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Almost Automorphic Groups and Semigroups (**)

Abstract. — Equivalence between almost periodicity and almost automorphy, even in a weak sense, for orbits of a strongly continuous group $\mathcal{U}$ of linear bounded operators on a Banach space X is proved; the semigroup case is treated as well. Moreover, a Stepanov type definition of almost automorphy is introduced. Stability of asymptotic almost automorphy under Miyadera-Voigt perturbations is finally investigated.

Gruppi e semigruppi quasi automorfi

Sunto. — Si dimostra che le nozioni di quasi periodicità e quasi automorfia, anche in senso debole, sono equivalenti per orbite di un gruppo fortemente continuo $\mathcal{U}$ di operatori lineari limitati in uno spazio di Banach X; si considera anche il caso dei semigruppi. Si introduce, inoltre, una nozione di quasi automorfia del tipo di Stepanov. Infine, si studia la stabilità della quasi automorfia asintotica rispetto a perturbazioni nella classe di Miyadera-Voigt.

The notion of almost automorphy, introduced by S. Bochner in 1955 for complex-valued functions defined on the real line, is related to and more general than that of almost periodicity; the curious name of almost automorphy is due to the fact that this notion was firstly encountered in the differential geometric study of automorphic functions on real and complex manifolds [5]. Since then, this notion has been generalized into different directions. First of all, W. A. Veech developed the theory of almost automorphic maps in the more general setting of topological groups [18]; other contributions in this sense came from A. Reich and R. Terras [12], [17]. More recently, vector-valued almost automorphic maps have been introduced by S. Zaidman [22] and M. Zaki [23]. Moreover, the definition of Bochner has been extended to continuous maps defined only on a half-line [10], [11]; in analogy to the classical theory of M.

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Fréchet in the almost periodic case, functions, with almost automorphic behaviour only on a half-line, are called asymptotically almost automorphic. Finally, almost automorphic dynamics in skew-product semiflows, arising in the qualitative study of non autonomous differential equations, is analyzed in a recent work of W. Shen and Y. Yi [14].

Let now $\mathcal{U} := (U(t))_{t\in\mathbb{R}}$ be a strongly continuous group of linear bounded operators acting on a complex Banach space X. Although the notion of almost automorphy is, in general, weaker than that of almost periodicity, it is shown in this note that these notions are equivalent for orbits of $\mathcal{U}$, that is for the maps $t \mapsto U(t)x_0$, x_0 being an element of X. Moreover, by applying the notion of weak almost automorphy introduced in [23], in Section 3 it is proved that a group $\mathcal{U}$ is weakly almost automorphic if and only if it is weakly almost periodic.

Analogous equivalences between asymptotic almost periodicity and asymptotic almost automorphy are proved for orbits of a strongly continuous semigroup $\mathcal{T} := (T(t))_{t\geq 0}$ on X, that is for maps $t \mapsto T(t)\,x_0$, $x_0 \in X$.

Afterwards, a Stepanov type notion of almost automorphy is defined. In 1933 W. Stepanov [15], [16] introduced a class of generalized almost periodic functions, for which continuity may fail; in this note, an analogous class of almost automorphic maps, which are only required to be measurable and locally Lebesgue integrable, is introduced. In Section 4 the equivalence between all these different notions of almost periodicity and almost automorphy for single orbits of $\mathcal{U}$ and $\mathcal{T}$ is proved; this extends and completes some well-known results of H. Henriquez [8], showing that for a strongly continuous semigroup $\mathcal{T}$ the definitions of asymptotic almost periodicity given by M. Fréchet and by W. Stepanov are equivalent.

In the final section stability of asymptotic almost automorphy under Miyadera-Voigt perturbations is analyzed; some recent results [7] on the permanence of asymptotic properties under an additive perturbation of the infinitesimal generator of $\mathcal{T}$ are improved, so to include the almost automorphic case.

1. - Notations

Throughout the paper X will denote a complex Banach space and X' the topological dual of X. The symbol $\mathcal{L}(X)$ will denote the Banach algebra of all linear bounded operators on X.

Let $\mathbb{J}$ denote either $\mathbb{R}$ or $\mathbb{R}_+$ and let $\mathcal{C}_b(\mathbb{J}, X)$ denote the Banach space of all continuous bounded functions from $\mathbb{J}$ to X endowed with the uniform norm

$$\|f\|_{\mathcal{C}_b(\mathbb{J},\,X)} := \sup_{t\in\mathbb{J}} \|f(t)\|_X.$$

The closed subspace of uniformly continuous functions will be denoted by $\mathcal{C}_{ub}(\mathbb{J}, X)$.

Recall that a subset Λ of $\mathbb{R}$ is called *relatively dense* in $\mathbb{R}$ if there exists a number

$l > 0$ such that every interval of length l in $\mathbb{R}$ contains at least one number from Λ. A real number $\tau \in \mathbb{R} \setminus \{0\}$ is called an *ε-period* for $f: \mathbb{R} \to X$ if

$$\|f(t+\tau) - f(t)\| \leq \varepsilon \ \text{ for every } \ t \in \mathbb{R}\ .$$

The function f is called *almost periodic* if for every $\varepsilon > 0$ the set of all ε-periods (which will be denoted by $\mathfrak{T}(f, \varepsilon)$) is relatively dense in $\mathbb{R}$.

According to a classical result of Bochner, a function $f: \mathbb{R} \to X$ is almost periodic if and only if for every sequence $\{\alpha'_n\} \subseteq \mathbb{R}$ there exists a subsequence $\{\alpha_n\}$ such that $\{f(\cdot + \alpha_n)\}$ converges uniformly on $\mathbb{R}$.

A function $f: \mathbb{R}_+ \to X$ is said to be *asymptotically almost periodic* if for every sequence $\{\alpha'_n\} \subseteq \mathbb{R}_+$ there exists a subsequence $\{\alpha_n\}$ such that $\{f(\cdot + \alpha_n)\}$ converges uniformly on $\mathbb{R}_+$.

