Rendiconti
Accademia Nazionale delle Scienze detta dei XL
*Memorie di Matematica e Applicazioni*
118° (2000), Vol. XXIV, fasc. 1, pagg. 111-120

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# On the Rosenblatt coefficient for normalized sums of real random variables (**)

ABSTRACT. — For a given sequence $(X_n)_{n \geq 1}$ of independent identically distributed real random variables, we consider the normalized sums $U_n$ defined by $U_n = (X_1 + \ldots + X_n)/\sqrt{n}$, and we give some results on $\mathbf{Cov}(I_A(U_p), I_B(U_q))$ with $p$, $q$ integers and $A$, $B$ Borel sets in $\mathbb{R}$.

## Sul coefficiente di Rosenblatt per somme normalizzate di variabili aleatorie reali

SUNTO. — Per un'assegnata successione $(X_n)_{n \geq 1}$ di variabili aleatorie reali, indipendenti e identicamente distribuite, si considerano le somme normalizzate $U_n$, definite da $U_n = (X_1 + \ldots + X_n)/\sqrt{n}$, e si dimostrano alcuni risultati riguardanti le covarianze del tipo $\mathbf{Cov}(I_A(U_p), I_B(U_q))$, con $(p, q)$ coppia d'interi e $(A, B)$ coppia d'insiemi boreliani di $\mathbb{R}$.

## 0. - INTRODUCTION

Let $(a_{k,n})_{k \geq 1, n \geq 1}$ be a matrix of real numbers. A classical problem of Probability Theory is the asymptotic behaviour, as $n$ go to infinity, of weighted partial sums such as

$$Z_n = \sum_{k=1}^{n} a_{k,n} Y_{k,n},$$

where $(Y_{k,n})$ is a sequence of real random variables, satisfying suitable assumptions (see for instance Stout [1], chap. 4). In the last years many authors (see for instance

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(**) Memoria presentata il 20 aprile 2000 da Giorgio Letta, uno dei XL.

Brosamler [2], Schatte [3], Lacey-Philipp [4]) have considered the case

$$Y_{k,n} = I_{A_{k,n}}(U_k);$$

here $A_{k,n}$ are Borel sets in $\mathbb{R}$, and $U_n$ denotes the random variable

$$(X_1 + \ldots + X_n)/\sqrt{n},$$

where $(X_n)_{n \geq 1}$ is a sequence of independent identically distributed real random variables, with $\boldsymbol{E}[X_1^2] = 1$ and $\boldsymbol{E}[X_1] = 0$, defined on a probability space $(\Omega, \mathcal{A}, P)$. This leads in a natural way to the problem of evaluating

$$\mathbf{Cov}(I_A(U_p), I_B(U_q)),$$

with $p$, $q$ integers and $A$, $B$ Borel sets in $\mathbb{R}$. In the present paper we give some bounds for the *Rosenblatt coefficient*

$$\sup_{A,x} |\mathbf{Cov}(I_A(U_p), I_{]-\infty,x]}(U_q))|,$$

(where $A$ varies among all Borel sets in $\mathbb{R}$ and $x$ in $\mathbb{R}$), and for

$$\sup_{A,B} |\mathbf{Cov}(I_A(U_p), I_B(U_q))|,$$

(where $A$, $B$ vary among all Borel sets in $\mathbb{R}$, with $B$ included in a fixed set of finite measure). We obtain some results (Theorems 1, 2 and 3) which can be set into the framework of the so-called *almost-orthogonal* random variables; for this kind of variables some interesting laws of large numbers exist (see for ex. Lacey-Philipp [4], pag. 203, Atlagh-Weber [5], pag. 52, and, for a detailed study, Weber [6], sect. 7.4).

## 2. - Notations and first results

We start by recalling a result on the concentration function of a sum of random variables. Let $Q_n$ be the concentration function of $U_n$, namely the function defined on $\mathbb{R}_+$ by

$$Q_n(\lambda) = \sup_x P\{x \leq U_n \leq x + \lambda\}.$$

If the law $\mu$ of $X_1$ is not degenerate, then there is a constant $C$, depending on $\mu$ only, such that, for every real number $\varepsilon > 0$, the inequality

$$Q_n(\varepsilon) \leq C(\varepsilon + 1/\sqrt{n})$$

holds good. See Petrov [7], pag. 49 for a proof. We recall also a result on the difference of the distribution functions of two random variables $Y$, $Z$. Let $\phi_Y$ and $\phi_Z$ be the characteristic functions of $Y$ and $Z$ respectively; then we have the inequality

$$|P\{Y \leqslant x\} - P\{Z \leqslant x\}| \leqslant \frac{1}{\pi} \int_{\mathbb{R}} \frac{|\phi_Y(t) - \phi_Z(t)|}{|t|} \, \mathrm{d}t \, .$$

Without loss of generality, we assume in the sequel $p \leqslant q$.

