{d}{dt} C_{t, x}(\xi) = C_{t, x} (\bar{X}(\xi)(\phi_x(s))) \]
\[ = C_{t, x} (Z_{\phi_x(s)}(\xi)) \]
for all $x \in K_x(T)$ and all $\xi \in \mathcal{K}$.
For $t \geq 0$, let
\[ A(t) : \mathcal{O}(A(t)) \subset \mathcal{L}(C(K_x(T), \mathcal{E}), \mathcal{E}) \to \mathcal{L}(C(K_x(T), \mathcal{E}), \mathcal{E}) \]
be the linear operator defined on
\[ \mathcal{O}(A(t)) = \mathcal{L}(\bar{X}(\mathcal{K}), \mathcal{E}) \]
by
\[ (A(t) R)(\xi) = R(\bar{X}(\xi)) , \]
i.e.
\[ ((A(t) R)(\xi))_x = (R(\bar{X}(\xi)))_x \]
\[ = R_x(Z_{\phi_x(s)}(\xi)) , \]
where $R \in \mathcal{L}(\bar{X}(\mathcal{K}), \mathcal{E})$.
Let $C_t \in C(M, \mathcal{L}(\mathcal{E}))$ be defined by
\[ C_t : x \mapsto C_{t, x}. \]
Then (28) yields the initial value problem
\[
\begin{cases}
\frac{d}{dt} C_t = A(t) C_t \\
C_0 = I ,
\end{cases}
\]
\[ \left\{ \begin{array}{l}
\left( \frac{d}{dt} C_t \right)_x = C_{t, x} (Z_{\phi_t, x}(\xi)) \\
C_{0, x} = I
\end{array} \right. \]

for all \( t \in \mathbb{R}_+ , x \in K_x(T), \xi \in K \).

As before, let \( K \) be strictly convex and let \( T : \mathbb{R} \to \mathcal{L}(C(M, \mathcal{E})) \) be a strongly continuous group of linear isometries of \( C(M, \mathcal{E}) \). Then \( K_\alpha(T) = M \), and \( T \) is expressed by

\[
(T(t) f)(x) = C_{t, x}(f(\phi_t(x)))
\]

for all \( f \in C(M, \mathcal{E}), x \in M, t \in \mathbb{R} \), where \( \phi : t \mapsto \phi_t \) is a continuous flow on \( M \), and \( C_{t, x} \in \mathcal{L}(\mathcal{E}) \) is a surjective isometry such that

\[
C_{t+s, x} = C_{t, x} \circ C_{s, \phi_t(x)} \quad \forall \, t, s \in \mathbb{R}, \, x \in M.
\]

Suppose now that \( M \) is a compact differentiable (i.e. \( C^\infty \)) manifold, and that the flow \( \phi \) is determined by a \( C^\infty \) vector field \( v \) on \( M \). For any \( f \in C^1(M, \mathcal{E}) \) we define \( v(f) \in C(M, \mathcal{E}) \) componentwise; that is to say, setting for \( x \in M \) and \( \lambda \in \mathbb{E}' \),

\[
\langle (v(f))(x), \lambda \rangle = \langle v(f(\cdot)), \lambda \rangle(x).
\]

Clearly

\[
f \in C^\infty(M, \mathcal{E}) \implies v(f) \in C^\infty(M, \mathcal{E}).
\]

If \( L : \mathbb{R} \to \mathcal{L}(C(M, \mathcal{E})) \) is the group defined by (23) for all \( t \in \mathbb{R} \) and all \( g \in C(M, \mathcal{E}) \), and if \( D \) is its infinitesimal generator, then

\[
C^\infty(M, \mathcal{E}) \subset \mathcal{O}(D)
\]

and

\[
D(f) = v(f) \quad \forall f \in C^\infty(M, \mathcal{E}).
\]

**Lemma 18:** If the map \( x \mapsto C_{t, x} \) of \( M \) into \( \mathcal{L}(\mathcal{E}) \) is of class \( C^\infty \) for all \( t \in \mathbb{R} \), the map \( t \mapsto C_{t, x} \) is of class \( C^\infty \) on \( \mathbb{R} \) for all \( x \in M \).