The following decomposition theorem, due, in the general case, to W. M. Ruess and to W. H. Summers, holds.

THEOREM 1.1 [13]: *A function $f \in \mathcal{C}_b(\mathbb{R}_+, X)$ is asymptotically almost periodic if, and only if, one the following two equivalent conditions is satisfied:*

a) there exist a unique almost periodic function $g: \mathbb{R} \to X$ and a unique $h \in \mathcal{C}_b(\mathbb{R}_+, X)$, vanishing at infinity, such that $f = h + g_{|\mathbb{R}_+}$;

b) for every $\varepsilon > 0$ there exist $\Lambda > 0$ and $K \geq 0$ such that every interval of lenght Λ contains some τ for which

$$\|f(t+\tau) - f(t)\| \leq \varepsilon$$

holds whenever $t, t+\tau \geq K$.

The maps g and h are called, respectively, the *principal term* and the *correction term* of f.

Given a function $f: \mathbb{R} \to X$ and a sequence $\alpha := \{\alpha_n\} \subseteq \mathbb{R}$ such that $\{f(\cdot + \alpha_n)\}$ converges pointwise on $\mathbb{R}$, the value of the limit shall be denoted by $T_\alpha f$.

A continuous map $f: \mathbb{R} \to X$ is called *almost automorphic* if for every sequence $\{\alpha'_n\} \subseteq \mathbb{R}$ there exist a subsequence $\{\alpha_n\}$ and a map $g: \mathbb{R} \to X$, such that

$$T_\alpha f = g \quad \text{ and } \quad T_{\alpha^{-1}} g = f\,, \tag{1.1}$$

where $T_{\alpha^{-1}} g$ denotes the pointwise limit for $n \to +\infty$ of $g(\cdot - \alpha_n)$.

Observe that an almost automorphic function is necessarily bounded. Moreover, since the convergence in (1.1) is only pointwise, the function g, which is bounded, need not be continuous in general. If for every sequence $\{\alpha'_n\}$ the corresponding limit function g is continuous, f is called a *continuous almost automorphic* function. It was proved by Veech that every continuous almost automorphic function is uniformly continuous.

At the light of point *a*) of Theorem 1.1 a continuous function $f: \mathbb{R}_+ \to X$ is called *asymptotically almost automorphic* if there exist a function $h \in \mathcal{C}_b(\mathbb{R}_+, X)$, vanishing at infinity, and an almost automorphic function $g \in \mathcal{C}_b(\mathbb{R}, X)$, such that $f(t) = h(t) + g_{|\mathbb{R}_+}(t)$ for every $t \geqslant 0$.

If f is asymptotically almost automorphic, it follows from the definition that for every sequence $\{\alpha'_n\} \subseteq \mathbb{R}_+$, such that $\alpha'_n \to +\infty$, there exists a subsequence $\{\alpha_n\}$ such that $\{f(\cdot + \alpha_n)\}$ converges pointwise on $\mathbb{R}_+$.

2. - Almost automorphic groups and semigroups

By definition an almost periodic function is necessarily almost automorphic. The converse is, in general, not true [14], [19]; nonetheless, it will be shown that these two notions are equivalent for orbits of a group.

Proposition 2.1: *Let $\mathcal{U} := (U(t))_{t \in \mathbb{R}}$ be a strongly continuous group of linear bounded operators on X. Let x_0 be in X.*

If $t \mapsto U(t)x_0$ is almost automorphic and $\mathcal{U}$ is uniformly bounded, then the map is almost periodic.

Proof: Choose any sequence $\{\alpha'_n\}$ in $\mathbb{R}$. By definition there exist a subsequence $\{\alpha_n\}$ and a function $g : \mathbb{R} \to X$ such that $U(\cdot + \alpha_n)\, x_0 \to g(\cdot)$ and $g(\cdot - \alpha_n) \to U(\cdot) x_0$ pointwise on $\mathbb{R}$.

Let $M > 0$ be such that $\|U(t)\| \leqslant M$ for all $t \in \mathbb{R}$. Since

$$\|U(t + \alpha_n)\, x_0 - U(t + \alpha_m)\, x_0\| = \|U(t)\,(U(\alpha_n)\, x_0 - U(\alpha_m)\, x_0)\| \leqslant M\|U(\alpha_n)\, x_0 - U(\alpha_m)\, x_0\|$$

for every $m, n \in \mathbb{N}$, the subsequence $\{U(\cdot + \alpha_n)\, x_0\}$ converges uniformly on $\mathbb{R}$, whence the thesis follows. ■

An analogous result can be established for orbits of a semigroups.

Proposition 2.2: *Let $\mathcal{T} := (T(t))_{t \geqslant 0}$ be a strongly continuous semigroup of linear bounded operators on X. Let x_0 be in X.*

(1) *If the map, from $\mathbb{R}_+$ to X, $t \mapsto T(t)\, x_0$ is asymptotically almost periodic, then it is asymptotically almost automorphic.*

(2) *If $t \mapsto T(t) x_0$ is asymptotically almost automorphic and $\mathcal{T}$ is uniformly bounded, then the map is asymptotically almost periodic.*

Proof: (1) If the orbit $t \mapsto T(t)\, x_0$ is asymptotically almost periodic, then there exist an almost periodic (and therefore almost automorphic) function $g \in \mathcal{C}_b(\mathbb{R}, X)$ and a function $h \in \mathcal{C}_b(\mathbb{R}_+, X)$, vanishing at infinity, such that $T(t)\, x_0 = h(t) + g_{|\mathbb{R}_+}(t)$ for every $t \geqslant 0$. By definition, $t \mapsto T(t)\, x_0$ is asymptotically almost automorphic.

