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Linear Isometries of Vector-Valued Functions (**)

Summary. — Let M be a compact Hausdorff space and let $C(M)$ be the Banach space of all complex-valued continuous functions on M. The classical Banach-Stone theorem, which associates to any surjectice linear isometry $A: C(M) \to C(M)$ a homeomorphism of M, was generalized by W. Holsztyński to the case in which the linear isometry A is not necessarily surjective. Holsztyński's result — which was further extended by M. Cambern to Banach spaces of continuous vector-valued functions on M — associates to A a subset $K(A)$ of M and a continuous surjective map $\psi: K(A) \to M$. In this paper, a maximal ψ-invariant subset of M is constructed in terms of the iterates of A. Actually, the construction of the invariant subset is carried out replacing the discrete subgroup of the iterates of A by a strongly continous semigroup of linear isometries.

Isometrie lineari di funzioni a valori vettoriali

Sunto. — Sia M uno spazio compatto di Hausdorff, e sia $C(M)$ lo spazio di Banach delle funzioni continue a valori complessi su M. Il classico teorema di Banach-Stone, che associa ad ogni isometria lineare $A: C(M) \to C(M)$ un omeomorfismo di M, è stato generalizzato da W. Holsztyński al caso in cui l'isometria lineare A non è necessariamente surgettiva. Il risultato di Holsztyński — esteso da M. Cambern a spazi di Banach di funzioni a valori vettoriali, continue su M — associa a A un sottoinsieme $K(A)$ di M ed una applicazione continua ψ di $K(A)$ su M. In questo lavoro, si costruisce un sottoinsieme ψ-invariante massimale di M definito mediante le iterate di A. Di fatto, il sottoinsieme invariante viene costruito sostituendo al semigruppo discreto delle iterate di A un sottogruppo fortemente continuo di isometrie lineari.

In one of the final chapters of [2], S. Banach made the important observation that two compact metric spaces M and N are homeomorphic if, and only if, the uniform spaces of all continuous, real-valued functions on M and N are isometric. As a byproduct of his proof, if A is such an isometry, there are a homeomorphism ψ of N onto M

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and a continuous function α, with modulus one at all points of N, such that

$$(Af)(y) = \alpha(y)(f(\psi(y))) \tag{1}$$

at all $y \in N$ and for any real-valued, continuous function f on M. This ground-breaking result was the starting point of a research field which is quite alive today. In [13] M. Stone extended Banach's theorem to continuous, complex-valued functions on compact (not necessarily metric) Hausdorff spaces and set the stage, within the framework of Boolean algebras, of what would later be called the Banach-Stone problem (see [3] also for exhaustive historical references until 1979), involving continuous vector-valued functions.

In [9], W. Holsztyński considered the case in which the linear isometry A is not surjective (1), and proved that (1) still holds, but gives only a partial description of A in the sense that ψ is then a continuous map of a closed subset $K(A)$ of N onto M and $y \in K(A)$. As was shown in [15], the case $K(A) = N$ can be characterized in terms of the behaviour of A on the extreme points of the closed unit ball of the space of all continuous, complex-valued functions on M.

In [4] M. Cambern proved that Holsztyński's result extends *mutatis mutandis* to Banach spaces of continuous vector-valued functions from M to a complex Banach space $\mathcal{E}$ and from N to a strictly convex complex Banach space $\mathcal{F}$.

In the case in which $M = N$ the question arises, for both Holsztyński's and Cambern's theorems, whether there exists a subset $K(A) \subset M$ that is invariant under the action of A and on which the action of A is therefore completely described by (1) or by a generalization thereof. In this paper, a maximal invariant set will be constructed in terms of the iterates of A. However, instead of considering these iterates, a more general situation will be investigated, replacing A by a strongly continuous semigroup of linear isometries.

After a first section devoted to the set of all extreme points of the closed unit ball of the Banach space of all continuous maps from M to $\mathcal{E}$, and of the closed unit ball of the dual space, n. 2 investigates the set $K(A) \subset N$, establishing a necessary and sufficient condition for $K(A)$ to coincide with N, and a sufficient condition for $K(A)$ to be closed, retrieving, as a consequence, a result of M. Cambern whereby $K(A)$ is closed when $\mathcal{E}$ has finite dimension.

In n. 3, A is replaced — under the hypotheses $M = N$ and $\mathcal{E} = \mathcal{F}$ — by a semigroup T of linear isometries, which, in particular, may coincide with the family of all iterates of A. Under rather weak hypotheses on T (that are fulfilled when $\mathcal{E}$ has finite dimension), a maximal «invariant» set $K_\infty(T) \subset M$ will be shown to exist, on which the action of T is determined by a semiflow ϕ acting on $K_\infty(T)$ and by an operator-valued cocycle associated to ϕ. If $K_\infty(T)$ is closed and the semigroup T is assumed to be strongly continuous — as will be done in nn. 5 and 6 — the semiflow ϕ is continuous,

(1) According to the Mazur-Ulam theorem ([2], pp. 166-168) surjective isometries are linear over the reals. The case of non-linear isometries was briefly investigated in [15].

and the infinitesimal generator of the semigroup defined by T in $K_\infty(T)$ is a bounded perturbation of the infinitesimal generator of the semigroup determined by ϕ.

Finally, in n. 7 the particular case of scalar-valued continuous functions will be considered, extending to semigroups of general linear isometries some results established in [17] under additional conditions.

1. Let $\mathcal{E}$ be a complex Banach space with norm $\| \|_{\mathcal{E}}$. If M is a compact Hausdorff space, $C(M, \mathcal{E})$ will stand for the complex Banach space of all continuous functions $f: M \to \mathcal{E}$, with the uniform norm $\|f\|_{C(M, \mathcal{E})} = \sup\{\|f(x)\|_{\mathcal{E}}: x \in M\}$. For any complex Banach space $\mathcal{E}$, $\mathcal{E}'$ will stand for the strong dual of $\mathcal{E}$; $B_{\mathcal{E}}$, $B_{\mathcal{E}'}$, $\overline{B_{\mathcal{E}}}$, $\overline{B_{\mathcal{E}'}}$ will indicate respectively the unit ball of $\mathcal{E}$, the unit ball of $\mathcal{E}'$ and their closures.

PROPOSITION 1: *Let* $\mathcal{A} \neq \{0\}$ *be a closed linear subspace of* $C(M, \mathcal{E})$. *If* $f \in \mathcal{A}$,

$$\|f\|_{C(M, \mathcal{E})} = \sup\{|\langle f, \Lambda\rangle| : \Lambda \text{ extreme point of } \overline{B_{\mathcal{A}'}}\}.$$

PROOF: Obviously,

$$\|f\|_{C(M, \mathcal{E})} \geqslant \sup\{|\langle f, \Lambda\rangle| : \Lambda \text{ extreme point of } \overline{B_{\mathcal{A}'}}\}. \tag{2}$$

Let now $\|f\|_{C(M, \mathcal{E})} = 1$.

Since M is compact, there is some $x_0 \in M$ such that $1 = \|f\|_{C(M, \mathcal{E})} = \|f(x_0)\|_{\mathcal{E}}$.

For any $\lambda \in \partial B_{\mathcal{E}'}$, with $\|\lambda\|_{\mathcal{E}'} = 1$, the continuous linear form on $\mathcal{A}$

$$\delta_{x_0} \otimes \lambda : f \mapsto \langle f(x_0), \lambda\rangle$$

has norm one, showing that the closed set

$$S := \{\Lambda \in \overline{B_{\mathcal{A}'}} : \langle f, \Lambda\rangle = 1\} \subset \mathcal{A}'$$

is not empty. Since, for $\Lambda_1, \Lambda_2 \in S$ and $0 < t < 1$,

$$\langle f, t\Lambda_1 + (1-t)\Lambda_2\rangle = t + 1 - t = 1,$$

S is also convex, and therefore is compact for the weak-star topology of $\mathcal{A}'$. By the Kreĭn-Milman theorem, S has one extreme point at least.

Let Λ_0 be one of these extreme points, and let $\Lambda_1, \Lambda_2 \in \overline{B_{\mathcal{A}'}}$, $0 < t < 1$ be such that

$$\Lambda_0 = t\Lambda_1 + (1-t)\Lambda_2.$$

Since $\Lambda_0 \in S$,

$$t\langle f, \Lambda_1\rangle + (1-t)\langle f, \Lambda_2\rangle = 1, \tag{3}$$

whence

$$\begin{aligned} 1 &\leqslant t|\langle f, \Lambda_1\rangle| + (1-t)|\langle f, \Lambda_2\rangle| \\ &\leqslant t\|f\|_\infty \|\Lambda_1\|_{\mathcal{A}'} + (1-t)\|f\|_\infty \|\Lambda_2\|_{\mathcal{A}'} \\ &\leqslant t + (1-t) = 1\,, \end{aligned}$$

and therefore

$$|\langle f, \Lambda_1\rangle| = |\langle f, \Lambda_2\rangle| = 1\,;$$

(3) yields then

$$\langle f, \Lambda_1\rangle = \langle f, \Lambda_2\rangle = 1\,,$$

i.e. $\Lambda_1, \Lambda_2 \in S$.

Hence

$$1 = \|f\|_{C(M,\,\mathcal{E})} = \langle f, \Lambda_0\rangle,$$

and this fact, together with (2) completes the proof of the proposition (2) ∎

LEMMA 1: *Let the closed linear subspace $\mathcal{A}$ of $C(M, \mathcal{E})$ be such that, for every $x \in M$ and every open neighbourhood U of x in M there is $g \in \mathcal{A}\setminus\{0\}$ with* Supp $g \subset U$. *If $f \in \mathcal{A}$ is a complex extreme point of $\overline{B_{\mathcal{A}}}$, then $\|f(x)\|_{\mathcal{E}} = 1$ for all $x \in M$.*

PROOF: If $\|f(x_0)\|_{\mathcal{E}} < 1$ for some $x_0 \in M$, there exist an open neighbourhood U of x_0 and some $\varepsilon > 0$ for which

$$\|f(x)\|_{\mathcal{E}} < 1 - \varepsilon \qquad \forall x \in U\,.$$

Let $g \in \mathcal{A}\setminus\{0\}$ be such that Supp $g \subset U$ and $\|g\|_{C(M,\,\mathcal{E})} \leqslant \varepsilon$. Given any $\zeta \in \Delta = \{\tau \in \mathbb{C} : |\tau| < 1\}$,

$$\begin{aligned} \|f(x) + \zeta g(x)\|_{\mathcal{E}} &\leqslant \|f(x)\|_{\mathcal{E}} + |\zeta|\,\|g(x)\|_{\mathcal{E}} \\ &\leqslant \|f(x)\|_{\mathcal{E}} + \|g(x)\|_{\mathcal{E}} \\ &< 1 - \varepsilon + \varepsilon = 1 \end{aligned}$$

if $x \in U$, and

$$\|f(x) + \zeta g(x)\|_{\mathcal{E}} = \|f(x)\|_{\mathcal{E}}$$

if $x \in M\setminus U$. Thus,

$$\|f + \zeta g\|_{C(M,\,\mathcal{E})} \leqslant 1$$

(2) The proof follows the ideas in [7], pp. 145-146.

for all $\zeta \in \Delta$, contradicting the hypothesis whereby f is a complex extreme point of $\overline{B_{\mathcal{A}}}$. ■

Lemma 1 and the following lemma characterize all extreme points of $\overline{B_{C(M,\mathcal{E})}}$, where $\mathcal{E}$ is strictly convex.

