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A Result in Nonlinear Subelliptic Potential Theory (**)

ABSTRACT. — We give a sufficient condition for a positive measure to be in $H^{-1,q}(\Omega, X)$, $q > 1$; Ω is a bounded open set and $X = (X_1, \ldots, X_m)$, where X_i, $i = 1, \ldots, m$ are vector fields satisfying an Hörmander condition.

Un risultato in teoria del potenziale non lineare subellittica

SUNTO. — Nella presente nota diamo una condizione sufficiente per l' appartenenza di una misura positiva a $H^{-1,q}(\Omega, X)$, $q > 1$; Ω denota un aperto limitato e X_i, $i = 1, \ldots, m$ sono campi vettoriali soddisfacenti la condizione di Hörmander.

1. - INTRODUCTION

Let Ω be a bounded open set in R^N; a positive bounded Radon measure μ on Ω belongs to $H^{-1,p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$ $p \in (1, N]$, iff

$$\int_\Omega \mu(dx) \int_0^{4R} \left(\frac{\mu(B(x, \varrho))}{|B(x, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} < +\infty . \tag{1.1}$$

Moreover

$$\left[\int_\Omega \mu(dx) \int_0^{4R} \left(\frac{\mu(B(x, \varrho))}{|B(x, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \right]^{\frac{p}{p-1}} \approx \|\mu\|_{H^{-1,p'}(\Omega)} \tag{1.2}$$

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where diam $(\Omega) = R$, $|E|$ denotes the Lebesgue measure of the set E and we denote again by μ the extension of μ to R^N by 0 (i.e. a set E is μ measurable if $E \cap \Omega$ is μ measurable and $\mu(E) = \mu(E \cap \Omega)$). We refer to [1] for the proofs of (1.1) and (1.2).

If $p > N$ we have that (1.1) holds for every bounded Radon measure on Ω and the integral in (1.2) is equivalent to $\mu(\Omega)$, then is greater than $\|\mu\|_{H^{-1,p'}(\Omega)}$.

The results of (1.1) and (1.2) have been generalized to vector fields defining a polynomial Lie group, [16] (as a typical example we may consider the Heisenberg group); in such a case a definition of intrinsic dimension of the Lie group is again possible.

Let X_i, $i = 1, \ldots, m$ be vector fields on R^N satisfying the Hörmander condition and let $d(x, y)$ be the intrinsic distance defined on R^N by the vector fields, [7][8][12][15]. Denote by $B_X(x, r)$ the ball of center x and radius r defined by the distance $d(x, y)$. We observe that a local duplication duplication property holds for the balls $B_X(x, r)$, i.e.

$$|B_X(x, 2r)| \leq c_0 |B_X(x, r)| \tag{1.3}$$

for $x \in \Omega$ and $r \leq R_0$, where $c_0 > 1$ and $R_0 > 0$ are constants depending on Ω and we can choose c_0 as the best constant such that (1.3) holds, [8] [15]. The inequality (1.3) allows us to give an estimate of the intrinsic dimension of our problem by $\nu = log_2 c_0$.

We denote by $H^{1,p}(\Omega, X)$, $p \geq 1$, the closure of $C^\infty(\overline{\Omega})$ for the norm

$$\|u\|_{H^{1,p}(\Omega, X)} = (\|u\|^p_{L^p(\Omega)} + \|Xu\|^p_{L^p(\Omega)})^{\frac{1}{p}}$$

and by $H^{1,p}_0(\Omega, X)$ the closure of $C^\infty_0(\Omega)$. By $H^{-1,p'}(\Omega, X)$, $\frac{1}{p} + \frac{1}{p'} = 1$, we denote the dual space of $H^{1,p}_0(\Omega, X)$.

Moreover a Poincaré inequality holds for $u \in H^{1,p}(B_X(x, r), X)$, $p \geq 1$,

$$\int_{B_X(x, r)} |u - u_r|^p \, dx \leq c_1 r^p \int_{B_X(x, r)} |Xu|^p \, dx \tag{1.4}$$

$x \in \Omega$ and $r \leq R_0$, where $c_1 > 0$ and $R_0 > 0$ are constants depending on Ω and u_r denotes the average of u on $B_X(x, r)$, [9] [11] [12].