(2) The proof is very similar to that of Proposition 2.1. ■

In analogy to the almost periodic case, a strongly continuous group (semigroup) is called *strongly almost automorphic* (*strongly asymptotically almost automorphic*) if every orbit is almost automorphic (asymptotically almost automorphic). A standard application of Banach-Steinhaus principle yields the following

COROLLARY 2.3: *A strongly continuous group* $\mathcal{U}$ *is strongly almost periodic if and only if it is strongly almost automorphic.*

A strongly continuous semigroup $\mathcal{T}$ *is strongly asymptotically almost periodic if and only if it is strongly asymptotically almost automorphic.*

REMARK 2.5: Let $(A, D(A))$ be the infinitesimal generator of a semigroup $\mathcal{T}$, such that the orbit $t \mapsto T(t)x_0$ is asymptotically almost automorphic for some $x_0 \in X$. A trivial consequence of the just proved results is that the (possibly mild) solution of the abstract Cauchy problem

$$\begin{cases} x'(t) = Ax(t), & t \geq 0, \\ x(0) = x_0 \end{cases}$$

is asymptotically almost periodic.

Such a result cannot be expected in the inhomogeneous case. Consider the Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + f(t), & t \geq 0, \\ x(0) = x_0, \end{cases}$$

where $(A, D(A))$ generates a strongly almost automorphic group $\mathcal{U}$ on X, $f: \mathbb{R} \to X$ is almost automorphic and $x_0 \in X$. Under these assumptions, the mild solution $U(\cdot)\, x_0 + \int_0^t U(\cdot - s)\, f(s)\, ds$, which is almost automorphic as proved by Zaki [23], is not almost periodic, in general.

3. - WEAKLY ALMOST AUTOMORPHIC GROUPS

A function $f: \mathbb{R} \to X$ is said to be *weakly almost automorphic* (*w.a.a.*) if for every sequence $\{\alpha_n'\} \subseteq \mathbb{R}$ there exist a subsequence $\{\alpha_n\}$ and a map $g: \mathbb{R} \to X$, such that

$$\langle f(t+\alpha_n), \lambda \rangle \to \langle g(t), \lambda \rangle \quad \text{and} \quad \langle g(t-\alpha_n), \lambda \rangle \to \langle f(t), \lambda \rangle \tag{3.1}$$

for every $\lambda \in X'$ and $t \in \mathbb{R}$, when $n \to +\infty$.

A strongly continuous group $\mathcal{U}$ on X is said to be *weakly almost automorphic* if every orbit $t \mapsto U(t)\, x$, $x \in X$, is weakly almost automorphic.

Recall that a function $f: \mathbb{R} \to X$ is said to be *weakly almost periodic* if for every

$\lambda \in X'$ the map, from $\mathbb{R}$ to $\mathbb{C}$, $t \mapsto \langle f(t), \lambda \rangle$ is almost periodic. A strongly continuous group $\mathcal{U}$ is said to be *weakly almost periodic* if for every $x \in X$ the orbit $t \mapsto U(t)\, x$ is weakly almost periodic.

Through the following criterion W. A. Veech characterized complex-valued almost periodic functions on $\mathbb{R}$ in terms of almost automorphic functions.

THEOREM 3.1 [18]: *A function $f: \mathbb{R} \to \mathbb{C}$ is almost periodic if and only if it is almost automorphic and the limit $T_\alpha f$, $\alpha := \{\alpha_n\}$ being any sequence in $\mathbb{R}$, is almost automorphic, whenever it exists.*

THEOREM 3.2: *A strongly continuous group $\mathcal{U}$ of linear bounded operators on X is weakly almost periodic if and only if it is weakly almost automorphic.*

PROOF: Take $x_0 \in X$. Suppose that the map $t \mapsto U(t)\, x_0$ is w.a.a.; this means that for every sequence $\{\alpha'_n\} \subseteq \mathbb{R}$ there exist a subsequence $\{\alpha_n\}$ and a map $g: \mathbb{R} \to X$, such that

$$(3.2) \qquad \langle U(t+\alpha_n)\, x_0, \lambda \rangle \to \langle g(t), \lambda \rangle \quad \text{and} \quad \langle g(t-\alpha_n), \lambda \rangle \to \langle U(t)\, x_0, \lambda \rangle$$

for every $\lambda \in X'$, pointwise on $\mathbb{R}$, when $n \to +\infty$.

It will be shown that $t \mapsto U(t)\, x_0$ is w.a.p. by means of Theorem 3.1.

By hypothesis the function $t \mapsto \langle U(t)\, x_0, \lambda \rangle$ is almost automorphic for every $\lambda \in X'$. Fix any $\lambda_0 \in X'$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} \langle U(t+\alpha_n)\, x_0, \lambda_0 \rangle$ exists; in particular, suppose $\lim_{n\to\infty} \langle U(t+\alpha_n)\, x_0, \lambda_0 \rangle = f(t)$ for some $f: \mathbb{R} \to \mathbb{C}$. Possibly passing to a subsequence, in view of (3.2) one can suppose $\lim_{n\to\infty} \langle U(t+\alpha_n)\, x_0, \lambda_0 \rangle = \langle g(t), \lambda_0 \rangle$, pointwise on $\mathbb{R}$, and $\lim_{n\to\infty} \langle g(t-\alpha_n), \lambda_0 \rangle = \langle U(t)\, x_0, \lambda_0 \rangle$.

It will be proved that $t \mapsto \langle g(t), \lambda_0 \rangle$ is almost automorphic, *i.e.* that for every sequence $\{\beta'_j\} \subseteq \mathbb{R}$ there exist a subsequence $\{\beta_j\}$ and a map $s: \mathbb{R} \to \mathbb{C}$, such that

$$\langle g(t+\beta_j), \lambda_0 \rangle \to s(t) \quad \text{and} \quad s(t-\beta_j) \to \langle g(t), \lambda_0 \rangle$$

pointwise on $\mathbb{R}$, when $j \to \infty$. Observe that g depends on $\{\alpha'_n\}$; in the following, g will be denoted therefore by the symbol $g_{\alpha'}$.