LEMMA 2: *Let $\mathcal{E}$ be strictly convex. If, and only if,*

$$\|f(x)\|_{\mathcal{E}} = 1 \qquad \forall x \in M ,$$

$f \in C(M, \mathcal{E})$ is an extreme point of $\overline{B_{C(M,\mathcal{E})}}$.

PROOF: Let $g \in C(M, \mathcal{E})$ and let $t \in (0, 1) \setminus \{0\}$ be such that

$$\|f + tg\|_{C(M,\mathcal{E})} \leq 1 .$$

Then

$$\|f(x) + tg(x)\|_{\mathcal{E}} \leq 1 \qquad \forall x \in M .$$

Since $f(x) \in \partial B_{\mathcal{E}}$ is an extreme point of $\overline{B_{\mathcal{E}}}$, then $g(x) = 0$ for all $x \in M$. ■

Let

$$\Theta(\mathcal{A}) = \{ g \in \overline{B_{\mathcal{A}}} : g \text{ extreme point of } \overline{B_{\mathcal{A}}} \} .$$

Lemma 1 and Lemma 2 yield

THEOREM 1: *If $\mathcal{E}$ is strictly convex and $\mathcal{A} \neq \{0\}$ is a closed linear subspace of $C(M, \mathcal{E})$ such that, for every $x \in M$ and every open neighbourhood of x in M there is $g \in \mathcal{A} \setminus \{0\}$ with* $\operatorname{Supp} g \subset U$, *then*

$$\Theta(\mathcal{A}) = \{ g \in \mathcal{A} : \|g(x)\|_{\mathcal{E}} = 1 \;\; \forall x \in M \} .$$

In particular, if $\mathcal{E}$ is strictly convex, then

$$\Theta(C(M, \mathcal{E})) = \{ f \in C(M, \mathcal{E}) : \|f(x)\|_{\mathcal{E}} = 1 \;\; \forall x \in M \} . \tag{4}$$

We will now describe $\Theta(C(M, \mathcal{E})')$.

Let

$$C := \{ \delta_x \otimes \lambda : x \in M, \lambda \in \overline{B'_{\mathcal{E}}} \} \subset \overline{B_{C(M,\mathcal{E})'}} .$$

LEMMA 3: *The set C is weak-star closed in $C(M, \mathcal{E})'$.*

PROOF: If Ω is contained in the weak-star closure of C, there is a generalized sequence $\{\delta_{x_j} \otimes \lambda_j\}$, with $x_j \in M$ and $\lambda_j \in \overline{B'_{\mathcal{E}}}$, converging to Ω, *i.e.*, such that

$$\langle f, \Omega \rangle = \lim \langle f(x_j), \lambda_j \rangle \qquad \forall f \in C(M, \mathcal{E}) . \tag{5}$$

Up to replacing this generalized sequence by a generalized subsequence, there is no restriction in assuming that $\{x_j\}$ converges to a point $x_0 \in M$, and that $\{\lambda_j\}$ converges to $\lambda_0 \in \overline{B_{\mathcal{E}'}}$ for the weak-star topology. Hence, (5) yields

$$\langle f, \Omega \rangle = \langle f(x_0), \lambda_0 \rangle \qquad \forall f \in C(M, \mathcal{E}) . \qquad \blacksquare$$

LEMMA 4: *If $\Omega \in C(M, \mathcal{E})'$ is an extreme point of $\overline{B_{C(M, \mathcal{E})'}}$, there exist $x_0 \in M$ and λ_0 extreme point of $\overline{B_{\mathcal{E}'}}$ such that $\Omega = \delta_{x_0} \otimes \lambda_0$.*

PROOF: The closure $\overline{\text{co}(C)}$ of the convex hull $\text{co}(C)$ of C coincides with the closed convex hull $\overline{\text{co}}(C)$, which is closed in $\overline{B_{\mathcal{E}'}}$.

If $\Omega \notin \overline{\text{co}}(C)$, there exist, ([6], p. 417), $f \in C(M, \mathcal{E})$, $c \in \boldsymbol{R}$ and $\varepsilon > 0$ such that

$$\Re \langle f, \Omega \rangle \geqslant c$$

and

$$\Re \langle f, \Lambda \rangle \leqslant c - \varepsilon \qquad \forall \Lambda \in C ,$$

i.e.,

$$\Re \langle f(x), \lambda \rangle \leqslant c - \varepsilon \qquad \forall x \in M, \lambda \in \overline{B_{\mathcal{E}'}} .$$

Since

$$\|f(x)\|_{\mathcal{E}} = \sup \{ |\langle f(x), \lambda \rangle| : \lambda \in \overline{B_{\mathcal{E}'}} \} ,$$

then

$$\|f(x)\|_{\mathcal{E}} \leqslant c - \varepsilon \qquad \forall x \in M ,$$

and therefore

$$\|f\|_{C(M, \mathcal{E})} \leqslant c - \varepsilon .$$

If $\|\Omega\| \leqslant 1$, then

$$\begin{aligned} c \leqslant \Re \langle f, \Omega \rangle &\leqslant |\langle f, \Omega \rangle| \\ &\leqslant \|f\|_{C(M, \mathcal{E})} \|\Omega\| \leqslant \|f\|_{C(M, \mathcal{E})} \leqslant c - \varepsilon . \end{aligned}$$

This contradiction shows that

$$\Omega \notin \overline{\text{co}}(C) \Rightarrow \Omega \notin \overline{B_{C(M, \mathcal{E})'}} ,$$

i.e.,

$$\overline{B_{C(M, \mathcal{E})'}} \subset \overline{\text{co}}(C) \subset \overline{B_{C(M, \mathcal{E})'}} ,$$

and therefore

$$\overline{\text{co}}(C) = \overline{B_{C(M, \mathcal{E})'}} .$$

Since the extreme points of $\overline{\mathrm{co}}\,(C)$ are contained in C (see, *e.g.*, [6], pp. 440-441), there are $x_0 \in M$ and $\lambda_0 \in \overline{B_{\mathcal{E}'}}$ such that $\Omega = \delta_{x_0} \otimes \lambda_0$.

If λ_0 is not an extreme point of $\overline{B_{\mathcal{E}'}}$, there are $\lambda_1, \lambda_2 \in \overline{B_{\mathcal{E}'}}$ and $t \in (0, 1)$ such that $\lambda_0 = t\lambda_1 + (1-t)\lambda_2$, and therefore

$$\Omega = \delta_{x_0} \otimes \lambda_0 = t\delta_{x_0} \otimes \lambda_1 + (1-t)\delta_{x_0} \otimes \lambda_2 . \qquad \blacksquare$$

In conclusion, the following theorem holds

THEOREM 2: *A linear form $\Lambda \in C(M, \mathcal{E})'$ is an extreme point of $\overline{B_{C(M,\mathcal{E})'}}$ if, and only if, there exist $x \in M$ and an extreme point λ of $\overline{B_{\mathcal{E}'}}$ such that $\Lambda = \delta_x \otimes \lambda$.*

2. Let M and N be compact Hausdorff spaces and let $\mathcal{E}$ and $\mathcal{F}$ be complex Banach spaces, with $\mathcal{F}$ strictly convex. In [4], M. Cambern has characterized all linear isometries of $C(M, \mathcal{E})$ into $C(N, \mathcal{F})$, proving the following theorem, which extends previous results established by W. Holsztyński in [9] for the case $\mathcal{E} = \mathcal{F} = \mathbf{C}$.

THEOREM 3: *Let $A \in \mathcal{L}(C(M, \mathcal{E}), C(N, \mathcal{F}))$ be a linear isometry. If $\mathcal{F}$ is strictly convex, there exist:*

a set $K(A) \subset N$;

a continuous, surjective map $\psi : K(A) \to M$;

a map $N \ni y \mapsto C_y \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, which is continuous for the strong operator topology in $\mathcal{L}(\mathcal{E}, \mathcal{F})$, such that

$$(Af)(y) = C_y(f \circ \psi(y)) \tag{6}$$

for all $y \in K(A)$ and all $f \in C(M, \mathcal{E})$.

The set $K(A)$ and the map ψ are described as follows.
For $x \in M$, $\xi \in \partial B(M, \mathcal{E})$, let

$$F(\xi, x) = \{f \in C(M, \mathcal{E}) : f(x) = \|f\|_{C(M,\mathcal{E})}\, \xi\},$$

$$K_A(\xi, x) = \{y \in N : \|(Af)(y)\|_{\mathcal{F}} = \|f\|_{C(M,\mathcal{E})} \quad \forall f \in F(\xi, x)\},$$

$$K_A(x) = \bigcup \{K(\xi, x) : \xi \in \partial B(M, \mathcal{E})\},$$

$$K(A) = \bigcup \{K_A(x) : x \in M\}.$$

In [4], Cambern shows that $K_A(\xi, x) \neq \emptyset$ for all $x \in M$, and

$$x_1 \neq x_2 \Rightarrow K_A(x_1) \cap K_A(x_2) = \emptyset .$$

Hence, for every $y \in K(A)$ there is a unique $x \in M$ such that $y \in K_A(x)$. The map $\psi : K(A) \to M$ is defined by setting $x = \psi(y)$.