**Proof:** For \( t_0 \in \mathbb{R} \) and \( r > 0 \), let \( \varrho : \mathbb{R} \to [0, 1] \) be a \( C^\infty \) function for which

\[
\varrho(t) = 1 \quad \text{if} \quad |t - t_0| \leq r
\]

\[
0 < \varrho(t) < 1 \quad \text{if} \quad r < |t - t_0| < 2r
\]

\[
\varrho(t) = 0 \quad \text{if} \quad |t - t_0| \geq 2r.
\]
Then
\[ + \int_{-\infty}^{+\infty} q(s) C_{t+s, x} ds = C_{t, x} \left( + \int_{-\infty}^{+\infty} q(s) C_{t, \phi(s)} ds \right), \]
i.e.,
\[ + \int_{-\infty}^{+\infty} q(s-t) C_{t, x} ds = C_{t, x} \left( + \int_{-\infty}^{+\infty} q(s) C_{t, \phi(s)} ds \right). \]
A neighbourhood \( U \) of \( t_0 \) in \( R \) and \( r > 0 \) can be so chosen that
\[ \int_{-\infty}^{+\infty} q(s) C_{t, \phi(s)} ds \neq 0 \]
whenever \( t \in U \).

Differentiation with respect to \( t \in U \) shows that the function \( t \mapsto C_{t, x} \) is of class \( C^1 \) on \( U \) for all \( x \in M \), and
\[ - \int_{-\infty}^{+\infty} \left( \frac{d}{dt} \right) (s-t) C_{t, x} ds = 2 \int_{-\infty}^{+\infty} q(s) C_{t, \phi(s)} ds + \]
\[ + C_{t, x} \left( + \int_{-\infty}^{+\infty} q(s) v(C_{t, \phi(s)}) ds \right). \]

Iteration of this computation completes the proof of the lemma. ■

Thus, \( Z_x \in \mathcal{L}(\delta) \) for all \( x \in M \), and
\[ (29) \quad Z_x = \frac{d}{dt} C_{t, x}. \]

By the same argument leading to Theorem 4 of [17] one proves then

**Theorem 10:** If the strongly continuous group \( T : R \rightarrow \mathcal{L}(C(M, \delta)) \) of linear isometries is such that
\[ T(t) C^\infty(M, \delta) \subset C^\infty(M, \delta) \quad \forall t \in R, \]
then: \( \mathcal{O}(D) = \mathcal{O}(X) \); (27) holds for all \( g \in \mathcal{O}(X) \) and all \( x \in M \), where \( Z_x \) is expressed by (29), and \( C^\infty(M, \delta) \) is a core for \( X \).

7. If \( \dim \delta < \infty \) and \( \dim \mathcal{F} < \infty \), the sets \( K(A) \) and \( K(T(t)) \) for all \( t \geq 0 \) are closed, \( K_{\infty}(T) \) is closed and non-empty, the linear isometries \( C_{A, x} \) and \( C_{t, x} \) are invertible for all \( t \geq 0 \).
If the semigroup \( T \) (or the semigroup \( T_A \)) is strongly continuous, the isometries \( C_t \), \( x \) are continuous functions of \((t, x) \in \mathbb{R}_+ \times M\) (or of \((t, x) \in \mathbb{R}_+ \times K_{\infty}(T)\) respectively).

In the case in which \( \mathcal{C} = \mathcal{F} = \mathcal{C}, [9], \) \( C_t \) is represented by a continuous function \( \alpha : M \to \mathfrak{D}; \) (4) and Theorem 2 yield

\[
\Theta(C(M)) = \{ b \in C(M) : |b(x)| = 1 \ \forall x \in M \},
\]

\[
\Theta(C(M)') = \{ c_\alpha : c \in \mathfrak{D}, x \in M \}.
\]

**Lemma 19:** [15] If \( \lambda \in C(M)' \), then \( \lambda \in \Theta(C(M)') \) if, and only if,

\[
|\langle b, \lambda \rangle| = 1
\]

for all \( b \in \Theta(C(M)) \).

Theorem 4 generalizes the second part of the following

**Theorem 11:** [15] If either

\[(30) \quad A(\Theta(C(M))) \subset \Theta(C(N)), \]

or

\[(31) \quad A'(\Theta(C(N)')) \subset \Theta(C(M)'), \]

then \( K(A) = N \), i.e.,

\[(32) \quad (Af)(y) = \alpha(y)(f \circ \psi(y)) \quad \forall y \in K(A), \quad f \in C(M). \]

**Proof:** The theorem is equivalent to the following chain of implications:

\[(30) \Rightarrow (31) \Rightarrow (32) \Rightarrow (30). \]

If (31) holds, for every \( y \in N \) there are a unique \( x \in M \) and a unique \( c \in \mathfrak{D} \) for which

\[
A' \delta_y = c \delta_x,
\]

i.e.,

\[(Af)(y) = cf(x)\]

for all \( f \in C(M) \). Setting \( c = \alpha(y) \) and \( x = \psi(y) \), (32) follows.