Fix $t \in \mathbb{R}$. As a consequence of the hypotheses on $\mathcal{U}$, the function $\tau \mapsto U(\tau)\, g_{\alpha'}(t)$ is weakly almost automorphic, $g_{\alpha'}(t)$ being an element of X. Thus there exist a subsequence $\{\beta_j\}$ of any given sequence $\{\beta'_j\} \subseteq \mathbb{R}$ and a function $m: \mathbb{R} \to X$, such that

$$(3.3) \qquad \langle U(\tau+\beta_j)\, g_{\alpha'}(t), \lambda \rangle \to \langle m(\tau), \lambda \rangle \quad \text{and} \quad \langle m(\tau-\beta_j), \lambda \rangle \to \langle U(\tau)\, g_{\alpha'}(t), \lambda \rangle$$

for every $\tau \in \mathbb{R}$ and $\lambda \in X'$, when $j \to \infty$. Observe that the function m depends on t, on $\{\alpha'_n\}$ and on $\{\beta'_j\}$; from now on, the notation $m_{\alpha', \beta', t}$ will be adopted. For every $\tau \in \mathbb{R}$ the symbol $U'(\tau)$ will denote the adjoint operator of $U(\tau)$.

If $j \in \mathbb{N}$, it holds

$$\begin{aligned}\langle g_{\alpha'}(t+\beta_j), \lambda_0\rangle &= \lim_{n\to\infty}\langle U(t+\alpha_n+\beta_j)\, x_0, \lambda_0\rangle \\ &= \lim_{n\to\infty}\langle U(t+\alpha_n)\, x_0, U'(\beta_j)\,\lambda_0\rangle \\ &= \langle g_{\alpha'}(t), U'(\beta_j)\lambda_0\rangle \\ &= \langle U(\beta_j)\, g_{\alpha'}(t), \lambda_0\rangle,\end{aligned}$$

where, in particular, (3.2) has been applied. Now (3.3) yields

$$\lim_{j\to\infty}\langle g_{\alpha'}(t+\beta_j), \lambda_0\rangle = \lim_{j\to\infty}\langle U(\beta_j)\, g_{\alpha'}(t), \lambda_0\rangle = \langle m_{\alpha', \beta', t}(0), \lambda_0\rangle .$$

Define $s : \mathbb{R} \to \mathbb{C}$ by

$$s(t) := \langle m_{\alpha', \beta', t}(0), \lambda_0\rangle .$$

Observe that this definition makes sense, since s is not required to be unique and s may depend on $\{\alpha'_n\}$ and $\{\beta'_j\}$.

Finally, one has to prove that

$$\text{(3.4)} \qquad s(t-\beta_j) \to \langle g_{\alpha'}(t), \lambda_0\rangle \quad \text{when } j\to\infty, \ \text{for every } t\in\mathbb{R}\,.$$

Observe first of all that arguing as above one can prove

$$\langle g_{\alpha'}(t-\beta_j), \lambda\rangle = = \langle U(-\beta_j)\, g_{\alpha'}(t), \lambda\rangle$$

for every $\lambda \in X'$. By definition, it results for a fixed $j \in \mathbb{N}$

$$\begin{aligned}s(t-\beta_j) &= \lim_{k\to+\infty}\langle g(t+\beta_k-\beta_j), \lambda_0\rangle \\ &= \lim_{k\to+\infty}\langle U(\beta_k)\, g_{\alpha'}(t-\beta_j), \lambda_0\rangle \\ &= \lim_{k\to+\infty}\langle g_{\alpha'}(t-\beta_j), U'(\beta_k)\,\lambda_0\rangle \\ &= \lim_{k\to+\infty}\langle U(-\beta_j)\, U(\beta_k)\, g_{\alpha'}(t), \lambda_0\rangle \\ &= \lim_{k\to+\infty}\langle U(\beta_k)\, g_{\alpha'}(t), U'(-\beta_j)\,\lambda_0\rangle .\end{aligned}$$

(3.3) yields now

$$\begin{aligned}\lim_{k\to+\infty}\langle U(\beta_k)\, g_{\alpha'}(t), U'(-\beta_j)\,\lambda_0\rangle &= \langle m_{\alpha', \beta', t}(0), U'(-\beta_j)\,\lambda_0\rangle \\ &= \langle U(-\beta_j)\, m_{\alpha', \beta', t}(0), \lambda_0\rangle \\ &= \langle m_{\alpha', \beta', t}(-\beta_j), \lambda_0\rangle ,\end{aligned}$$

where the last equality follows from the following identity

$$\begin{aligned}\langle m(\tau-\beta_j), \lambda\rangle &= \lim_{k\to+\infty}\langle U(\tau+\beta_k-\beta_j)\, g_{\alpha'}(t), \lambda\rangle \\ &= \lim_{k\to+\infty}\langle U(\tau+\beta_k)\, g_{\alpha'}(t), U'(-\beta_j)\,\lambda\rangle \\ &= \langle m_{\alpha',\beta',t}(\tau), U'(-\beta_j)\,\lambda\rangle \\ &= \langle U(-\beta_j)\, m_{\alpha',\beta',t}(\tau), \lambda\rangle\end{aligned}$$

holding for every $\tau\in\mathbb{R}$, $j\in\mathbb{N}$, $\lambda\in X'$. Finally, (3.3) entails that

$$\lim_{j\to+\infty} s(t-\beta_j) = \lim_{j\to+\infty}\langle m_{\alpha',\beta',t}(-\beta_j), \lambda_0\rangle = \langle U(0)\, g_{\alpha'}(t), \lambda_0\rangle = \langle g_{\alpha'}(t), \lambda_0\rangle,$$

proving (3.4). ■

Some recent results of E. Vesentini, concerning periodicity and almost periodicity on Markov lattice semigroups, may be extended by means of Theorem 3.2, so to include the almost automorphic case.