Any $\xi \in \mathcal{E}$ defines a function $\underline{\xi} \in C(M, \mathcal{E})$ as follows:

$$\underline{\xi}(x) = \xi \qquad \forall x \in M .$$

For $y \in N$, the operator $C_y \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is given by

$$C_y(\xi) = A(\underline{\xi}) .$$

Since, for any $y \in N$,

$$\begin{aligned} \|C_y \xi\|_{\mathcal{F}} &= \|(A\underline{\xi})(y)\|_{\mathcal{F}} \leq \|A\| \, \|\underline{\xi}\|_{C(M,\mathcal{E})} \\ &= \|\underline{\xi}\|_{C(M,\mathcal{E})} = \|\xi\|_{\mathcal{E}}, \end{aligned}$$

then

$$\|C_y\| \leq 1 \qquad \forall y \in N .$$

Being $\underline{\xi} \in F(\xi, x)$ for all $x \in M$, then

$$\|C_y \xi\|_{\mathcal{F}} = \|\xi\|_{\mathcal{E}} \qquad \forall \xi \in \mathcal{E}, \qquad \forall y \in K(A) .$$

Since $y \mapsto C_y \xi$ is continuous for all $\xi \in \mathcal{E}$, that proves

Lemma 5: *For any $y \in \overline{K(A)}$, C_y is a linear isometry of $\mathcal{E}$ into $\mathcal{F}$.*

In [4] M. Cambern shows that, if $y \in K_A(x)$, then

$$(Af)(y) = C_y(f(x)) \qquad \forall f \in C(M, \mathcal{E}) .$$

By the construction of ψ, that yields (6).

Proposition 2: *If the map $C: y \mapsto C_y$ of N into $\mathcal{L}(\mathcal{E}, \mathcal{F})$ is continuous for the uniform operator topology of $\mathcal{L}(\mathcal{E}, \mathcal{F})$, the set $K(A)$ is closed.*

Proof: Let $y_0 \in \overline{K(A)}$.
For any $f \in \overline{B_{C(M,\mathcal{E})}}$ and for $n = 1, 2, \ldots$ there is some $y_n \in K(A)$ such that

$$\|(Af)(y_0) - (Af)(y_n)\|_{\mathcal{F}} < \frac{1}{n} ,$$

i.e.,

$$\|(Af)(y_0) - C_{y_n}(f(\psi(y_n)))\|_{\mathcal{F}} < \frac{1}{n} ,$$

and moreover

$$\|C_{y_0} - C_{y_n}\| < \frac{1}{n} .$$

Suppose that the set $\{\psi(y_n)\}$ is infinite. Because M is compact, the set $\{\psi(y_n)\}$ has at least one cluster point x_0. For any $\varepsilon > 0$ there is an open neighbourhood U of x_0 in M such that

$$\|f(x) - f(x_0)\|_{\mathcal{E}} < \varepsilon \qquad \forall\, x \in U\,.$$

Let $n_0 > 0$ be so large that $\frac{1}{n_0} < \varepsilon$, and let $n > n_0$ be such that $x_n \in U$. Then

$$\begin{aligned}
\|(Af)(y_0) - C_{y_0}(f(x_0))\|_{\mathcal{F}} &\leq \|(Af)(y_0) - C_{y_n}(f(x_n))\|_{\mathcal{F}} + \\
&\quad + \|(C_{y_n} - C_{y_0})(f(x_n))\|_{\mathcal{F}} + \\
&\quad + \|C_{y_0}(f(x_n) - f(x_0))\|_{\mathcal{F}} \\
&\leq \|(Af)(y_0) - C_{y_n}(f(x_n))\|_{\mathcal{F}} + \\
&\quad + \|C_{y_n} - C_{y_0}\|\,\|f(x_n)\|_{\mathcal{E}} + \\
&\quad + \|C_{y_0}\|\,\|f(x_n) - f(x_0)\|_{\mathcal{E}} \\
&< \frac{1}{n} + \frac{1}{n} + \varepsilon < 3\,\varepsilon\,.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, that shows that

$$(Af)(y_0) = C_{y_0}(f(x_0))\,.$$

Obviously, the same conclusion holds when the set $\{\psi(y_n)\}$ is finite; in which case $x_0 \in \{\psi(y_n)\}$ can be chosen such that $\psi(y_{n_j}) = x_0$ for $n_1 < n_2 < \ldots$.

Let now u_0 be another cluster point of the set $\{\psi(y_n)\}$ when this latter set is infinite, or such that $\psi(y_{m_j}) = u_0$ for $m_1 < m_2 < \ldots$. By the same argument as before, one shows that

$$(Af)(y_0) = C_{y_0}(f(u_0))\,.$$

Hence,

$$C_{y_0}(f(x_0) - f(u_0)) = 0\,,$$

and therefore

$$f(x_0) = f(u_0) \qquad \forall\, f \in C(M, \mathcal{E})$$

because C_{y_0} is injective. If $x_0 \neq u_0$, given any two vectors ξ_1 and ξ_2 in $\mathcal{E}$, there is a function $f \in C(M, \mathcal{E})$ such that

$$f(x_0) = \xi_1, \qquad f(u_0) = \xi_2.$$

Thus $x_0 = u_0$, and $y_0 \in \psi_A(x_0)$. ■

In view of the definition of C_y, the hypothesis of Proposition 2 can be rephrased by requiring that the restriction of A to the closed subspace of $C(M, \mathcal{E})$ consisting of all $\mathcal{E}$-valued constant functions on M be continuous for the uniform operator topology.

COROLLARY 1: [4] *If* $\dim \mathcal{E} < \infty$, $K(A)$ *is closed in* N.

LEMMA 6: *Let* $\mathcal{F}$ *be strictly convex and* $\mathcal{E}$ *reflexive. If* $y \in N$ *and there is* $\mu \in \partial B_{\mathcal{F}'}$ *such that*

$$A'(\delta_y \otimes \mu) = \delta_x \otimes \lambda$$

for some $x \in M$ *and* $\lambda \in \partial B_{\mathcal{E}'}$, *then* $y \in K(A)$.

PROOF: Since $\mathcal{E}$ is reflexive, there exists $\xi \in \mathcal{E}$ such that $\langle \xi, \lambda \rangle = 1$. If $f \in C(M, \mathcal{E})$ is such that $f(x) = \|f\|_{C(M, \mathcal{E})}\, \xi$, then

$$\begin{aligned} \langle (Af)(y), \mu \rangle &= \langle Af, \delta_y \otimes \mu \rangle = \langle f, A'(\delta_y \otimes \mu) \rangle \\ &= \langle f, \delta_x \otimes \lambda \rangle = \langle f(x), \lambda \rangle \\ &= \|f\|_{C(M, \mathcal{E})} \langle \xi, \lambda \rangle = \|f\|_{C(M, \mathcal{E})}. \end{aligned}$$

Since

$$\begin{aligned} \|f\|_{C(M, \mathcal{E})} &= \langle (Af)(y), \mu \rangle \leqslant \|(Af)(y)\|_{\mathcal{F}} \|\mu\|_{\mathcal{F}'} \\ &= \|(Af)(y)\|_{\mathcal{F}} \leqslant \|Af\|_{C(M, \mathcal{E})} = \|f\|_{C(M, \mathcal{E})}, \end{aligned}$$

then

$$\|(Af)(y)\|_{\mathcal{F}} = \|f\|_{C(M, \mathcal{E})},$$

and therefore $f \in K(A)$. ■

On the other hand, if $y \in K(A)$, for any $\mu \in \partial B_{\mathcal{F}'}$ and all $f \in C(M, \mathcal{E})$

$$\begin{aligned} \langle f, A'(\delta_y \otimes \mu) \rangle &= \langle Af, \delta_y \otimes \mu \rangle \\ &= \langle (Af)(y), \mu \rangle = \langle C_y(f(\psi(y))), \mu \rangle \\ &= \langle f(\psi(y)), C_y'(\mu) \rangle = \langle f, \delta_{\psi(y)} \otimes C_y'(\mu) \rangle. \end{aligned}$$

In conclusion, in view of Theorem 2, the following theorem holds

THEOREM 4: *If* $\mathcal{F}$ *is strictly convex, and* $\mathcal{E}$ *is uniformly convex, then* $K(A) = N$ *if, and only if,*

$$A'(\Theta(C(N, \mathcal{F})')) \subset \Theta(C(M, \mathcal{E})').$$

3. Let M be, as before, a compact Hausdorff space, let $\mathcal{E}$ be a strictly convex complex Banach space, and let $T: \boldsymbol{R}_+ \to \mathcal{L}(C(M, \mathcal{E}))$ be a semigroup of linear isometries $T(t): C(M, \mathcal{E}) \to C(M, \mathcal{E})$.

According to Theorem 3, for every $t \geqslant 0$ there exist:

a subset $K(T(t))$ of M;

a continuous surjective map ϕ_t: $K(T(t)) \to M$;

a map $x \mapsto C_{t,x}$ of M into $\mathcal{L}(\mathcal{E})$, continuous for the strong operator topology in $\mathcal{L}(\mathcal{E})$, such that

$$(T(t)\, f)(x) = C_{t,x}(f(\phi_t(x))) \qquad \forall f \in C(M, \mathcal{E}),\ \forall x \in K(T(t)). \tag{7}$$

If $t = 0$, then $K(I) = M$, $\phi_0 = I$ and $C_{0,x} = I$ for all $x \in M$.

If $t \geqslant 0$, for all $x \in M$ $\|C_{t,x}\| \leqslant 1$, and, if $x \in \overline{K(T(t))}$, $C_{t,x}$ is a linear isometry of $\mathcal{E}$.

Lemma 7: *Let* $t, s \geqslant 0$ *and* $x \in M$. *If* $x \in K(T(t))$ *and* $\phi_t(x) \in K(T(s))$, *then* $x \in K(t+s)$. *If* $x \in K(T(t)) \cap K(T(t+s))$, *then* $\phi_t(x) \in K(T(s))$.