We recall that a variational $p-$ *capacity* of a set E relative to the open set Ω, $E \subset\subset \Omega$ may be defined (generalizing the usual Newtonian definition) as

$$p - \text{cap}(E, \Omega) = \inf_{\{v \in C_0^\infty(\Omega),\ v = 1 \text{ in a neighbourhood of } E\}} \left\{ \int_\Omega |Xv|^p \, dx \right\}$$

and we refer for the properties of such a type of capacity to [6] [13] [14]. We denote $p - \text{cap}(E) = p - \text{cap}(E, R^N)$ and we recall that, if $\overline{E} \subset \Omega$, $p - \text{cap}(E, \Omega) = 0$ iff $p - \text{cap}(E) = 0$.

Using the same methods as in [6] Theorem 4.1 and the estimates on the cut-off functions between balls, see [5], we obtain easily

$$p - \operatorname{cap}(B(x_0, r), B(x_0, 2r)) \simeq \frac{r^p}{m(B(x_0, r))}$$

for $\overline{B(x_0, 4r)} \subseteq \Omega$, where Ω is a bounded open set and the constants may depend on Ω. A function u defined in an open set O with $\overline{O} \subseteq \Omega$ is *quasi* $-$ *continuous* if for every $\varepsilon > 0$ there exists an open set V with $p - \operatorname{cap}(V, \Omega) \leq \varepsilon$ such that u is continuous on $E \backslash V$; a function u in $H^{1,p}(O, X)$ has a quasi-continuous q.e. representative (i.e. there exists a quasi-continuous function $\tilde{u}$ such that $\tilde{u} = u$ up to sets of zero p-capacity); the proof follows the same methods as in [10] for the case $p = 2$. In the following we identify u with its quasi-continuous representative.

Moreover for $u \in H_0^{1,p}(\Omega, X)$ and μ Radon measure in $H^{-1,p'}(\Omega, X)$ we have

$$\langle \mu, u \rangle = \int_\Omega u \mu(dx)$$

where $\langle .\,,. \rangle$ denotes the coupling between $H^{-1,p'}(\Omega, X)$ and $u \in H_0^{1,p}(\Omega, X)$ (the proof follows the same methods as in [10] for the case $p = 2$).

We observe that in the case of vector fields we may only hope to generalize the results that hold in euclidean case for every $p > 1$, since we have no precise definition for the intrinsic dimension, i.e. we have only one side information on the measure of intrinsic balls.

Results of such a type or connected with have been given in [2] [3]; here we give a new proof of the result founded on methods from partial differential equations without the use of tools, as maximal functions, typical of real analysis.

We now state our result:

THEOREM 1.1: *Let μ be a positive bounded Radon measure on Ω such that*

$$\int_\Omega \mu(dx) \int_0^{4R} \left(\frac{\mu(B_X(x, \varrho))}{|B_X(x, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} < +\infty \tag{1.5}$$

where R is the diameter of Ω for the distance d and we denote again by μ the extension of μ to R^N by 0.

Moreover we have

$$\|\mu\|_{H^{-1,p'}(\Omega, X)} \leq C \left[\int_\Omega \mu(dx) \int_0^{4R} \left(\frac{\mu(B_X(x, \varrho))}{|B_X(x, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \right]^{\frac{p-1}{p}} . \tag{1.6}$$

REMARK 1.1: We observe that by a covering argument we may prove the result of Theorem 1.1 for open sets Ω that are balls B_X with a radius small enough; so we may assume that B_X is such that for every ball in $200B_X$ a Poincaré inequality holds.

REMARK 1.2: We remark that we may assume that μ has compact support in Ω, otherwise an approximation result gives the result.

REMARK 1.3: We observe that we may prove the result only for $p \in (1, \nu]$, where ν derives from the constant c_0 in (1.3) relative to a ball $2B_X$ with $\Omega \subseteq B_X$. The Morrey-Campanato inequalities give the result for $p > \nu$.

We end this section describing the methods used in the proof of the Theorem 1.1. We take into account the observatioons of Remarks 1.1, 1.2, 1.3 and we consider the case where Ω is a ball B_X such that a Poincaré inequality hold for every ball in $200B_X$, moreover we assume that μ has compact support in B_X and $p \in (1, \nu]$. In the proof we prove at first (1.5) for measures $\mu \in H^{-1,p'}(\Omega, X)$ using an estimate given in [4], then we end the proof by an approximation argument.