Let K be a compact metric space and let $C(K)$ denote the Banach space (with respect to the uniform norm) of all complex-valued continuous functions on K. A continuous flow $\Phi : \mathbb{R}\times K\to K$ (with $\varphi_t(\cdot) := \Phi(t, \cdot)$) defines a strongly continuous Markov lattice group $U : \mathbb{R}\to\mathcal{L}(C(K))$, expressed by

$$U(t)\, f = f\circ\varphi_t$$

for every $t\in\mathbb{R}$ and $f\in C(K)$.

Let d be a distance defining the metric topology of K. A point $x_0\in K$ is said to be *asymptotically stable* if for every $\varepsilon>0$ and $T>0$ there exists some $t\geqslant T$ such that

$$d(\varphi_t(x_0), x_0)\leqslant\varepsilon .$$

Corollary 1 in [20] and Theorem 3.2 yield the following

PROPOSITION 3.3: *If the group U is weakly almost automorphic, every point of K is asymptotically stable.*

E. Vesentini proved in [20] that the existence of almost periodic orbits of $\mathcal{U}$ and of asymptotically stable points of K imposes constraints on the spectral structure of the infinitesimal generator $(A, D(A))$ of $\mathcal{U}$. This remains true in some cases when almost periodicity is substituted by the weaker notion of almost automorphy. An example is given in the following proposition which follows immediately from Proposition 2 in [20] and Theorem 3. 2. Recall that φ_t is said to have *topological discrete spectrum* for some $t>0$ if all eigenfunctions of $U(t)$ span a dense linear subspace of $C(K)$.

Proposition 3.4: *If the group U is weakly almost automorphic and if φ_t has a topological discrete spectrum for some $t > 0$, then the intersection between $i\mathbb{R}$ and the point spectrum of A' is not empty.*

4. - Stepanov almost automorphic groups and semigroups

Let f be a $L^p_{loc}(\mathbb{R}; X)$ function, for some $1 \leq p < +\infty$. This means that $f(t) \in X$ for almost all $t \in \mathbb{R}$, is Bochner integrable and $\|f(\cdot)\|^p$ is Lebesgue integrable on every compact interval of $\mathbb{R}$. Given $\varepsilon > 0$, a *Stepanov ε − period* for f is a real number τ_ε such that

$$\left\{ \int_0^1 \|f(t+s+\tau_\varepsilon) - f(t+s)\|^p \, ds \right\}^{1/p} \leq \varepsilon \text{ for every } t \in \mathbb{R} .$$

A function f in $L^p_{loc}(\mathbb{R}; X)$ is said to be *almost periodic in the sense of Stepanov* (or *S^p-almost periodic*) if for every $\varepsilon > 0$ there is a relatively dense set of Stepanov ε-periods for f.

For a continuous function $f \in L^p_{loc}(\mathbb{R}; X)$ the almost periodicity in the sense of Bohr implies that of Stepanov. The converse is false, except, as is well known, in the case of uniformly continuous functions. For more details on almost periodicity in the sense of Stepanov the reader is referred to [1].

As pointed out by S. Bochner, the almost periodicity in the sense of Stepanov can be reduced to that of Bohr. In fact, let $f \in L^p_{loc}(\mathbb{R}; X)$. Denote by $\tilde{f}$ the map, from $\mathbb{R}$ to $L^p([0, 1]; X)$,

$$\tilde{f}(s) := f(t+s), \; t \in [0, 1] . \tag{4.1}$$

Bochner proved that f is almost periodic in the sense of Stepanov if and only if the (continuous) function $\tilde{f}: \mathbb{R} \to L^p([0, 1]; X)$ is almost periodic in the sense of Bohr.

A Stepanov type definition of almost automorphy will now be introduced.

A function $f \in L^p_{loc}(\mathbb{R}; X)$, where $1 \leq p < +\infty$, is said to be *almost automorphic in the sense of Stepanov* (S^p-a.a.) if the associated map $\tilde{f}$, from $\mathbb{R}$ to $L^p([0, 1]; X)$, defined by (4.1), is almost automorphic. This means that for every sequence $\{\alpha'_n\} \subseteq \mathbb{R}$ there exist a subsequence $\{\alpha_n\}$ and a map $g : \mathbb{R} \to L^p([0, 1]; X)$ (1), such that $T_\alpha \tilde{f} =$

(1) A map $g : \mathbb{R} \to L^p([0, 1]; X)$ shall be represented as $g(t) = g(t, s)$ with $t \in \mathbb{R}$ and $s \in [0, 1]$.

$=g$ and $T_{\alpha^{-1}} g = \tilde{f}$, *i.e.*

$$\left(\int_0^1 \| f(t+\alpha_n + s) - g(t, s) \|^p \, ds \right)^{1/p} \to 0 \quad \text{and}$$

$$\left(\int_0^1 \| g(t-\alpha_n, s) - f(t+s) \|^p \, ds \right)^{1/p} \to 0$$

when $n \to +\infty$, pointwise on $\mathbb{R}$.

A function $f \in L^p_{\text{loc}}(\mathbb{R}; X)$ is said to be S^p*-bounded* if

$$\|f\|_{S^p} = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(s)\|^p \, ds \right)^{1/p} < +\infty \, .$$

The space, which will be denoted by S^p_s, of all S^p-bounded functions is a Banach space for the norm $\|\cdot\|_{S^p}$. Basic properties of L^p spaces yield the following

Proposition 4.1: *Let* $1 \leq p < q < +\infty$. *If* $f \in L^q_{\text{loc}}(\mathbb{R}; X)$ *is* S^q*-almost automorphic, then f is* S^p*-almost automorphic.*

The relationship between almost automorphy and S^p-almost automorphy will now be investigated.