Proof: If $\phi_t(x) \in K(T(s))$, then $x \in K(T(t)) \cap \phi_t^{-1}(K(T(s)))$ and, for all $f \in C(M, \mathcal{E})$,

$$\begin{aligned}(T(t+s)\, f)(x) &= (T(t) \circ T(s)\, f)(x) = C_{t,x}((T(s)\, f)(\phi_t(x))) = \\ &= C_{t,x} \circ C_{s,\phi_t(x)}(f(\phi_s \circ \phi_t(x))) \\ &= C_{t,x} \circ C_{s,\phi_t(x)}(f(z)),\end{aligned} \tag{8}$$

where $z = (\phi_s \circ \phi_t)(x)$. If $f(z) = \|f\|_{C(M,\mathcal{E})}\, \xi$, with $\|\xi\|_{\mathcal{E}} = 1$, then

$$\|T(t+s)\, f(x)\|_{\mathcal{E}} = \|f(z)\|_{\mathcal{E}} = \|f\|_{C(M,\mathcal{E})} = \|T(t+s)\, f\|_{C(M,\mathcal{E})}.$$

Therefore $x \in K(T(t+s))$ and

$$T(t+s)\, f(x) = C_{t+s,x}(f(\phi_{t+s}(x))). \tag{9}$$

Choosing $f = \underline{\xi}$, for any $\xi \in \mathcal{E}$, (8) and (9) yield

$$\begin{aligned}C_{t+s,x}(\xi) &= T(t+s)\, \underline{\xi}(x) \\ &= C_{t,x} \circ C_{s,\phi_t(x)}(\xi),\end{aligned}$$

whence

$$C_{t+s,x} = C_{t,x} \circ C_{s,\phi_t(x)} \qquad \forall t, s \in \boldsymbol{R}_+, \tag{10}$$

and therefore

$$f(\phi_{t+s}(x)) = f(\phi_s \circ \phi_t(x)) \qquad \forall f \in C(M, \mathcal{E}).$$

If $x \in K(T(t)) \cap K(T(t+s))$, then

$$C_{t+s,x}(f(\phi_{t+s}(x))) = (T(t+s)\,f)(x) = (T(t) \circ T(s)\,f)(x)$$
$$= C_{t,x}((T(s)\,f)(\phi_t(x))).$$

Letting $z = \phi_{t+s}(x)$, if $f(z) = \|f\|_{C(M,\mathcal{E})}\,\xi$, with $\|\xi\|_{\mathcal{E}} = 1$, then

$$\|(T(s)\,f)(\phi_t(x))\|_{\mathcal{E}} = \|C_{t+s,x}(f(\phi_{t+s}(x)))\|_{\mathcal{E}} = \|(T(t+s)\,f)(x)\|_{\mathcal{E}}$$
$$= \|f(z)\|_{\mathcal{E}} = \|f\|_{C(M,\mathcal{E})} = \|T(t+s)\,f\|_{C(M,\mathcal{E})},$$

and therefore $\phi_t(x) \in K(T(s))$. ■

Corollary 2: *If* $t, s \geq 0$,

$$K(T(t)) \cap K(T(t+s))) = \phi_t^{-1}(K(T(s))),$$

and $\phi_{t+s} = \phi_s \circ \phi_t$ *on* $\phi_t^{-1}(K(T(s)))$.

In general, the family $\{K(T(t)) : t > 0\}$ is not increasing, as the following lemma shows.

Lemma 8: *If*

(11) $$K(T(t)) \subset K(T(t+s))$$

for some $t \geq 0$ *and some* $s > 0$, *then* $K(T(r)) = M$ *for all* $r \geq 0$.

Proof: If (11) holds for some $t \geq 0$ and some $s > 0$, then

$$K(T(t)) = K(T(t)) \cap K(T(t+s)) = \phi_t^{-1}(K(T(s))),$$

and therefore

$$M = \phi_t(K(T(t))) = K(T(s)).$$

Hence, if $0 < l < s$ and $r = s - l$, then

$$K(T(r)) = K(T(r)) \cap K(T(s)) = K(T(r)) \cap K(T(r+l))$$
$$= \phi_r^{-1}(K(T(l))),$$

and therefore

$$M = \phi_r(K(T(r))) = K(T(l)),$$

showing that, if $K(T(s)) = M$ for some $s > 0$, then $K(T(r)) = M$ for all $r \in [0, s]$.

Let

$$s_0 = \sup\{s \geq 0 : K(T(s)) = M\}.$$

If $0 < s_0 < \infty$, there are t, s, with $0 < t < s_0$ and $0 < s < s_0$, such that $t + s > s_0$.

Then $K(T(t)) = M = K(T(s))$, and therefore

$$K(T(t+s)) = K(T(t)) \cap K(T(t+s)) = \phi_t^{-1}(K(T(s)))$$
$$= \phi_t^{-1}(M) = K(T(t)) = M\,.$$

This contradiction shows that either $s_0 = 0$ or $s_0 = +\infty$, and completes the proof of the lemma. ■

If (11) holds for some $t \geqslant 0$ and some $s > 0$, (7) holds for all $t \geqslant 0$, $f \in C(M)$, $x \in M$.

Let $n > 1$ and let $t_j > 0$ for $j = 1, 2, \ldots, n$. Then

(12) $$K(T(t_1)) \cap K(T(t_1 + t_2)) \cap \ldots \cap K(T(t_1 + t_2 + \ldots + t_n)) =$$
$$(K(T(t_1)) \cap K(T(t_1 + t_2))) \cap (K(T(t_1)) \cap K(T(t_1 + t_2 + t_3))) \cap \ldots \cap$$
$$(K(T(t_1))) \cap K(T(t_1 + t_2 + \ldots + t_n))) = \phi_{t_1}^{-1}(K(T(t_2))) \cap$$
$$\phi_{t_1}^{-1}(K(T(t_2 + t_3))) \cap \ldots \cap \phi_{t_1}^{-1}(K(T(t_2 + \ldots + t_n))) =$$
$$\phi_{t_1}^{-1}(K(T(t_2)) \cap K(T(t_2 + t_3)) \cap \ldots \cap K(T(t_2 + \ldots + t_n))) =$$
$$\phi_{t_1}^{-1} \circ \phi_{t_2}^{-1}(K(T(t_3)) \cap \ldots \cap K(T(t_3 + \ldots + t_n))) = \ldots =$$
$$\phi_{t_1}^{-1} \circ \phi_{t_2}^{-1} \circ \ldots \circ \phi_{t_{n-1}}^{-1}(K(T(t_n))) \neq \emptyset.$$

Lemma 9: *The set*

$$\bigcap \{\overline{K(T(t))} : t \geqslant 0\}$$

is compact and non-empty.

Proof: By the chain of equalities above, the family $\{\overline{K(T(t))} : t \geqslant 0\}$ of closed subsets of the compact *space* M has the finite intersection property. ■

Corollary 3: *If $K(T(t))$ is closed for all $t \in \boldsymbol{R}_+$, the set*

(13) $$K_\infty(T) = \bigcap \{K(T(t)) : t \geqslant 0\}$$

is compact and non-empty.

The fact that the set $K_\infty(T)$ is non-empty follows from weaker conditions.

Theorem 5: *If there is some $s > 0$ such that $K(T(t))$ is closed whenever $0 \leqslant t \leqslant s$, the set $K_\infty(T)$ defined by* (13) *is non-empty.*

PROOF: Consider the set (12), where $t_p > 0$ for $p = 1, 2, \ldots, n$. Letting $t_p = q_p s + r_p$, with $q_p \in \boldsymbol{Z}_+$ and $0 \leqslant r_p < s$ for $p = 1, 2, \ldots, n$, the set (12) contains the set

$$G(t_1, \ldots, t_n) := K(T(t_1)) \bigcap_{p=2}^{n} \left(\bigcap_{j=0}^{q_p} K(T(t_1 + \ldots + t_{p-1} + js)) \bigcap K(T(t_1 + \ldots + t_p)) \right),$$

which — as was noticed before — is not empty. Since

$$K(T(t_1 + \ldots + t_{p-1} + (j-1)\, s)) \bigcap K(T(t_1 + \ldots + t_{p-1} + js)) =$$

$$\phi_{t_1 + \ldots + t_{p-1} + (j-1)s}{}^{-1}(K(T(s)))$$

and

$$K(T(t_1 + \ldots + t_{p-1} + q_p s) \bigcap K(T(t_1 + \ldots + t_p) =$$

$$K(T(t_1 + \ldots + t_{p-1} + q_p s) \bigcap K(T(t_1 + \ldots + t_{p-1} + q_p s + r_p) =$$

$$\phi_{t_1 + \ldots + t_{p-1} + q_p s}{}^{-1}(K(T(r_p))),$$

the set $G(t_1, \ldots, t_n)$ is closed. By the finite intersection property, the intersection of all sets $G(t_1, \ldots, t_n)$ is not empty. Hence $K_\infty(T)$ is not empty. ∎

As a consequence of Proposition 2, the following lemma holds.

LEMMA 10: *If there is some $t_0 > 0$ such that the map $x \mapsto C_{t,x}$ of M into $\mathcal{L}(\mathcal{E})$ is continuous for the uniform operator topology whenever $t \in [0, t_0]$, then $K_\infty(T) \neq \emptyset$. If the hypothesis holds for all $t > 0$, $K_\infty(T)$ is also closed.*

Corollary 1 yields

COROLLARY 4: *If $\dim_C \mathcal{E} < \infty$, $K_\infty(T)$ is closed and non-empty.*

Let $K_\infty(T)$ be non-empty.
Since $K(T(s)) = \phi_s^{-1}(M)$, for all $s \geqslant 0$

$$\phi_t^{-1}(K_\infty(T)) = \phi_t^{-1}(\bigcap\{K(T(s)) : s \geqslant 0\}) = \bigcap\{\phi_t^{-1}(K(T(s))) : s \geqslant 0\}$$

$$= \bigcap\{K(T(t+s)) : s \geqslant 0\} = \bigcap\{K(T(s)) : s \geqslant t\} \supset$$

$$\supset \bigcap\{K(T(s)) : s \geqslant 0\}) = K_\infty(T),$$

and therefore

$$\phi_t(K_\infty(T)) \subset K_\infty(T) \qquad \forall\, t \geqslant 0\,. \tag{14}$$

REMARK: The set $K_\infty(T)$ — if non-empty — is the largest subset of M which is ϕ_t-invariant for all $t \geqslant 0$. Let $x \in M$. Then $x \in \phi_t^{-1}(K_\infty(T)) \setminus K_\infty(T)$ for some $t > 0$ if, and only if,

$$x \in K(T(t)) \cap K(T(t+s)) \qquad \forall s \geqslant 0\,,$$

i.e.,

$$x \in K(T(s)) \qquad \forall s \geqslant t\,,$$

and moreover

$$x \notin K(T(r)) \qquad \text{for some } r \in (0, t)\,.$$

Hence

$$\phi_t^{-1}(K_\infty(T)) \setminus K_\infty(T) \subset \bigcap \{K(T(s)) : s \geqslant t\} \setminus K(T(r)) \tag{15}$$

for some $r \in (0, t)$.

If

$$K(T(t)) \subset K_\infty(T) \tag{16}$$

for some $t > 0$, then $K(T(s)) \supset K(T(t))$ for all $s > 0$, and Lemma 8 yields

THEOREM 6: *If, and only if,* (16) *holds for some* $t > 0$, *then* $K_\infty(T) = M$, *and* (7) *holds for all* $t \geqslant 0$.

Let $K_\infty(T)$ be closed and non-empty. In view of the ϕ_t-invariance of $K_\infty(T)$, one defines a semigroup $\tilde{T}$: $\boldsymbol{R}_+ \to \mathcal{L}(C(K_\infty(T), \mathcal{E}))$ of linear contractions of $C(K_\infty(T), \mathcal{E})$, by

$$(\tilde{T}(t)\, g)(x) = C_{t,x}(g(\phi_t(x)))$$

for all $t \geqslant 0$, $g \in C(K_\infty(T), \mathcal{E})$, $x \in C(K_\infty(T))$.