LEMMA 1: *For every Borel set A in* $\mathbb{R}$ *and for every Lipschitz function f, with Lipschitz constant L, we have*

$$|\mathbf{Cov}\,(I_A(U_p),\ f(U_q))| \leqslant L\sqrt{2p/q}\, .$$

PROOF: We can assume that the event $H = \{U_p \in A\}$ is not negligible. Denote by $\boldsymbol{E}_H[\cdot]$ the expectation with respect to the conditional probability measure $P(\cdot|H)$, and let $(X_n')_n$ be an independent copy of the sequence $(X_n)_n$. Put

$$V_q = (X_1' + \ldots + X_p' + X_{p+1} + \ldots + X_q)/\sqrt{q}. \tag{1}$$

Then we have

$$\boldsymbol{E}_H[f(V_q)] = \boldsymbol{E}[f(U_q)]\, ,$$

hence

$$\begin{aligned}
|\mathbf{Cov}\,(I_A(U_p),\ f(U_q))| &= \left| \int_H f(U_q)\, \mathrm{d}P - P(H) \int f(U_q)\, \mathrm{d}P \right| \\
&= P(H)\, |\boldsymbol{E}_H[f(U_q)] - \boldsymbol{E}[f(U_q)]| \\
&= P(H)\, |\boldsymbol{E}_H[f(U_q)] - \boldsymbol{E}_H[f(V_q)]| \\
&\leqslant P(H)\, L\boldsymbol{E}_H[\,|U_q - V_q|\,] \\
&\leqslant L\boldsymbol{E}[\,|U_q - V_q|\,]\, .
\end{aligned}$$

By using the second moment, we get

$$\begin{aligned}
|\mathbf{Cov}(I_A(U_p),\ f(U_q))| &\leqslant L(\mathbf{Var}\,[U_q - V_q])^{1/2} \\
&= L\left( q^{-1} \sum_{k=1}^{p} \mathbf{Var}\,[X_k - X_k'] \right)^{1/2} \\
&= L\sqrt{2p/q}.
\end{aligned}$$

The lemma is thus proved.

LEMMA 2: *Let ε be a strictly positive real number. For every Borel set A in ℝ we have*

$$|\mathbf{Cov}(I_A(U_p), I_{]-\infty,x]}(U_q))| \leq \frac{2}{\varepsilon}\sqrt{\frac{p}{q}} + 2Q_q(\varepsilon).$$

PROOF: Let the real numbers $\varepsilon$ and $x$ be fixed, and denote by $f_\varepsilon$ the Lipschitz function defined as

$$f_\varepsilon(t) = I_{]-\infty,x]}(t) + g_\varepsilon(t) = I_{]-\infty,x]}(t) + \left(1 + \frac{x-t}{\varepsilon}\right) I_{]x,x+\varepsilon]}(t).$$

One verifies immediately that $f_\varepsilon$ has Lipschitz constant $1/\varepsilon$. Let $H$ be the event $\{U_p \in A\}$; we can assume again that $H$ is not negligible. If $Q$ denotes the conditional probability law $P(\cdot|H)$, we have

$$|\mathbf{Cov}(I_A(U_p), I_{]-\infty,x]}(U_q))| = P(H)\,|Q\{U_q \leq x\} - P\{U_q \leq x\}|.$$

Moreover

$$\begin{aligned} |Q\{U_q \leq x\} - P\{U_q \leq x\}| &= |\boldsymbol{E}^Q[(f_\varepsilon - g_\varepsilon)(U_q)] - \boldsymbol{E}^P[(f_\varepsilon - g_\varepsilon)(U_q)]| \\ &= |\boldsymbol{E}^Q[(f_\varepsilon - g_\varepsilon)(U_q)] - \boldsymbol{E}^Q[(f_\varepsilon - g_\varepsilon)(V_q)]| \\ &= |\boldsymbol{E}^Q[f_\varepsilon(U_q) - f_\varepsilon(V_q)] - \boldsymbol{E}^Q[g_\varepsilon(U_q) - g_\varepsilon(V_q)]|, \end{aligned}$$

where $V_q$ are the random variables defined in (1). By arguing as in Lemma 1, we get

$$|\boldsymbol{E}^Q[(f_\varepsilon(U_q) - f_\varepsilon(V_q)]| \leq \frac{2}{\varepsilon P(H)}\sqrt{\frac{p}{q}}. \tag{2}$$