Proposition 4.2: *Let* $f: \mathbb{R} \to X$ *be a continuous function and* $1 \leq p < \infty$. *If f is almost automorphic, then f is* S^p*-almost automorphic.*

Proof: Take a sequence $\{\alpha'_n\} \subseteq \mathbb{R}$. Let $\{\alpha_n\}$ and h be respectively a subsequence of $\{\alpha'_n\}$ and a map from $\mathbb{R}$ to X, such that $T_\alpha f = h$ and $T_{\alpha^{-1}} h = f$.

Define $\tilde{f}$ as in (4. 1) and $g(t) := h(t+s)$, $s \in [0, 1]$, $t \in \mathbb{R}$. Observe that $g(t)$ belongs to $L^p([0, 1]; X)$ for all $t \in \mathbb{R}$, since $h(t+\cdot)$ is strongly measurable and bounded.

It will now be shown that $T_\alpha \tilde{f} = g$ and $T_{\alpha^{-1}} g = \tilde{f}$, the convergence being in $L^p([0, 1]; X)$, that is

$$\left(\int_0^1 \| f(t+\alpha_n + s) - h(t+s) \|^p \, ds \right)^{1/p} \to 0 \quad \text{and}$$

$$\left(\int_0^1 \| h(t-\alpha_n + s) - f(t+s) \|^p \, ds \right)^{1/p} \to 0$$

when $n \to \infty$, for all $t \in \mathbb{R}$. Consider the first integral, which can be written as

$$\left(\int_t^{t+1} \|f(\alpha_n + s) - h(s)\|^p \, ds \right)^{1/p} .$$

Fix $t \in \mathbb{R}$ and set

$$\varphi_n(s) := \|f(\alpha_n + s) - h(s)\|^p , \; s \in [t, t+1] .$$

Since f is almost automorphic, f and h are bounded, so that φ_n is uniformly bounded on $[t, t+1]$. Moreover, $\varphi_n(s) \to 0$ for every $s \in [t, t+1]$, since $f(t + \alpha_n) \to h(t)$ pointwise on $\mathbb{R}$. As a consequence of the Lebesgue dominated convergence theorem,

$$\int_t^{t+1} \varphi_n(s) \, ds \to 0 .$$

Finally, the fact that $T_{\alpha^{-1}} g = \tilde{f}$ can be proved in an analogous way, and therefore f is S^p-almost automorphic. ■

It has been observed in Section 1 that an almost periodic function is necessarily almost automorphic. The same is true when Stepanov type conditions are considered.

PROPOSITION 4.3: *If $f \in L^p_{\text{loc}}(\mathbb{R}; X)$ is S^p-almost periodic for some $1 \leq p < +\infty$, then f is S^p-almost automorphic.*

PROOF: If $f \in L^p_{\text{loc}}(\mathbb{R}; X)$ is S^p-almost periodic, for some $1 \leq p < +\infty$, the function $\tilde{f}$, defined by (4.1), is almost periodic, and therefore almost automorphic. Thus f is S^p-almost automorphic. ■

The results just proved for a generic function f shall now be applied in order to investigate the behaviour of a single orbit of a group.

Let $\mathcal{U}$ be a strongly continuous group of linear bounded operators on X. Let x_0 be in X and let $1 \leq p < +\infty$. Consider the following statements:

(1) the map $t \mapsto U(t)\, x_0$ is S^p-almost automorphic,
(2) the map $t \mapsto U(t)\, x_0$ is almost automorphic;
(3) the map $t \mapsto U(t)\, x_0$ is S^p-almost periodic;
(4) the map $t \mapsto U(t)\, x_0$ is almost periodic.

Then (2) implies (1) as a consequence of Proposition 4.2, (3) implies (1) in view of Proposition 4.3, (4) implies (2) in virtue of Proposition 2.1 and (4) implies (3) in view of the properties of Stepanov almost periodic functions.

Suppose now that $\mathcal{U}$ is uniformly bounded. Thus (2) and (4) are equivalent in view of what has been proved in Section 2 and (3) and (4) are equivalent, since the map $t \mapsto U(t)x_0$ is in this case uniformly continuous. In fact, the following statement can be proved.

PROPOSITION 4.4: *If the strongly continuous group $\mathcal{U}$ is uniformly bounded, then statements* (1)...(4) *are equivalent.*

PROOF: It suffices to prove that (1) entails (3). Let $f(t) := U(t)\, x_0$ be S^p-a.a. for some $1 \leq p < +\infty$. Then the map $\tilde{f}$, from $\mathbb{R}$ to $L^p([0,1]; X)$, is almost automorphic.

By choosing any sequence $\{\alpha'_n\}$ in $\mathbb{R}$ and by applying the definition, one can find a subsequence $\{\alpha_n\}$ such that $\{\tilde{f}(\cdot + \alpha_n)\}$ converges if $n \to +\infty$, pointwise on $\mathbb{R}$; in particular, $\{\tilde{f}(\alpha_n)\}$ converges when $n \to +\infty$.

Moreover, if $M > 0$ is such that $\|U(t)\| \leq M$ for all $t \in \mathbb{R}$, then

$$\sup_{t\in\mathbb{R}} \|\tilde{f}(t+\alpha_m) - \tilde{f}(t+\alpha_n)\|_{L^p([0,1];X)}$$

$$= \sup_{t\in\mathbb{R}} \left(\int_0^1 \|U(t+\alpha_m+s)\, x_0 - U(t+\alpha_n+s)\, x_0\|^p\, ds \right)^{1/p}$$

$$= \sup_{t\in\mathbb{R}} \left(\int_0^1 \|U(t)(U(\alpha_m+s)\, x_0 - U(\alpha_n+s)\, x_0)\|^p\, ds \right)^{1/p}$$

$$\leq M \left(\int_0^1 \|U(\alpha_m+s)\, x_0 - U(\alpha_n+s)\, x_0\|^p\, ds \right)^{1/p} = M\|\tilde{f}(\alpha_m) - \tilde{f}(\alpha_n)\|_{L^p([0,1];X)}$$

for every $m, n \in \mathbb{N}$. Thus the function $\tilde{f}$ is almost periodic, so that the map $t \mapsto U(t)x_0$ is S^p-almost periodic, showing that (1) implies (3). ■

If $1 \leq p < +\infty$, a group $\mathcal{U}$ is called *strongly S^p – almost periodic* (*automorphic*) if every map, from $\mathbb{R}$ to X, $t \mapsto U(t)x$ is S^p-almost periodic (automorphic). In [8] Henriquez proved that a strongly S^p-almost periodic group is strongly almost periodic (in particular, it is uniformly bounded). Thus the following result can be stated.