4. Let M, N, P be compact Hausdorff spaces, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ be complex Banach spaces, with $\mathcal{F}$, $\mathcal{G}$ strictly convex, and let

$$A \in \mathcal{L}(C(M, \mathcal{E}), C(N, \mathcal{F}))\,, \qquad B \in \mathcal{L}(C(N, \mathcal{F}), C(P, \mathcal{G}))$$

be linear isometries. Then $B \circ A$ is a linear isometry of $C(M, \mathcal{E})$ into $C(P, \mathcal{G})$.

Arguing as in the proof of Lemma 7, one shows that

$$K(B) \cap K(B \circ A) = \psi_B^{-1}(K(A)) \tag{17}$$

and

$$\psi_{B\circ A} = \psi_B \circ \psi_A \qquad \text{on} \qquad \psi_B^{-1}(K(A)).$$

If $M = P$ and $\mathcal{E} = \mathcal{G}$, and if $B \circ A$ is the identity on M, then $K(B \circ A) = P$, and (17) becomes

$$\psi_B(K(B)) = K(A),$$

whence $K(A) = N$. That implies M. Jerison's extension, [10], of the classical Banach-Stone theorem to vector-valued, continuous functions.

Let now $M = N$ and $\mathcal{E} = \mathcal{F}$. By similar arguments to those developed in n. 3, one can handle the discrete case, in which the semigroup T is replaced by the iterates $\{A^n : n \in \mathbf{N}\}$ of an isometry $A \in \mathcal{L}(C(M, \mathcal{E}))$, and the Banach space $\mathcal{E}$ is strictly convex. Assuming in Theorem 3 $N = M$, $\mathcal{E} = \mathcal{F}$, and replacing A by A^n, $K(A)$ by $K(A^n)$, C_y by $C_{A^n, y}$, ψ by ψ_{A^n}, one shows, as in n. 3, that

$$K(A^p) \cap K(A^{p+q}) = \psi_{A^p}^{-1}(K(A^q)).$$

Let $n_1, n_2, \ldots, n_p$ be positive integers. As in n. 3 one proves that

$$K(A^{n_1}) \cap K(A^{n_1+n_2}) \ldots \cap K(A^{n_1+\ldots+n_p}) = \psi_{A^{n_1}}^{-1} \circ \ldots \circ \psi_{A^{n_{p-1}}}^{-1}(K(A^{n_p})) \neq \emptyset, \tag{18}$$

and this shows that

$$\bigcap \{\overline{K(A^n)} : n \in \mathbf{Z}_+\} \neq \emptyset .$$

Since the left-hand side of (18) contains the set

$$\bigcap_{m=1}^{n_1+\ldots+n_p} K(A^m) = \psi_A^{-1} \circ \psi_{A^2}^{-1} \circ \ldots \circ \psi_{A^{n_1+\ldots+n_{p-1}}}^{-1}(K(A))$$

which is (non-empty and) closed when $K(A)$ is closed, the following proposition holds.

Proposition 3: *If $K(A)$ is closed, the set*

$$K_\infty(A) := \bigcap \{K(A^n) : n \in \mathbf{Z}_+\}$$

is non-empty.

Similar arguments as those developed in the proof of Lemma 8 lead to

LEMMA 11: *If*

$$K(A^p) \subset K(A^{p+q})$$

for two positive integers p *and* q, *then* $K(A) = M$.

Arguing as in Theorem 6 one proves

THEOREM 7: *If, and only if,*

$$K(A^p) \subset K_\infty(A)$$

for some $p \geq 0$, *then* $K(A) = M$.

If $\tilde{A} \in \mathcal{L}(C(K_\infty(A), \mathcal{E}))$ is defined by

$$(\tilde{A}g)(x) = C_{A,x}(g(\psi_A(x)))$$

for all $x \in K_\infty(A)$ and all $g \in C(K_\infty(A), \mathcal{E})$, then $\tilde{A}$ is a contraction of $C(K_\infty(A), \mathcal{E})$.

If $A\underline{\xi} = \zeta\underline{\xi}$ for some $\zeta \in \boldsymbol{C}$ and $\xi \in \mathcal{E}\setminus\{0\}$, then $|\zeta| = 1$ and $\tilde{A}\underline{\xi} = \zeta\underline{\xi}$, *i.e.*,

$$C_{A,x}(\xi) = \zeta\xi \qquad \forall x \in K_\infty(A),$$

and viceversa. That proves

LEMMA 12: *Let* $K_\infty(A) \neq \emptyset$. *If, and only if,* ζ *is an eigenvalue of* $C_{A,x}$ *with an eigenvector* $\xi \in \mathcal{E}\setminus\{0\}$ *for all* $x \in K_\infty(A)$, *then* $|\zeta| = 1$ *and* ζ *is an eigenvalue of* $\tilde{A}$ *with an eigenvector* $\underline{\xi}$.

Let now

$$(Af)(y) = \zeta f(y) \qquad \forall f \in C(M, \mathcal{E}) \tag{19}$$

and for some $y \in M$ and $\zeta \in \boldsymbol{C}$. Then $|\zeta| \leq 1$. If $f \in F(\xi, y)$ for some $\xi \in \mathcal{E}$ with $\|\xi\|_{\mathcal{E}} = 1$, then

$$\|(Af)(y)\|_{\mathcal{E}} = |\zeta|\,\|f\|_{C(M,\mathcal{E})} = |\zeta|\,\|Af\|_{C(M,\mathcal{E})}.$$

Thus

$$\zeta \in \partial\Delta \Rightarrow y \in K(A),$$

and therefore

$$C_{A,y}(f(\psi_A(y))) = (Af)(y) = \zeta f(y) \qquad \forall f \in C(M, \mathcal{E}).$$

Because $C_{A,y}$ is an isometry, that implies that

$$\|f(\psi_A(y)\|_{\mathcal{E}} = \|f(y)\|_{\mathcal{E}}$$

for all $f \in C(M, \mathcal{E})$, and therefore $\psi_A(y) = y$, proving thereby

PROPOSITION 4: *If* $y \in M$ *and* $\zeta \in \partial\Delta$ *satisfy* (19), *then* $y \in K(A)$, $\psi_A(y) = y$ *and* $C_{A,y} = \zeta I$.

We shall conclude this section with a result on the compression spectrum of A in the case in which $M = N$, $\mathcal{E} = \mathcal{F} = \boldsymbol{C}$ and A is a linear isometry of $C(M)$ onto $C(N)$. Now $K(A) = M$, and A is expressed by (1) for all $y \in M$ and all $f \in C(M)$, with $\alpha \in \Theta(C(M))$ and ψ a homeomorphism of M onto itself.

The compression spectrum of A is, by definition, the point spectrum $p\sigma(A')$ of the dual operator A' of A. If $\zeta \in p\sigma(A')$, there is some $\lambda \in C(M)' \setminus \{0\}$ such that

$$\langle Af, \lambda\rangle = \zeta\langle f, \lambda\rangle \qquad \forall f \in C(M)\,, \tag{20}$$

i.e.,

$$\int \alpha(x)\ f(\psi(x))\ d\lambda(x) = \zeta \int f(x)\ d\lambda(x)$$

for all $f \in C(M)$, where λ has been identified with its representative Borel measure.

This implies, first of all, that $\zeta \neq 0$.

Let $x_0 \in \operatorname{Supp}\lambda$ be such that $\psi(x_0) \notin \operatorname{Supp}\lambda$. Le U be an open neighbourhood of x_0 in M, disjoint from $\operatorname{Supp}\lambda$, and let $V = \psi^{-1}(U)$.

For any $f \in C(M)$ such that $\operatorname{Supp} f \subset U$,

$$\int f(x)\ d\lambda(x) = 0\ ,$$

and therefore

$$\int \alpha(x)\ f(\psi(x))\ d\lambda(x) = 0\ . \tag{21}$$

If $g \in C(M)$ is such that $\operatorname{Supp} g \subset V$, then, setting $f = g \circ \psi^{-1}$, $\operatorname{Supp} f \subset U$, and (21) yields

$$\int \alpha(x)\ g(x)\ d\lambda(x) = 0\ ,$$

showing that $x_0 \notin \operatorname{Supp}\lambda$: which is a contradiction.

Hence, $\psi(\operatorname{Supp}\lambda) \subset \operatorname{Supp}\lambda$, and therefore $\psi(\operatorname{Supp}\lambda) = \operatorname{Supp}\lambda$ because ψ is a homeomorphism. That proves

THEOREM 8: *If* $A \in \mathcal{L}(C(M))$ *is a bijective isometry and if* $\zeta \in p\sigma(A')$, *then*

$\zeta \neq 0$. *Furthermore, the support of any* $\lambda \in C(M)' \setminus \{0\}$ *satisfying* (20), *is* ψ*-invariant.*

As a consequence, if $\operatorname{Supp} \lambda = \{x_0\}$, then x_0 is fixed by ψ. In that case, $\zeta = f(x_0)$.

5. – Applying some of the results of n. 4 to $T(t)$, for any $t > 0$, we see that, if $K(T(t))$ is closed, the set

$$K_\infty(T(t)) := \bigcap \{K(T(nt)) : n \in \mathbf{N}\}$$

is non-empty and $\widetilde{T(t)}$ is a contraction of $C(K_\infty(T(t)), \mathcal{E})$.

LEMMA 13: *If* $(T(\tau)\, f)(x) = \zeta f(x)$ *for some* $\tau > 0$, $x \in M$ *and* $\zeta \in \partial \Delta$, *and for all* $f \in C(M, \mathcal{E})$, *then* $x \in K(T(\tau))$, $\phi_\tau(x) = x$ *and* $C_{\tau, x} = \zeta I$.

COROLLARY 5: *Let* $K(T(\tau))$ *be closed. If* $x \in K_\infty(T)$ *and* $\tau > 0$ *are such that*

$$(\tilde{T}(\tau)\, g)(x) = g(x) \qquad \forall g \in C(K_\infty(T), \mathcal{E})$$

and if, for every $t \in (0, \tau)$ *there is some* $k \in C(K_\infty(T), \mathcal{E})$ *for which*

$$(\tilde{T}(t)\, k)(x) \neq k(x),$$

then $C_{\tau, x} = I$ *and the semiflow* ϕ *is periodic with period* τ *at the point* x.

So far, no hypothesis on the topological structure of the semigroups T and $\tilde{T}$ has been introduced.

Throughout this and the following sections, $K_\infty(T)$ *will be assumed to be closed and non-empty.*

For any $t \geq 0$ and any $x \in K_\infty(T)$,

$$(T(t)\, f)(x) = C_{t, x}(f(\phi_t(x))) = (\tilde{T}\, f_{|K_\infty(T)})(x)$$

for all $f \in C(K_\infty(T), \mathcal{E})$.