Since we have trivially

$$|\boldsymbol{E}^Q[(g_\varepsilon(U_q) - g_\varepsilon(V_q)]| \leq \frac{2Q_q(\varepsilon)}{P(H)}, \tag{3}$$

from relations (2) and (3) it follows that

$$|Q\{U_q \leq x\} - P\{U_q \leq x\}| \leq \frac{2}{\varepsilon P(H)}\sqrt{\frac{p}{q}} + \frac{2Q_q(\varepsilon)}{P(H)},$$

hence the statement of the lemma.

3. - THE MAIN RESULTS.

THEOREM 1: *There is a constant $K$, depending on the law of $X_1$ only, such that, for every pair $p$, $q$ of integers, we have*

$$\sup_{A,\,x} |\mathbf{Cov}\,(I_A(U_p),\, I_{]-\infty,\,x]}(U_q))| \leqslant K \sqrt[4]{\frac{p}{q}}\,. \tag{4}$$

PROOF: The statement being trivial if $X_1$ is degenerate, we can assume that $X_1$ is not degenerate; then, by Lemma 2 and Petrov's theorem (stated in the preceding section), we have for $\varepsilon > 0$,

$$\sup_{A,\,x} |\mathbf{Cov}\,(I_A(U_p),\, I_{]-\infty,\,x]}(U_q))| \leqslant \frac{2}{\varepsilon}\sqrt{\frac{p}{q}} + 2C\left(\varepsilon + \frac{1}{\sqrt{q}}\right).$$

The conclusion then follows since the minimum of the function

$$\varepsilon \mapsto (2/\varepsilon)\,\sqrt{p/q} + 2C(\varepsilon + 1/\sqrt{q})$$

is given by $4\sqrt{C}\sqrt[4]{\frac{p}{q}} + \frac{2C}{\sqrt{q}}$, which is obviously less than $(4\sqrt{C} + 2C)\sqrt[4]{\frac{p}{q}}$. This proves the statement of the theorem.

REMARK When the $X_n$ are gaussian random variables, we have

$$\sup_{A,\,x} |\mathbf{Cov}\,(I_A(U_p),\, I_{]-\infty,\,x]}(U_q))| = K_0\sqrt{\frac{p}{q}}\,. \tag{5}$$

Hence we are faced with the question of what conditions on the law of $X_1$ can guarantee an «optimal» relation as (5); in other words we are wondering when we are allowed to put the square root of $p/q$, in place of the fourth one, in relation (4). In what follows we are going to give two sufficient conditions for this to happen. We need two lemmas.

LEMMA 3: *Let $\phi$ be the characteristic function of $X_1$ and $l$ a member of $\{0, 1\}$. Assume that there exists an integer $r$ such that the function $t \mapsto |t^l \phi^r(t)|$ is integrable. Then the relation*

$$\sup_{p/q \leqslant 1/2} \int |t|^l \left|\phi\left(\frac{t}{\sqrt{q}}\right)\right|^{q-p} dt < \infty$$

*holds good.*

Proof: Let $L$ be the real function defined by

$$L(t) = \boldsymbol{E}\left[X_1^2\left(1 \wedge |t| \frac{|X_1|}{3}\right)\right].$$

It is easily seen that $L$ is symmetric, increasing on $[0, \infty[$, bounded by 1 and has limit 0 in $t = 0$, so that there exists a real number $\bar{t}$ on $]0, 1[$ such that $L(\bar{t}) < 1/4$. By the inequality

$$\left| e^{itx} - 1 - itx + \frac{1}{2} t^2 x^2 \right| \leqslant t^2 x^2 \left(1 \wedge |t| \frac{|x|}{3}\right),$$

(Kallenberg [8], pag. 69) we get $\left| \phi\left(\frac{t}{\sqrt{n}}\right) - 1 + \frac{1}{2}\frac{t^2}{n} \right| \leqslant \frac{t^2}{n} L\left(\frac{t}{\sqrt{n}}\right)$, for every real number $t$; it follows

$$\left| \phi\left(\frac{t}{\sqrt{n}}\right) \right|^{n/2} \leqslant \left[ \left| 1 - \frac{1}{2}\frac{t^2}{n} \right| + \frac{t^2}{n} L\left(\frac{t}{\sqrt{n}}\right) \right]^{n/2}.$$