2. - PROOF OF THEOREM 1.1

Let B_X be a ball of radius R; taking into account Remark 1.1, we may assume that $16R \leq R_0$, where R_0 is the same constant appearing in the duplication property and in the Poincaré inequality relative to $16B_X$.

We begin by proving the following result:

LEMMA 2.1: *Let $B_X(x, r)$ be a ball in $16\, B_X$ and $u \in H^{1,p}(B_X(x, r), X)$; then*

$$\int_{B_X(x,r)} |u|^p\, dx \leq Cr^p \frac{|B_X(x,r)|}{|\{y \in B_X(x,r);\, u(y) = 0\}|} \int_{B_X(x,r)} |Xu|^p\, dx\,.$$

PROOF: From the Poincaré inequality we have

$$\int_{B_X(x,r)} |u - u_r|^p\, dx \leq c_1\, r^p \int_{B_X(x,r)} |Xu|^p\, dx$$

where u_r is the average of u on $B_X(x, r)$. Then

$$\int_{\{y \in B_X(x,r);\, u(y) = 0\}} |u_r|^p\, dx \leq c_1\, r^p \int_{B_X(x,r)} |Xu|^p\, dx\,. \tag{2.1}$$

From (2.1) we obtain

$$|u_r|^p \leqslant c_1 r^p \frac{1}{|\{y \in B_X(x, r); u(y) = 0\}|} \int_{B_X(x, r)} |Xu|^p \, dx \tag{2.2}$$

and (2.2) gives easily the result. ■

Corollary 2.2: *Let* $B_X(x, r)$ *be a ball in* $16\, B_X$ *and* $u \in H_0^{1, p}(B_X(x, r), X)$; *then*

$$\int_{B_X(x, r)} |u|^p \, dx \leqslant \int_{B_X(x, r)} |Xu|^p \, dx .$$

Proof: We observe that $u \in H^{1, p}(B_X(x, 2r), X)$, where we denote again by u the extension by 0 of u to R^N There exists a ball $\tilde{B}_X$ such that $\tilde{B}_X \subseteq B_X(x, 2r) - B_X(x, r) \subseteq B_X(x, 2r) \subseteq 4\, \tilde{B}_X \subseteq B_X(x, 6r)$, then using the duplication property we obtain that $|B_X(x, 2r) - B(x, r)| \geqslant C|B_X(x, r)|$ and Lemma 2.1 gives the result. ■

We prove at first (1.6) for measures $\mu \in H^{-1, p}(B_X, X)$:

Lemma 2.3: *Let* μ *be a positive bounded Radon measure in* $H^{-1, p'}(B_X, X)$. *We denote again by* μ *the extension of* μ *to* R^N *by* 0. *Assume that the following condition holds*

$$\int_{B_X} \mu(dx) \int_0^{4R} \left(\frac{\mu(B_X(x, \varrho))}{|B_X(x, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} < +\infty .$$

Then

$$\|\mu\|_{H^{-1, p'}(\Omega, X)} \leqslant C \left[\int_{B_X} \mu(dx) \int_0^{4R} \left(\frac{\mu(B_X(x, \varrho))}{|B_X(x, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \right]^{\frac{p-1}{p}} .$$

Proof: Let w be the solution of the problem

$$\int_{B_X} |Xw|^{p-2} Xw\, Xv \, dx = \int_{B_X} v \mu(dx) \tag{2.4}$$

$$\forall v \in H_0^{1, p}(B_X, X), \; w \in H_0^{1, p}(B_X, X) .$$

We observe that w is positive and w (extended by 0) is a subsolution of the subelliptic p-Laplace operator relative to μ in R^N. Then from Theorem 1.1 in [4] we have that the following inequality holds q.e. in $x \in B_X$ (for the natural capacity associated to the subelliptic p-Laplace operator)

$$w(x) \leqslant$$

$$\leqslant C\left[\left(\frac{1}{|B_X(x,4R)|}\int_{B_X}|w|^p\,dx\right)^{\frac{1}{p}} + \int_0^{4R}\left(\frac{\mu(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho}\right] \leqslant$$