Let the semigroup $\tilde{T}$ be strongly continuous.

Since, for any $\xi \in \mathcal{E}$,

$$C_{t, x}(\xi) = (\tilde{T}(t)\, \underline{\xi})(x),$$

the map $(t, x) \mapsto C_{t, x}$ of $\boldsymbol{R}_+ \times K_\infty(T)$ into $\mathcal{L}(\mathcal{E})$ is continuous for the strong operator topology in $\mathcal{L}(\mathcal{E})$.

We will show now that $\phi : t \mapsto \phi_t$ is a continuous semiflow in $K_\infty(T)$, *i.e.*, $(t, x) \mapsto \phi_t(x)$ is a continuous map of $\boldsymbol{R}_+ \times K_\infty(T)$ into $K_\infty(T)$.

If that is not the case, there exist $t_0 \geq 0$, $x_0 \in K_\infty(T)$ and an open neighbourhood U

of $\phi_{t_0}(x_0)$ such that, for every $\delta > 0$ and for every open neighbourhood V of x_0 there are $t \in \boldsymbol{R}_+ \cap (t_0 - \delta, t_0 + \delta)$ and $x \in V$ for which $\phi_t(x) \notin U$. In view of the compactness of $K_\infty(T)$, there are generalized sequences $\{t_j\}$ in $\boldsymbol{R}_+$ and $\{x_j\}$ in $K_\infty(T)$ converging to t_0 and to x_0, such that $\phi_{t_j}(x_j) \notin U$ and that $\{\phi_{t_j}(x_j)\}$ converges to some

(22) $$y_0 \in K_\infty(T) \setminus U .$$

Hence, for any $f \in C(K_\infty(T), \mathcal{E})$,

$$C_{t_0, x_0}(f(\phi_{t_0}(x_0)) = C_{t_0, x_0}(f(y_0)) .$$

The injectivity of C_{t_0, x_0} implies then that $f(\phi_{t_0}(x_0)) = f(y_0)$ for all $f \in C(K_\infty(T), \mathcal{E})$, and therefore $\phi_{t_0}(x_0) = y_0$, contradicting (22) and proving thereby that the semiflow ϕ is continuous.

If $L : \boldsymbol{R}_+ \to \mathcal{L}(C(K_\infty(T), \mathcal{E}))$ is the semigroup defined by the continuous semiflow $t \mapsto \phi_t$ on $K_\infty(T)$; *i.e.*

(23) $$L(t)\, g = g \circ \phi_t$$

for all $t \geqslant 0$ and all $g \in C(K_\infty(T), \mathcal{E})$, then

(24) $$(\tilde{T}(t)\, g)(x) = C_{t, x}((L(t) g)(x)) \qquad \forall\, t \geqslant 0,\ g \in C(K_\infty(T), \mathcal{E}),\ x \in K_\infty(T) .$$

The map $\tilde{T}(t)$ is a linear isometry if, and only if, ϕ_t is surjective. It is easily seen, [18], that the set of all $t > 0$ for which $\tilde{T}(t)$ is an isometry is either $\boldsymbol{R}_+^*$ or the empty set.

If the semigroup T is strongly continuous, Corollary 5 may yield more information on the global behaviour of ϕ_t and $C_{t, x}$. As an example, assume now that M is the unit circle: $M = \partial\Delta$. According to Proposition 3 of [19], if the continuous semiflow ϕ has a periodic point with period $\tau > 0$, then ϕ is periodic with period τ. Hence, the following theorem holds.

THEOREM 9: *Let the semigroup T be strongly continuous. If M is the unit circle and x and τ satisfy the hypotheses of Corollary 5, then ϕ is the restriction to $\boldsymbol{R}_+$ of a continuous periodic flow, and T is the restriction to $\boldsymbol{R}_+$ of a strongly continuous periodic group $\boldsymbol{R} \times C(\partial\Delta, \mathcal{E}) \to C(\partial\Delta, \mathcal{E})$ of surjective linear isometries of $C(\partial\Delta, \mathcal{E})$.*

For any $t \in \boldsymbol{R}$ and $g \in C(\partial\Delta, \mathcal{E})$, $x \in \partial\Delta$, $T(t)\, g$ is expressed by

$$(T(t)\, g)(x) = C_{t, x}(g(\phi_t(x))) ,$$

where, $C_{t, x}$ is invertible in $\mathcal{L}(C(M, \mathcal{E}))$ for all $t \in \boldsymbol{R}$, and, if $t \leqslant 0$, $C_{t, x}$ is expressed by

$$C_{t, x} = C_{-t, \phi_t(x)}{}^{-1} .$$

Going back to the general case of $C(M, \mathcal{E})$, since $K_\infty(T)$ is closed and non-empty, the contraction semigroup $\tilde{T}$ acting on the Banach space $C(K_\infty(T), \mathcal{E})$ is strongly continuous, its infinitesimal generator $\tilde{X}: \mathcal{D}(\tilde{X}) \subset C(K_\infty(T), \mathcal{E}) \to C(K_\infty(T), \mathcal{E})$ is m-dissipative.

If the semigroup T is strongly continuous — in which case its infinitesimal generator $X : \mathcal{D}(X) \subset C(M, \mathcal{E}) \to C(M, \mathcal{E})$ is conservative and m-dissipative, [16] — also $\tilde{T}$ is strongly continuous.

The space $\tilde{\mathcal{A}}$ consisting of the restrictions to $K_\infty(T)$ of the elements of $\mathcal{D}(X)$ is contained in $\mathcal{D}(\tilde{X})$. Hence, if Y is the linear operator with domain $\mathcal{D}(Y) = \tilde{\mathcal{A}}$ defined on the restriction to $K_\infty(T)$ of any $f \in \mathcal{D}(X)$ by

$$(Yf_{|K_\infty(T)})(x) = (Xf)(x) \qquad \forall\, x \in K_\infty(T),$$

then $Y \subset \tilde{X}$.

Because $T(t)\,\mathcal{D}(X) \subset \mathcal{D}(X)$, then

$$\tilde{T}(t)\ \mathcal{D}(Y) \subset \mathcal{D}(Y).$$

Since $\mathcal{D}(X)$ is dense in $C(M, \mathcal{E})$, if the space $C(M, \mathcal{E})_{|K_\infty(T)}$ of the restrictions to $K_\infty(T)$ of all $f \in C(M, \mathcal{E})$ is dense in $C(K_\infty(T), \mathcal{E})$, then $\tilde{\mathcal{A}}$ is dense in $C(K_\infty(T), \mathcal{E})$. Thus $\tilde{\mathcal{A}} = \mathcal{D}(Y)$ is a core of $\tilde{X}$, and the following lemma holds.

LEMMA 14: *If $C(M, \mathcal{E})_{|K_\infty(T)}$ is dense in $C(K_\infty(T), \mathcal{E})$, the operator $\tilde{X}$ is the closure of Y.*

If $\tilde{T}$ is strongly continuous, also the semigroup L is strongly continuous. Denoting by $D : \mathcal{D}(D) \subset C(K_\infty(T), \mathcal{E}) \to C(K_\infty(T), \mathcal{E})$, the infinitesimal generator of L, then, for any $\xi \in \mathcal{E}$, $\underline{\xi} \in \mathcal{D}(D)$ and $D\underline{\xi} = 0$.

The space $C(K_\infty(T), \mathcal{E})$ is a module over the ring $C(K_\infty(T))$ of all complex-valued continuous functions on $K_\infty(T)$. The infinitesimal generator D_0 of the Markov lattice semigroup L_0 defined in $C(K_\infty(T))$ by the semiflow ϕ is a derivation $D_0 : \mathcal{D}(D_0) \subset C(K_\infty(T)) \to C(K_\infty(T))$. If $\varphi \in \mathcal{D}(D_0)$ and $f \in \mathcal{D}(D)$, then $\varphi f \in \mathcal{D}(D)$ and

$$D(\varphi f) = D_0\varphi \cdot f + \varphi \cdot Df.$$

Hence, if $\xi \in \mathcal{E}$,

$$D(\varphi\,\underline{\xi}) = D_0\varphi \cdot \underline{\xi}.$$

Since all non-trivial derivations in $C(K_\infty(T))$ are unbounded [3], and since D is closed, the following lemma holds.

LEMMA 15: *If $\mathcal{D}(D) = C(K_\infty(T), \mathcal{E})$, then $D = 0$.*

(3) See [12], or also [17] for a direct proof.

For all $t > 0$ and all $g \in C(K_\infty(T), \mathcal{E})$,

$$\frac{1}{t}(\widetilde{T}(t)\, g - g)(x) = \frac{1}{t}(C_{t,x} - I)((L(t)\, g)(x)) + \frac{1}{t}((L(t) - I)\, g)(x)\,.$$

Hence, if $g \in \mathcal{D}(\widetilde{X}) \cap \mathcal{D}(D)$, the limit

$$\lim_{t \downarrow 0} \frac{1}{t}(C_{t,x} - I)((L(t)\, g)(x)) = \lim_{t \downarrow 0} \frac{1}{t}(C_{t,x} - I)(g(x))\,,$$

exists for all $x \in K_\infty(T)$, and

$$(\widetilde{X}\, g)(x) = \lim_{t \downarrow 0} \frac{1}{t}(C_{t,x} - I)(g(x)) + (Dg)(x)\,. \tag{25}$$

In particular, letting

$$\mathcal{K} = \{\xi \in \mathcal{E} : \underline{\xi} \in \mathcal{D}(\widetilde{X})\}\,,$$

then

$$(\widetilde{X}\underline{\xi})(x) = \lim_{t \downarrow 0} \frac{1}{t}(\widetilde{T}(t)\, \underline{\xi} - \underline{\xi})(x) = \lim_{t \downarrow 0} \frac{1}{t}(C_{t,x} - I)(\xi) \tag{26}$$

for all $\xi \in \mathcal{K}$ and all $x \in K_\infty(T)$.

Since $\widetilde{X}$ is closed and also the image $\underline{\mathcal{K}}$ of $\mathcal{K}$ in $C(K_\infty(T), \mathcal{E})$ by the map $\xi \mapsto \underline{\xi}$ is a closed subspace of $\mathcal{D}(\widetilde{X})$, the operator $\widetilde{X}_{|\underline{\mathcal{K}}}$ is closed. As a consequence:

LEMMA 16: *If $\widetilde{T}$ is strongly continuous, for every $x \in K_\infty(T)$ the linear operator*

$$Z_x : \mathcal{D}(Z_x) = \mathcal{K} \subset \mathcal{E} \to \mathcal{E}$$

defined by

$$Z_x \xi = (\widetilde{X}\underline{\xi})(x)$$

is closed [4].