THEOREM 4.5: *Let $\mathcal{U}$ be a strongly continuous group of linear bounded operators acting on X. Let $1 \leq p < +\infty$. Then the following assertions are equivalent*:

(1) *$\mathcal{U}$ is strongly S^p-almost automorphic*;
(2) *$\mathcal{U}$ is strongly almost automorphic*;
(3) *$\mathcal{U}$ is strongly S^p-almost periodic*;
(4) *$\mathcal{U}$ is strongly almost periodic.*

PROOF: The only thing to prove is that, if $\mathcal{U}$ is strongly S^p-almost automorphic, it is uniformly bounded. Suppose that every orbit $t \mapsto U(t)x$, $x \in X$, is S^p-almost automorphic. Then every orbit is S^p-bounded, *i.e.* for every $x \in X$ there exists a constant M_x such that

$$\sup_{t \in \mathbb{R}} \int_0^1 \|U(t+s)\, x\|^p \, ds \leq M_x .$$

Under this assumption, Henriquez proved [8, Theorem 1] that the function, from $\mathbb{R}$ to X, $t \mapsto U(t)\, x$ is bounded. As a consequence of the uniform boundedness principle, $\mathcal{U}$ is uniformly bounded. The thesis follows by applying Proposition 4.4. ■

A function $f \in L^p_{\mathrm{loc}}(\mathbb{R}_+ ; X)$ is said to be *asymptotically almost periodic in the sense of Stepanov* (or *a. $-S^p-$a.p.*) if the function $\tilde{f}$: $\mathbb{R}_+ \to L^p([0, 1]; X)$, defined by (4.1), that is $\tilde{f}(s) := f(t+s)$, $t \in [0, 1]$, is asymptotically almost periodic.

Analogously a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}_+ ; X)$ is said to be *asymptotically almost automorphic in the sense of Stepanov* (or *a. S^p-a.a.*) if $\tilde{f}$ is asymptotically almost automorphic.

PROPOSITION 4.6: *Let $f : \mathbb{R}_+ \to X$ be a continuous asymptotically almost automorphic map. If $1 \leq p < \infty$, then f is a.S^p-a.a..*

PROOF: By hypothesis there exist an almost automorphic map $g : \mathbb{R} \to X$ and a function $h \in \mathcal{C}_0(\mathbb{R}_+ , X)$, such that $f = g_{|\mathbb{R}_+} + h$. In particular, g is S^p-almost automorphic by Proposition 4.2, that is $\tilde{g}$: $\mathbb{R} \to L^p([0, 1]; X)$ is almost automorphic.

Define $\tilde{h}(t) := h(t+s)$, $s \in [0, 1]$, $t \in \mathbb{R}$. It will now be shown that $\tilde{h}$ belongs to $\mathcal{C}_0(\mathbb{R}_+ , L^p([0, 1]; X))$. Since $h \in \mathcal{C}_0(\mathbb{R}_+ , X)$, for every $\varepsilon > 0$ there exists some $L > 0$ such that $\|h(t)\| \leq \varepsilon$ whenever $t \geq L$. If $t \geq L$, then

$$\int_0^1 \|h(t+s)\|^p \, ds = \int_t^{t+1} \|h(s)\|^p \, ds \leq \varepsilon^p ,$$

proving that $\tilde{h} \in \mathcal{C}_0(\mathbb{R}_+ , L^p([0, 1]; X))$.

Since $\tilde{f} = \tilde{g} + \tilde{h}$ on $\mathbb{R}_+$, with $\tilde{g}$ almost automorphic and $\tilde{h} \in \mathcal{C}_0(\mathbb{R}_+ , L^p([0, 1]; X))$, $\tilde{f}$ is asymptotically almost automorphic, and therefore f is a.S^p-a.a.. ■

The analogous of Proposition 4.3 on the half-line shall now be proved.

PROPOSITION 4.7: *If $f \in L^p_{\mathrm{loc}}(\mathbb{R}_+ ; X)$ is asymptotically S^p-almost periodic for some $1 \leq p < +\infty$, then f is asymptotically S^p-almost automorphic.*

Proof: Since by hypothesis $\tilde{f}: \mathbb{R}_+ \to L^p([0, 1]; X)$ is asymptotically almost periodic, there exist two functions $G: \mathbb{R} \to L^p([0, 1]; X)$ and $Q \in \mathcal{C}_0(\mathbb{R}_+, L^p([0, 1]; X))$ such that G is almost periodic and $\tilde{f} = G_{|\mathbb{R}_+} + Q$.

G is, in particular, almost automorphic, and therefore $\tilde{f}$ is asymptotically almost automorphic. Thus f is asymptotically S^p-almost automorphic. ■

Let $\mathcal{T}$ be a strongly continuous semigroup of linear bounded operators on X. Let x_0 be in X and let $1 \leq p < +\infty$.

Consider the following assertions

(1) the map $t \mapsto T(t)\, x_0$ is asymptotically S^p-almost automorphic,
(2) the map $t \mapsto T(t)\, x_0$ is asymptotically almost automorphic;
(3) the map $t \mapsto T(t)\, x_0$ is asymptotically S^p-almost periodic;
(4) the map $t \mapsto T(t)\, x_0$ is asymptotically almost periodic.