$$\leqslant C\left[\left(\frac{1}{|B_X|}\int_{B_X}|w|^p\,dx\right)^{\frac{1}{p}} + \int_0^{4R}\left[\left(\frac{\mu(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho}\right]^{\frac{p-1}{p}}\right] \leqslant$$

$$C\left[\left(\frac{R^p}{|B_X|}\int_{B_X}|Xw|^p\,dx\right)^{\frac{1}{p}} + \int_0^{4R}\left(\frac{\mu(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho}\right]$$

where here and in the following C denotes possibly different constants independent of R and we take into account the duplication property and Corollary 2.2.

Then taking into account (2.4) with $v = w$ we obtain

$$\int_{B_X}|Xw|^p\,dx \leqslant \int_{B_X} w\mu(dx) \leqslant C\mu(B_X)\left(\frac{R^p}{|B_X|}\int_{B_X}|Xw|^p\,dx\right)^{\frac{1}{p}} +$$

$$+\int_{B_X}\mu(dx)\int_0^{4R}\left(\frac{\mu(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho} \leqslant C\mu(B_X)\left(\mu(B_X)\frac{R^p}{|B_X|}\right)^{\frac{1}{p-1}} +$$

$$+\frac{1}{2}\int_{B_X}|Xw|^p\,dx + C\int_{B_X}\mu(dx)\int_0^{4R}\left(\frac{\mu(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho} \leqslant$$

$$\leqslant C\int_{B_X}\mu(dx)\int_{2R}^{4R}\left(\frac{\mu(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho} + \frac{1}{2}\int_{B_X}|Xw|^p\,dx +$$

$$+C\int_{B_X}\mu(dx)\int_0^{4R}\left(\frac{\mu(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho}.$$

Then finally we have

$$\int_{B_X} |Xw|^p\, dx \leqslant C \int_{B_X} \mu(dx) \int_0^{4R} \left(\frac{\mu(B_X(x, \varrho))}{|B_X(x, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} .$$

Then

$$\|\mu\|_{H^{-1, p'}(B_X, X)} \leqslant C \left(\int_{B_X} |Xw|^p\, dx \right)^{\frac{p-1}{p}} \leqslant$$

$$\leqslant C \left[\int_{B_X} \mu(dx) \int_0^{4R} \left(\frac{\mu(B_X(x, \varrho))}{|B_X(x, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \right]^{\frac{p-1}{p}}$$

so we have the result. ■

We are now in position to end the proof of Theorem (1.1). We have to prove that if

$$\int_{B_X} \mu(dx) \int_0^{4R} \left(\frac{\mu(B_X(x, \varrho))}{|B_X(x, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} < +\infty$$

then $\mu \in H^{-1, p'}(B_X, X)$. Taking into account Lemma 2.1, we may assume that μ has compact support in B_X. Assume that $200R \leqslant R_0$ where R_0 is as in the duplication property relative to $200 B_X$.

We consider a finite covering of B_X by a finite number of balls $B_X(x_i, R/n)$, $x_i \in B_X$, such that the balls $B_X(x_i, R/2n)$ are disjoint and every point of B_X is covered by at most M balls in the covering, where M is independent of n. We define

$$\mu_n = \sum_i \frac{\mu(B_X(x_i, R/n))}{|B_X(x_i, R/n)|} \mathbf{1}_{B_X(x_i, R/n)}\, dx$$

where dx denotes the Lebesgue measure.

We observe that for n big enough μ_n has compact support in B_X. Let $\varrho \geqslant R/2n$, then

$$\mu_n(B_X(x, \varrho)) \leqslant \sum_{x_i \in B_X(x, 4\varrho)} \frac{\mu(B_X(x_i, R/n))}{|B_X(x_i, R/n)|} \left| B_X\left(x_i, \frac{R}{n}\right) \right| =$$

$$= \sum_{x_i \in B_X(x, 4\varrho)} \mu\left(B_X\left(x_i, \frac{R}{n}\right)\right) \leqslant M\mu(B_X(x, 6\varrho)) .$$