[4] Here is a direct proof. Let $\xi \in \mathcal{D}(Z_x)$ and let $\{\xi_n\}$ be a sequence in $\mathcal{D}(Z_x)$, converging to ξ and such that $\{Z_x \xi_n\}$ converges to some $\eta \in \mathcal{E}$. Since the sequences $\{\underline{\xi_n}\}$ and $\{\underline{Z_x \xi_n}\} = \{\widetilde{X}\underline{\xi_n}\}$ in $C(M, \mathcal{E})$ converge respectively to $\underline{\xi}$ and to $\underline{\eta}$, then $\underline{\xi} \in \mathcal{D}(\widetilde{X})$ and $\underline{\eta} = \widetilde{X}\underline{\xi}$, *i.e.*, $\xi \in \mathcal{D}(Z_x)$ and $\eta = Z_x \xi$.

Let $g \in \mathcal{D}(\widetilde{X}) \cap \mathcal{D}(D)$. Since $g(x) \in \mathcal{K}$, (25) yields

$$(\widetilde{X} g)(x) = Z_x(g(x)) + (Dg)(x) \tag{27}$$

for all $x \in K_\infty(T)$.

If $\mathcal{K} = \mathcal{E}$, that is, if $\underline{\xi} \in \mathcal{D}(\widetilde{X})$ for all $\xi \in \mathcal{E}$, then $\underline{g(x)} \in \mathcal{D}(\widetilde{X})$, and the following lemma holds.

Lemma 17: *If* $\mathcal{K} = \mathcal{E}$, *then* $Z_x \in \mathcal{L}(\mathcal{E})$, $\mathcal{D}(D) = \mathcal{D}(\widetilde{X})$ *and* (27) *holds for all* $g \in \mathcal{D}(D)$ *and all* $x \in K_\infty(T)$.

Since the closed operator X is densely defined, conservative and m-dissipative, its spectrum $\sigma(X)$ is non-empty, [16] (5). Either $\sigma(X)$ is the closed left half-plane $\{\zeta \in \boldsymbol{C} : \Re\zeta \leqslant 0\}$, or $\sigma(X)$ is contained in the imaginary axis: in which case T is the restriction to $\boldsymbol{R}_+$ of a strongly continuous group of surjective linear isometries of $C(M, \mathcal{E})$ (and $K_\infty(T) = M$).

If T is an eventually differentiable semigroup, according to a theorem of A. Pazy (see [11], Theorem 4.7, pp. 54-57), there are $a \in \boldsymbol{R}$ and $b \in \boldsymbol{R}_+^*$ such that the resolvent set of X contains the set

$$\{\zeta \in \boldsymbol{C} : \Re\zeta \geqslant a - b \log |\Im\zeta|\}.$$

As a consequence, the first of the two possibilities listed above is ruled out, and $\sigma(X)$ turns out to be a compact subset of the imaginary axis. But then (see [5], Corollary 8.20), $X \in \mathcal{L}(C(M, \mathcal{E}))$. Hence $\mathcal{D}(X) = C(M, \mathcal{E})$, and (25) — which holds (with $\widetilde{X}$ replaced by X) for all $g \in C(M, \mathcal{E})$ and at all $x \in M$ — yields: $\mathcal{D}(D) = C(M, \mathcal{E})$. Thus, by Lemma 15 the following proposition holds.

Proposition 5: *If T is an eventually differentiable semigroup, there is a conservative operator* $X \in \mathcal{L}(C(M, \mathcal{E}))$ *such that T is the restriction to* $\boldsymbol{R}_+$ *of the group* $G : \boldsymbol{R} \to \mathcal{L}(C(M, \mathcal{E}))$ *of surjective linear isometries defined by*

$$(G(t)\, f)(x) = ((\exp tX)\, f))(x)$$

for all $f \in C(M, \mathcal{E})$, $t \in \boldsymbol{R}$ *and* $x \in M$.

Remark: The same argument as before shows, more in general, that any strongly continuous, eventually differentiable semigroup of linear isometries of a complex Banach space $\mathcal{F}$ is the restriction to $\boldsymbol{R}_+$ of a strongly continuous group of surjective linear isometries of $\mathcal{F}$.

(5) We correct a misprint in [16], where the inclusion $r(X) \subset \Pi_r$ displayed at p. 309, shall be replaced by $r(X) \supset \Pi_r$.

6. Since, for $t \geqslant 0$ and $h > 0$,

$$C_{t+h,\,x} = C_{t,\,x} \circ C_{h,\,\phi_t(x)},$$

then, for any $\xi \in \mathcal{H}$, (25) yields

$$\lim_{h \downarrow 0} \frac{1}{h}(C_{t+h,\,x} - C_{t,\,x})(\xi) = C_{t,\,x} \circ \lim_{h \downarrow 0} \frac{1}{h}(C_{h,\,\phi_t(x)} - I)(\xi)$$

$$= C_{t,\,x}((\widetilde{X}\underline{\xi})(\phi_t(x))) = C_{t,\,x}(Z_{\phi_t(x)}(\xi)).$$

Hence, the map $t \mapsto C_{t,\,x}(\xi)$ of $\boldsymbol{R}_+$ into $\mathcal{E}$ is of class C^1 on $\boldsymbol{R}_+$, and

$$\frac{d}{dt} C_{t,\,x}(\xi) = C_{t,\,x}(\widetilde{X}(\underline{\xi})(\phi_t(x))) = C_{t,\,x}(Z_{\phi_t(x)}(\xi)) \tag{28}$$

for all $x \in K_\infty(T)$ and all $\xi \in \mathcal{H}$.

For $t \geqslant 0$, let

$$A(t) : \mathcal{D}(A(t)) \subset \mathcal{L}(C(K_\infty(T), \mathcal{E}), \mathcal{E}) \longrightarrow \mathcal{L}(C(K_\infty(T), \mathcal{E}), \mathcal{E})$$

be the linear operator defined on

$$\mathcal{D}(A(t)) = \mathcal{L}(\widetilde{X}(\underline{\mathcal{H}}), \mathcal{E})$$

by

$$(A(t)\, R)(\xi) = R(\widetilde{X}(\underline{\xi})),$$

i.e.

$$((A(t)\, R)(\xi))_x = (R(\widetilde{X}(\underline{\xi})))_x = R_x(Z_{\phi_t(x)}(\xi)),$$

where $R \in \mathcal{L}(\widetilde{X}(\mathcal{H}), \mathcal{E}))$.

Let $C_t \in C(\overline{M}, \mathcal{L}(\mathcal{E}))$ be defined by

$$C_t \colon x \mapsto C_{t,\,x}.$$

Then (28) yields the initial value problem

$$\begin{cases} \dfrac{d}{dt} C_t = A(t)\, C_t \\ C_0 \quad = I, \end{cases}$$

i.e.,

$$\begin{cases} \left(\dfrac{d}{dt} C_t\right)_x = C_{t,x}(Z_{\phi_t(x)}(\xi)) \\ C_{0,x} \qquad = I \end{cases}$$

for all $t \in \boldsymbol{R}_+$, $x \in K_\infty(T)$, $\xi \in \mathcal{K}$.

As before, let $\mathcal{E}$ be strictly convex and let $T : \boldsymbol{R} \to \mathcal{L}(C(M), \mathcal{E})$ be a strongly continuous group of linear isometries of $C(M, \mathcal{E})$. Then $K_\infty(T) = M$, and T is expressed by

$$(T(t)\, f)(x) = C_{t,x}(f(\phi_t(x)))$$

for all $f \in C(M, \mathcal{E})$, $x \in M$, $t \in \boldsymbol{R}$, where $\phi : t \mapsto \phi_t$ is a continuous flow on M, and $C_{t,x} \in \mathcal{L}(\mathcal{E})$ is a surjective isometry such that

$$C_{t+s,x} = C_{t,x} \circ C_{s,\phi_t(x)} \qquad \forall\, t, s \in \boldsymbol{R},\ x \in M .$$

Suppose now that M is a compact differentiable (*i.e.* C^∞) manifold, and that the flow ϕ is determined by a C^∞ vector field v on M. For any $f \in C^1(M, \mathcal{E})$ we define $v(f) \in C(M, \mathcal{E})$ componentwise; that is to say, setting for $x \in M$ and $\lambda \in \mathcal{E}'$,

$$\langle (v(f))(x), \lambda \rangle = (v(\langle f(\cdot), \lambda \rangle))(x) .$$

Clearly

$$f \in C^\infty(M, \mathcal{E}) \Rightarrow v(f) \in C^\infty(M, \mathcal{E}) .$$

If $L : \boldsymbol{R} \to \mathcal{L}(C(M, \mathcal{E}))$ is the group defined by (23) for all $t \in \boldsymbol{R}$ and all $g \in C(M, \mathcal{E})$, and if D is its infinitesimal generator, then

$$C^\infty(M, \mathcal{E}) \subset \mathcal{D}(D)$$

and

$$D(f) = v(f) \qquad \forall f \in C^\infty(M, \mathcal{E}) .$$

LEMMA 18: *If the map $x \mapsto C_{t,x}$ of M into $\mathcal{L}(\mathcal{E})$ is of class C^∞ for all $t \in \boldsymbol{R}$, tha map $t \mapsto C_{t,x}$ is of class C^∞ on $\boldsymbol{R}$ for all $x \in M$.*

PROOF: For $t_0 \in \boldsymbol{R}$ and $r > 0$, let $\varrho : \boldsymbol{R} \to [0, 1]$ be a C^∞ function for which

$$\begin{aligned} &\varrho(t) = 1 && \text{if } |t - t_0| \leq r \\ &0 < \varrho(t) < 1 && \text{if } r < |t - t_0| < 2r \\ &\varrho(t) = 0 && \text{if } |t - t_0| \geq 2r . \end{aligned}$$

Then

$$\int_{-\infty}^{+\infty} \varrho(s)\, C_{t+s,\,x}\, ds = C_{t,\,x}\left(\int_{-\infty}^{+\infty} \varrho(s)\, C_{s,\,\phi_t(x)}\, ds\right),$$

i.e.,

$$\int_{-\infty}^{+\infty} \varrho(s-t)\, C_{s,\,x}\, ds = C_{t,\,x}\left(\int_{-\infty}^{+\infty} \varrho(s)\, C_{s,\,\phi_t(x)}\, ds\right).$$

A neighbourhood U of t_0 in $\boldsymbol{R}$ and $r > 0$ can be so chosen that

$$\int_{-\infty}^{+\infty} \varrho(s)\, C_{s,\,\phi_t(x)}\, ds \neq 0$$

whenever $t \in U$.