Then (2) implies (1) as a consequence of Proposition 4.6, (3) implies (1) in view of Proposition 4.7, (4) implies (2) as a consequence of Proposition 2.2; furthermore, (4) implies (3), since every asymptotically almost periodic function is asymptotically S^p-almost periodic. Moreover, in analogy to the group case, the following result can be proved.

Proposition 4.8: *If the semigroup $\mathcal{T}$ is uniformly bounded, statements* (1), …, (4) *are equivalent.*

Proof: Under this assumption, it has been shown in Proposition 2.2 that (2) and (4) are equivalent. Since it has been proved in [8, Theorem 2] that (3) implies (4), exactly as in Proposition 4.4 it suffices to prove that if $t \mapsto T(t)\, x_0$ is a.S^p-a.a. then it is a.S^p-a.p.. This can be done exactly as in Proposition 4.4. ■

If $1 \leq p < +\infty$, a semigroup $\mathcal{T}$ is called *strongly asymptotically S^p – almost periodic* (*automorphic*) if every map, from $\mathbb{R}_+$ to X, $t \mapsto T(t)\, x$ is asymptotically S^p-almost periodic (automorphic).

If $\mathcal{T}$ is strongly asymptotically S^p-almost automorphic, proceeding as in the proof of Theorem 4.5 one shows that $\mathcal{T}$ is uniformly bounded. Then Proposition 4.8 yields the following result.

Theorem 4.9: *Let $\mathcal{T}$ be a strongly continuous semigroup of linear bounded operators acting on X. Let $1 \leq p < +\infty$. The following assertions are equivalent*:

(1) *$\mathcal{T}$ is strongly asymptotically S^p-almost automorphic*;
(2) *$\mathcal{T}$ is strongly asymptotically almost automorphic*;
(3) *$\mathcal{T}$ is strongly asymptotically S^p-almost periodic*;
(4) *$\mathcal{T}$ is strongly asymptotically almost periodic.*

5. - Stability of asymptotic almost automorphy

The results obtained in the previous sections will now be applied in order to study the permanence of almost automorphy under perturbations.

More precisely, let $\mathcal{T} = (T(t))_{t \geq 0}$ be a strongly continuous semigroup of linear bounded operators on X with generator $(A, D(A))$. The symbol X_1 will denote the Sobolev space of order one associated to $\mathcal{T}$, that is the Banach space $(D(A), \|\cdot\|_1)$, where $\|x\|_1 := \|x\| + \|Ax\|$ for every $x \in D(A)$.

Consider now an operator $B \in \mathcal{L}(X_1, X)$, such that there exist a constant $0 < q < 1$ and some $t_0 > 0$ for which

$$\int_0^{t_0} \|BT(s)\, x\| ds \leq q\|x\|$$

for every $x \in D(A)$. It is well known ([9], [21]) that under this assumption the operator $(A + B, D(A))$ generates a strongly continuous semigroup $\mathcal{S} := (S(t))_{t \geq 0}$. The perturbed semigroup $\mathcal{S}$ does not inherit, in general, asymptotic properties of $\mathcal{T}$, as boundedness, asymptotic almost periodicity, uniform ergodicity and total uniform ergodicity, unless suitable additional hypotheses on $\mathcal{T}$ and B are imposed. In a recent work [7], S. Piazzera and the author found conditions on $\mathcal{T}$ and B, guaranteeing the stability of asymptotic properties under perturbations in the class of Miyadera-Voigt.

Analogous results to that of [7] for asymptotic (S^p-) almost automorphy may be obtained in view of the results proved in sections 2 and 4; for the sake of completeness the case of asymptotic S^p-almost periodicity is considered as well. The symbol $\mathcal{E}$ denotes the set either of all a.a.a., or of all a.-S^p-a.a. or of all a.-S^p-a.p. functions from $\mathbb{R}_+$ to X. The following result can be stated.

Theorem 5.1: *Let $\mathcal{T} = (T(t))_{t \geq 0}$ be a strongly continuous semigroup on X, generated by $(A, D(A))$ and let $B \in \mathcal{L}(X_1, X)$. Assume that there exists a constant $0 < q < 1$ such that*

$$\int_0^{t} \|BT(s)\, x\| ds \leq q\|x\|$$

for all $t \geq 0$ and $x \in D(A)$. Let $(S(t))_{t \geq 0}$ be the strongly continuous semigroup generated by $(A + B, D(A))$.

If $t \mapsto T(t)x$ belongs to $\mathcal{E}$ for every $x \in X$, then $t \mapsto S(t)\, x$ belongs to $\mathcal{E}$ for every $x \in X$.

Proof: By means of Theorem 4.9, $\mathcal{T}$ is strongly asymptotically almost periodic. Since the class of all a.a.p. maps from $\mathbb{R}_+$ to X is a closed, translation-invariant subspace of $\mathcal{C}_{ub}(\mathbb{R}_+, X)$, Theorem 3.5 in [7] may be applied, yielding that $t \mapsto S(t)\, x$ is a.a.p. for all $x \in X$; thus every orbit of the perturbed semigroup $\mathcal{S}$ belongs to $\mathcal{E}$. ■

In [7] stability results are then applied to the analysis of asymptotic behaviour of solutions of delay equations, *i.e* equations of the form

$$\begin{cases} u'(t) = Au(t) + \Phi u_t, \qquad t \geq 0 , \\ u(0) = x , \\ u_0 = f , \end{cases}$$

where $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on X, $x \in X$, $f \in L^p([-1, 0], X)$, $1 \leq p < \infty$ and Φ is a bounded linear operator from $W^{1,p}([-1, 0], X)$ to X. In view of Theorem 5.1, every result of Section 4 in [7] can be rephrased for asymptotic almost automorphy, asymptotic S^p-almost automorphy and asymptotic S^p-almost periodicity as well.

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