If (30) holds, then, for every \( y \in N \) and all \( b \in \Theta(M) \),

\[
1 = |(Ab)(y)| = |\langle Ab, \delta_y \rangle| = |\langle b, A' \delta_y \rangle|,
\]

and therefore, by Lemma 19, (31) holds.
Viceversa, if (32) is satisfied, with $\alpha \in \Theta(N)$ and $\psi$ a continuous surjective map of $N$ onto $M$, then (30) holds.

By the Tietze extension theorem, Lemma 14 yields

**Proposition 6:** If $\dim_c \mathcal{E} < \infty$, the operator $\bar{X}$ is the closure of $Y$.

We consider now the strongly continuous semigroup $T : \mathbb{R} \to \mathcal{L}(C(M))$ of linear isometries of $C(M)$, and the strongly continuous semigroup $\bar{T} : \mathbb{R} \to \mathcal{L}(C(K_\omega(T)))$ expressed on any $g \in C(K_\omega(T))$ by

$$\overline{T}(t) g(x) = \alpha_t(x) g(\phi_t(x)),$$

where $\alpha_t \in \Theta(C(K_\omega(T)))$ is a continuous function of $t$, and $\phi : t \mapsto \phi_t$ is a continuous semiflow on $K_\omega(T)$.

The existence of fixed points of the semiflow $\phi$ yields some information on the point spectrum $\rho \sigma(X)$ and the residual spectrum $\sigma(X)$ of $X$, as will be illustrated now in the case $\mathcal{E} = \mathcal{C}$.

If $x_0 \in K_\omega(T)$ is fixed by $\phi$, i.e.,

$$\phi_t(x_0) = x_0 \quad \forall t \geq 0,$$

then

$$\overline{T}(t) f(x_0) = \alpha_t(x_0) f(\phi_t(x_0)) = \alpha_t(x_0) f(x_0)$$

for all $f \in C(M)$, and

$$\alpha_{t+s}(x_0) = \alpha_t(x_0) \alpha_s(\phi_t(x_0)) = \alpha_t(x_0) \alpha_s(x_0)$$

for all $t, s \geq 0$.

Letting

$$\alpha_t(x_0) = \frac{1}{\alpha_t(x_0)} = \alpha_t(x_0),$$

we extend the map $R_+ \ni t \mapsto \alpha_t(x_0)$ to a continuous homomorphism of $\mathbb{R}$ into the multiplicative group $\mathfrak{D}$. Hence there is $a \in \mathbb{R}$ such that

$$\alpha_t(x_0) = e^{at}$$

for all $t \in \mathbb{R}$, and therefore (33) becomes

$$\overline{T}(t) f(x_0) = e^{at} f(x_0) \quad \forall t \in \mathbb{R}_+,$$

i.e.,

$$((\overline{T}(t) - e^{at} I, \delta_{x_0}) = 0 \quad \forall t \in \mathbb{R}_+.$$
For any \( f \in \mathcal{O}(X) \),
\[
(Xf)(x_0) = \langle Xf, \delta_{x_0} \rangle = \lim_{t \downarrow 0} \frac{1}{t} (T(t) - I) f, \delta_{x_0} \bigg|_{Xf} = \lim_{t \downarrow 0} \frac{1}{t} (\alpha_t(x_0) f(\phi_t(x_0)) - f(x_0)) = \lim_{t \downarrow 0} \frac{1}{t} (\alpha_t(x_0) - 1) f(x_0) = \lim_{t \downarrow 0} \frac{1}{t} (e^{it} - 1) f(x_0) = iaf(x_0) = \langle (X - iaI) f, \delta_{x_0} \rangle.
\]

Hence, \( ia \in \rho \sigma(X) \cup \sigma(X) \).

In conclusion, the following theorem holds.

**Theorem 12:** If \( x_0 \in K_\alpha(T) \) is fixed by the semiflow \( \phi \), there is \( a \in \mathbb{R} \) such that \( ia \in \rho \sigma(X) \cup \sigma(X) \), and (34) holds for all \( t \in \mathbb{R}_+ \).

If \( ia \) is an isolated point of \( \sigma(X) \), then ([14], p. 178) \( ia \in \rho \sigma(X) \).

**References**