Then

$$\left(\frac{\mu_n(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}} \leqslant \left(M\frac{\mu(B_X(x,6\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}} \leqslant \tag{2.5}$$

$$\leqslant C\left(\frac{\mu(B_X(x,6\varrho))}{|B_X(x,6\varrho)|}(6\varrho)^p\right)^{\frac{1}{p-1}}.$$

Let now $\varrho \leqslant R/2n$,

$$\mu_n(B_X(x,\varrho)) \leqslant M \sum_{x_i \in B_X(x,\,2R/n)} \frac{\mu(B_X(x_i,R/n))}{|B_X(x_i,R/n)|}|B_X(x,\varrho)| \leqslant$$

$$\leqslant C \sum_{x_i \in B_X(x,\,2R/n)} \frac{\mu(B_X(x,3R/n))}{|B_X(x_i,6R/n)|}|B_X(x,\varrho)| \leqslant$$

$$\leqslant C \sum_{x_i \in B_X(x,\,2R/n)} \frac{\mu(B_X(x,3R/n))}{|B_X(x,3R/n)|}|B_X(x,\varrho)|.$$

Then

$$\left(\frac{\mu_n(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}} \leqslant C\left(\frac{\mu(B_X(x,3R/n))}{|B_X(x,3R/n)|}\varrho^p\right)^{\frac{1}{p-1}}. \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$\int_0^{4R}\left(\frac{\mu_n(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho} \leqslant$$

$$\leqslant C\left[\int_{R/2n}^{4R}\left(\frac{\mu(B_X(x,6\varrho))}{|B_X(x,6\varrho)|}(6\varrho)^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho} + \left(\frac{\mu(B_X(x,3R/n))}{|B_X(x,3R/n)|}\left(\frac{3R}{n}\right)^p\right)^{\frac{1}{p-1}}\right] \leqslant$$

$$\leqslant C\left[\int_{3R/n}^{24R}\left(\frac{\mu(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho} + \int_{R/n}^{3R/n}\left(\frac{\mu(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho}\right] \leqslant$$

$$\leqslant C\int_{R/n}^{24R}\left(\frac{\mu(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho}.$$

Now we estimate the norm in $H^{-1,p'}(B_X, X)$ of the measure μ_n. We have for n big enough

$$\int_{B_X} \mu_n(dx) \int_0^{4R} \left(\frac{\mu_n(B_X(x, \varrho))}{|B_X(x, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \leqslant$$

$$\leqslant C \int_{B_X} \sum_i \frac{\mu(B_X(x_i, R/n))}{|B_X(x_i, R/n)|} \mathbf{1}_{B_X(x_i, R/n)} dx \int_{R/n}^{24R} \left(\frac{\mu(B_X(x, \varrho))}{|B_X(x, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \leqslant$$

$$\leqslant C \int_{B_X} \sum_i \frac{\mu(B_X(x_i, R/n))}{|B_X(x_i, R/n)|} \mathbf{1}_{B_X(x_i, R/n)} dx \int_{R/n}^{24R} \left(\frac{\mu(B_X(x, \varrho))}{|B_X(x, 6\varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \leqslant$$

$$\leqslant C \int_{B_X} \sum_i \frac{\mu(B_X(x_i, R/n))}{|B_X(x_i, R/n)|} \mathbf{1}_{B_X(x_i, R/n)} dx \int_{R/n}^{24R} \left(\frac{\mu(B_X(x, \varrho))}{|B_X(x_i, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \leqslant$$

$$\leqslant C \int_{B_X} \sum_i \frac{\mu(B_X(x_i, R/n))}{|B_X(x_i, R/n)|} \mathbf{1}_{B_X(x_i, R/n)} dx \int_{R/n}^{24R} \left(\frac{\mu(B_X(x_i, 3\varrho))}{|B_X(x_i, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \leqslant$$

$$\leqslant C \int_{B_X} \sum_i \frac{\mu(B_X(x_i, R/n))}{|B_X(x_i, R/n)|} \mathbf{1}_{B_X(x_i, R/n)} dx \int_{R/n}^{24R} \left(\frac{\mu(B_X(x_i, 3\varrho))}{|B_X(x_i, 3\varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \leqslant$$

$$\leqslant C \int_{B_X} \sum_i \frac{\mu(B_X(x_i, R/n))}{|B_X(x_i, R/n)|} \mathbf{1}_{B_X(x_i, R/n)} dx \int_{3R/n}^{72R} \left(\frac{\mu(B_X(x_i, \varrho))}{|B_X(x_i, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} .$$