Hence on the interval $J_n = [-\bar{t}\sqrt{n}, \bar{t}\sqrt{n}]$ we obtain

$$|t^l| \left| f\left(\frac{t}{\sqrt{n}}\right) \right|^{n/2} \leqslant |t^l| \left[1 - \frac{1}{2}\frac{t^2}{n} + \frac{1}{4}\frac{t^2}{n}\right]^{n/2} \leqslant |t^l| e^{-t^2/8},$$

while, on $J_n^c$, we have $\left| \phi\left(\frac{t}{\sqrt{n}}\right) \right| \leqslant \sup_{|u| \geqslant \bar{t}} |\phi(u)| = d < 1$, where, since $|\phi|$ is integrable, the last inequality follows from a well known result on characteristic functions (see Feller [9], pag. 501). Hence, for every pair of integers $p$, $q$, with $p \leqslant q/2$, one gets the inequalities

$$\begin{aligned}
\int |t|^l \left| \phi\left(\frac{t}{\sqrt{q}}\right) \right|^{q-p} dt &\leqslant \sup_n \int |t|^l \left| \phi\left(\frac{t}{\sqrt{n}}\right) \right|^{n/2} dt \\
&\leqslant \int |t|^l e^{-t^2/8} dt + \sup_n \int_{J_n^c} |t|^l \left| \phi\left(\frac{t}{\sqrt{n}}\right) \right|^{n/2} dt \\
&\leqslant C_1 + \sup_n n^{\frac{l+1}{2}} d^{\frac{n}{2} - r} \int |u^l \phi(u)|^r du,
\end{aligned}$$

where $C_1$ is an absolute constant. The lemma is proved.

Lemma 4: *Let $p$, $q$ be two integers, with $p \leqslant q$, and assume that the event $\{U_p \in A\}$ is not negligible. Denote by $\boldsymbol{\Phi}_q$ and $\widetilde{\boldsymbol{\Phi}}_q$ the characteristic functions of $U_q$ with respect to*

*P and* $P_{\{U_p \in A\}}$; *then we have*

$$|\mathbf{Cov}\,(I_A(U_p),\, I_{]-\infty,\,x]}(U_q))| \leq \frac{P\{U_p \in A\}}{\pi} \int_{\mathbb{R}} \frac{|\Phi_q(t) - \widetilde{\Phi}_q(t)|}{|t|}\, dt\,.$$

*If in addition the function* $t \mapsto |t|\,|\phi^r(t)|$ *is integrable, then for every bounded Borel set B and for every q greater than r, we have*

$$|\mathbf{Cov}\,(I_A(U_p),\, I_B(U_q))| \leq \frac{P\{U_p \in A\}}{2\pi}\, \text{meas}\,(B) \int_{\mathbb{R}} |\Phi_q(t) - \widetilde{\Phi}_q(t)|\, dt\,.$$

Proof: The first statement follows from the relation

$$\begin{aligned} |\mathbf{Cov}\,(I_A(U_p),\, I_{]-\infty,\,x]}(U_q))| &= P\{U_p \in A\}\, |P\{U_q \leq x\} - P_{\{U_p \in A\}}\{U_q \leq x\}| \\ &\leq \frac{P\{U_p \in A\}}{\pi} \int_{\mathbb{R}} \frac{|\Phi_q(t) - \widetilde{\Phi}_q(t)|}{|t|}\, dt\,, \end{aligned}$$

by the relation on the difference of two distribution function (see section 2). As to the second statement, just note that, for $q$ greater than $r$, we have

$$\begin{aligned} |\mathbf{Cov}\,(I_A(U_p),\, I_{]-\infty,\,x]}(U_q))| &= P\{U_p \in A\}\, |P\{U_q \in B\} - P_{\{U_p \in A\}}\{U_q \in B\}| \\ &= \frac{P\{U_p \in A\}}{2\pi} \left| \int_B dx \int_{\mathbb{R}} e^{-itx}[\Phi_q(t) - \widetilde{\Phi}_q(t)]\, dt \right|. \end{aligned}$$

This proves the lemma.