Differentiation with respect to $t \in U$ shows that the function $t \mapsto C_{t,\,x}$ is of class C^1 on U for all $x \in M$, and

$$-\int_{-\infty}^{+\infty} \left(\frac{d\varrho}{dt}\right)(s-t)\, C_{s,\,x}\, ds = \frac{\partial}{\partial t} C_{t,\,x}\left(\int_{-\infty}^{+\infty} \varrho(s)\, C_{s,\,\phi_t(x)}\, ds\right) +$$

$$+ C_{t,\,x}\left(\int_{-\infty}^{+\infty} \varrho(s)\, v(C_{s,\,\phi_t(x)})\, ds\right).$$

Iteration of this computation completes the proof of the lemma. ∎

Thus, $Z_x \in \mathcal{L}(\mathcal{E})$ for all $x \in M$, and

$$Z_x = \frac{d}{dt} C_{t,\,x}. \tag{29}$$

By the same argument leading to Theorem 4 of [17] one proves then

Theorem 10: *If the strongly continuous group* $T : \boldsymbol{R} \to \mathcal{L}(C(M, \mathcal{E})$ *of linear isometries is such that*

$$T(t)\, C^\infty(M, \mathcal{E}) \subset C^\infty(M, \mathcal{E}) \qquad \forall\, t \in \boldsymbol{R}\,,$$

then: $\mathcal{D}(D) = \mathcal{D}(X)$; (27) *holds for all* $g \in \mathcal{D}(X)$ *and all* $x \in M$, *where* Z_x *is expressed by* (29), *and* $C^\infty(M, \mathcal{E})$ *is a core for* X.

7. If $\dim \mathcal{E} < \infty$ and $\dim \mathcal{F} < \infty$, the sets $K(A)$ and $K(T(t))$ for all $t \geq 0$ are closed, $K_\infty(T)$ is closed and non-empty, the linear isometries $C_{A,\,x}$ and $C_{t,\,x}$ are invertible for all $t \geq 0$.

If the semigroup T (or the semigroup $\tilde{T}$) is strongly continuous, the isometries $C_{t,x}$ are continuous functions of $(t, x) \in \boldsymbol{R}_+ \times M$ (or of $(t, x) \in \boldsymbol{R}_+ \times K_\infty(T)$ respectively).

In the case in which $\mathcal{E} = \mathcal{F} = \boldsymbol{C}$, [9], C_y is represented by a continuous function $\alpha : M \to \partial\Delta$; (4) and Theorem 2 yield

$$\Theta(C(M)) = \{h \in C(M) : |h(x)| = 1 \ \ \forall x \in M\},$$

$$\Theta(C(M)') = \{c\delta_x : c \in \partial\Delta,\ x \in M\}.$$

LEMMA 19: [15] *If* $\lambda \in C(M)'$, *then* $\lambda \in \Theta(C(M)')$ *if, and only if,*

$$|\langle h, \lambda\rangle| = 1$$

for all $h \in \Theta(C(M))$.

Theorem 4 generalizes the second part of the following

THEOREM 11: [15] *If either*

$$A(\Theta(C(M))) \subset \Theta(C(N)), \tag{30}$$

or

$$A'(\Theta(C(N)')) \subset \Theta(C(M)'), \tag{31}$$

then $K(A) = N$, *i.e,*

$$(Af)(y) = \alpha(y) \cdot (f \circ \psi(y)) \qquad \forall y \in K(A), \qquad f \in C(M). \tag{32}$$

PROOF: The theorem is equivalent to the following chain of implications:

$$(30) \Rightarrow (31) \Rightarrow (32) \Rightarrow (30).$$

If (31) holds, for every $y \in N$ there are a unique $x \in M$ and a unique $c \in \partial\Delta$ for which

$$A'\delta_y = c\delta_x,$$

i.e.,

$$(Af)(y) = cf(x)$$

for all $f \in C(M)$. Setting $c = \alpha(y)$ and $x = \psi(y)$, (32) follows.

If (30) holds, then, for every $y \in N$ and all $h \in \Theta(M)$,

$$1 = |(Ah)(y)| = |\langle Ah, \delta_y\rangle| = |\langle h, A'\delta_y\rangle|,$$

and therefore, by Lemma 19, (31) holds.

Viceversa, if (32) is satisfied, with $\alpha \in \Theta(N)$ and ψ a continuous surjective map of N onto M, then (30) holds. ■

By the Tietze extension theorem, Lemma 14 yields

PROPOSITION 6: *If* $\dim_C \mathcal{E} < \infty$, *the operator* $\tilde{X}$ *is the closure of* Y.

We consider now the strongly continuous semigroup $T : \boldsymbol{R}_+ \to \mathcal{L}(C(M))$ of linear isometries of $C(M)$, and the strongly continuous semigroup $\tilde{T}: \boldsymbol{R}_+ \to \mathcal{L}(C(K_\infty(T)))$ expressed on any $g \in C(K_\infty(T))$ by

$$(\tilde{T}(t)\, g)(x) = \alpha_t(x)\, g(\phi_t(x)),$$

where $\alpha_t \in \Theta(C(K_\infty(T)))$ is a continuous function of t, and $\phi : t \mapsto \phi_t$ is a continuous semiflow on $K_\infty(T)$.

The existence of fixed points of the semiflow ϕ yields some information on the point spectrum $p\sigma(X)$ and the residual spectrum $r\sigma(X)$ of X, as will be illustrated now in the case $\mathcal{E} = \boldsymbol{C}$.

If $x_0 \in K_\infty(T)$ is fixed by ϕ, *i.e.*,

$$\phi_t(x_0) = x_0 \qquad \forall\, t \geq 0,$$

then

$$(T(t)\, f)(x_0) = \alpha_t(x_0)\, f(\phi_t(x_0)) = \alpha_t(x_0)\, f(x_0) \tag{33}$$

for all $f \in C(M)$, and

$$\alpha_{t+s}(x_0) = \alpha_t(x_0)\, \alpha_s(\phi_t(x_0)) = \alpha_t(x_0)\, \alpha_s(x_0)$$

for all $t, s \geq 0$.

Letting

$$\alpha_{-t}(x_0) = \frac{1}{\alpha_t(x_0)} = \overline{\alpha_t(x_0)},$$

we extend the map $\boldsymbol{R}_+ \ni t \mapsto \alpha_t(x_0)$ to a continuous homomorphism of $\boldsymbol{R}$ into the multiplicative group $\partial\Delta$. Hence there is $a \in \boldsymbol{R}$ such that

$$\alpha_t(x_0) = e^{iat} \tag{34}$$

for all $t \in \boldsymbol{R}$, and therefore (33) becomes

$$(T(t)\, f)(x_0) = e^{iat} f(x_0) \qquad \forall\, t \in \boldsymbol{R}_+,$$

i.e.,

$$\langle (T(t) - e^{iat} I, \delta_{x_0} \rangle = 0 \qquad \forall\, t \in \boldsymbol{R}_+.$$

For any $f \in \mathcal{D}(X)$,

$$(Xf)(x_0) = \langle Xf, \delta_{x_0} \rangle = \lim_{t \downarrow 0} \left\langle \frac{1}{t} (T(t) - I) f, \delta_{x_0} \right\rangle$$

$$= \lim_{t \downarrow 0} \frac{1}{t} (\alpha_t(x_0) f(\phi_t(x_0) - f(x_0)) = \lim_{t \downarrow 0} \frac{1}{t} (\alpha_t(x_0) - 1) f(x_0)$$

$$= \lim_{t \downarrow 0} \frac{1}{t} (e^{iat} - 1) f(x_0) = iaf(x_0) = \langle (X - iaI) f, \delta_{x_0} \rangle .$$

Hence, $ia \in p\sigma(X) \cup r\sigma(X)$.
In conclusion, the following theorem holds.

THEOREM 12: *If $x_0 \in K_\infty(T)$ is fixed by the semiflow ϕ, there is $a \in \boldsymbol{R}$ such that $ia \in p\sigma(X) \cup r\sigma(X)$, and* (34) *holds for all $t \in \boldsymbol{R}_+$.*

If *ia* is an isolated point of $\sigma(X)$, then ([14], p. 178) $ia \in p\sigma(X)$.

REFERENCES

[1] W. ARENDT, *Characterization of positive semigroups on $C_0(X)$*, in R. Nagel (ed), One parameter Semigroups of Positive Operators, Lecture Notes in Mathematics, n. 1184, Springer-Verlag, Berlin/Heidelberg/New york/Tokyo, 1980, 122-162.

[2] S. BANACH, *Théorie des opérations linéaires*, in «Monografje Matematyczne», PWN, Warsaw, 1932; Dover, New York, 1955.

[3] E. BEHRENDS, *M-Structure and the Banach-Stone-Theorem*, Springer-Verlag, Berlin/Heidelberg/New York, 1979.

[4] M. CAMBERN, *A Holsztyński theorem for spaces of continuous vector-valued functions*, Studia Math., **63** (1978), 213-217.

[5] E. B. DAVIES, *One parameter semigroups*, Academic Press, London, 1980.

[6] N. DUNFORD - J. P. SCHWARTZ, *Linear Operators*, I, Interscience, New York, 1958.

[7] K. HOFFMAN, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, 1962.

[8] E. HILLE - R. S. PHILLIPS, *Functional Analysis and Semigroups*, Amer. Math. Soc. Coll. Publ., Vol. **31**, Providence R.I., 1957.

[9] W. HOLSZTYNSKI, *Continuous mappings induced by isometries of spaces of continuous functions*, Studia Math., **26** (1966), 133-136.

[10] M. JERISON, *The space of bounded maps into a Banach space*, Ann. of Math., **52** (1950), 309-327.

[11] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York/Berlin/Heidelberg/Tokyo, 1983.

[12] S. SAKAI, *C*-algebras and W*-algebras*, Springer-Verlag, New York/Heidelberg/Berlin, 1971.

[13] M. H. STONE, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., **41** (1937), 375-481.

[14] J. van Neerven, *The Asymptotic Behaviour of Semigroups of Linear Operators*, Birkhäuser, Verlag, Basel/Boston/Berlin, 1996.
[15] E. Vesentini, *On the Banach-Stone theorem*, Advances in Math., **112** (1995), 135-146.
[16] E. Vesentini, *Conservative operators*, in: P. Marcellini, G. Talenti, E. Vesentini (ed), Partial differential equations and applications, Marcel Dekker, New York/Basel/Hong Kong, 1996, 303-311.
[17] E. Vesentini, *Semiflows and semigroups*, Rend. Mat. Acc. Lincei, (9) **7** (1996), 75-82.
[18] E. Vesentini, *Periodicity and Almost Periodicity in Markov Lattice Semigroups*, Ann. Sc. Norm. Sup. Pisa, (4) **25** (1997), 829-839.
[19] E. Vesentini, *Periodic points and non-wandering points of continuous dynamical systems*, Advances in Math., **134** (1998), 308-327.

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