Let now be $x \in B_X(x_i, R/n)$ and $\varrho \geqslant 3R/n$; then $B_X(x, \varrho/2) \subseteq B_X(x_i, \varrho) \subseteq B_X(x, 2\varrho)$ and

$$\int_{3R/n}^{72R} \left(\frac{\mu(B_X(x_i, \varrho))}{|B_X(x_i, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \leqslant \int_{3R/n}^{72R} \left(\frac{\mu(B_X(x, 2\varrho))}{|B_X(x, \varrho/2)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \leqslant$$

$$\leqslant C \int_{3R/n}^{72R} \left(\frac{\mu(B_X(x, 2\varrho))}{|B_X(x, 2\varrho)|} (2\varrho)^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \leqslant C \int_{6R/n}^{144R} \left(\frac{\mu(B_X(x, \varrho))}{|B_X(x, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} .$$

We obtain

$$\int_{3R/n}^{144R} \left(\frac{\mu_n(B_X(x_i, \varrho))}{|B_X(x_i, \varrho)|} \varrho^p \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho}$$

$$\leq C\left[\int_{6R/n}^{4R}\left(\frac{\mu(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho}+\left(\frac{\mu(B_X)}{|B_X|}R^p\right)^{\frac{1}{p-1}}\right];$$

then

$$\int_{B_X}\mu_n(dx)\int_0^{4R}\left(\frac{\mu_n(B_X(x,\varrho))}{|B_X(x,\varrho)|}\varrho^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho}\leq$$

$$\leq C\sum_i\frac{\mu(B_X(x_i,R/n))}{|B_X(x_i,R/n)|}\left|B_X\left(x_i,\frac{R}{n}\right)\right|\inf_{x\in B_X(x_i,R/n)}\int_{6R/n}^{4R}\left(\frac{\mu_n(B_X(x,\varrho))}{|B_X(x,\varrho)|}(\varrho)^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho}+$$

$$+C\left[\left(\sum_i\frac{\mu(B_X(x_i,R/n))}{|B_X(x_i,R/n)|}\left|B_X\left(x_i,\frac{R}{n}\right)\right|\left(\frac{\mu(B_X)}{|B_X|}R^p\right)^{\frac{1}{p-1}}\right)\right]\leq$$

$$\leq C\left[\int_{B_X}\mu(dx)\int_{6R/n}^{4R}\left(\frac{\mu(B_X(x,\varrho))}{|B_X(x,\varrho)|}(\varrho)^p\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho}+\mu(B_X)\left(\frac{\mu(B_X)}{|B_X|}R^p\right)^{\frac{1}{p-1}}\right].$$

Hence the sequence μ_n is bounded in $H^{-1,p'}(B_X, X)$, moreover the sequence is bounded as measures sequence since $\mu_n(B_X)\leq M\mu(B_X)$ at least for n big enough; then, up to extraction of subsequences, we have that μ_n converges to a measures ζ weakly in $H^{-1,p'}(B_X, X)$ and weakly* in the measures. Let now g be in $C(B_X)$ with compact support in B_X; we denotes by $\varepsilon(s)$ the modulus of uniform continuity of g. Then

$$\int_{B_X}g\mu_n(dx)\geq\sum_i\left(\sup_{x\in B_X(x_i,R/n)}g-\varepsilon\left(\frac{R}{n}\right)\right)\mu(B_X)\left(x_i,\frac{R}{n}\right)\geq$$

$$\geq\int_{B_X}g\mu(dx)-M\varepsilon\left(\frac{R}{n}\right)\mu(B_X).$$

Let now $n\to+\infty$, we obtain

$$\int_{B_X}g\zeta(dx)\geq\int_{B_X}g\mu(dx),$$

hence $0\leq\mu\leq\zeta$ in the measures; then, since $\zeta\in H^{-1,p'}(B_X, X)$, we have $\mu\in H^{-1,p'}(B_X, X)$. ■

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