Theorem 2: *Assume that there exists an integer r such that the function* $t \mapsto |\phi^r(t)|$ *is integrable. Then, for every pair of integers* $p, q$, *we have the relation*

$$\sup_{A,\,x} |\mathbf{Cov}\,(I_A(U_p),\, I_{]-\infty,\,x]}(U_q))| \leq K \sqrt{\frac{p}{q}}\,,$$

where *K is a constant depending on the law of* $X_1$ *only. Moreover, if the function* $t \mapsto |t|\,|\phi^r(t)|$ *is integrable too, then for every bounded Borel set B we have*

$$\mathbf{Cov}\,(I_A(U_p),\, I_B(U_q)) \leq K_1\ \text{meas}\,(B) \sqrt{\frac{p}{q}}\,,$$

*where* $K_1$ *is a constant depending on the law of* $X_1$ *only.*

Proof: Without loss of generality we can assume that $p \leq q/2$, since for $p \geq q/2$ we have trivially

$$|\mathbf{Cov}\,(I_A(U_p),\, I_B(U_q))| = \sqrt{\frac{p}{q}}\sqrt{\frac{q}{p}}\,|\mathbf{Cov}\,(I_A(U_p),\, I_B(U_q))| \leqslant 4\sqrt{\frac{p}{q}}\,.$$

As in the proof of Lemma 1, we assume also that the event $H = \{U_p \in A\}$ is not negligible. Denote by $Q$ the conditional probability measure $P(\cdot|H)$ and put

$$V_q = \frac{X_1' + \ldots + X_p' + X_{p+1} + \ldots + X_q}{\sqrt{q}}\,,$$

where $(X_n')$ is an independent copy of the sequence $(X_n)$; then we have

$$\begin{aligned} |\Phi_q(t) - \widetilde{\Phi}_q(t)| &= |\boldsymbol{E}^Q[e^{itU_q}] - \boldsymbol{E}^Q[e^{itV_q}]| \\ &= \left|\phi\left(\frac{t}{\sqrt{q}}\right)\right|^{q-p} |\boldsymbol{E}^Q[e^{it\frac{X_1+\ldots+X_p}{\sqrt{q}}}] - \boldsymbol{E}^Q[e^{it\frac{X_1'+\ldots+X_p'}{\sqrt{q}}}]| \\ &\leqslant \left|\phi\left(\frac{t}{\sqrt{q}}\right)\right|^{q-p} \frac{2|t|}{P(H)}\sqrt{\frac{p}{q}}\,. \end{aligned}$$

Lemmas 3 and 4 achieve the conclusion.

Finally, we have a «Berry-Esseen type» result. In detail:

THEOREM 3: *Assume that $X_1$ has finite absolute third moment. Then we have*

$$\sup_{A,\,x} |\mathbf{Cov}\,(I_A(U_p),\, I_{]-\infty,\,x]}(U_q))| \leqslant K_2\sqrt{\frac{p}{q}}\,,$$

*where the constant $K_2$ depends on the law of $X_1$ only.*

PROOF: As in the proof of the preceding theorem, we can assume $p \leqslant q/2$. Let $(Y_n)$ be a sequence of random variables and assume that the $(Y_n)$ are independent $\mathcal{N}(0,\,1)$ and independent on $(X_n)$. Put

$$U_q' = \frac{X_1 + \ldots + X_p + Y_{p+1} + \ldots + Y_q}{\sqrt{q}}\,.$$

Since $\mathbf{Cov}\,(I_A(U_p),\, I_{]-\infty,\,x]}(U_q))$ is equal to

$$\mathbf{Cov}(I_A(U_p),\, I_{]-\infty,\,x]}(U_q) - I_{]-\infty,\,x]}(U_q')) + \mathbf{Cov}\,(I_A(U_p),\, I_{]-\infty,\,x]}(U_q'))\,,$$

it will be enough to prove that the two terms in the above sum are bounded by a number of the form $K\sqrt{\frac{p}{q}}$. As in the above theorem, denote by $H$ the (non negligible) event $\{U_p \in A\}$ and by $Q$ the conditional probability measure $P(\cdot|H)$; then we have

$$|\mathbf{Cov}\,(I_A(U_p),\, I_{]-\infty,\,x]}(U'_q))| = P(H)\,|(Q\{U'_q \leqslant x\} - P\{U'_q \leqslant x\})\,|$$

$$\leqslant \frac{P(H)}{\pi}\int \frac{1}{|t|}\left|\int e^{itU'_q}\,dQ - \int e^{itV'_q}\,dQ\right| dt\,.$$

Here $V'_q$ denotes $\dfrac{X'_1 + \ldots + X'_p + Y_{p+1} + \ldots + Y_q}{\sqrt{q}}$, where $(X'_n)$ is a copy of $(X_n)$, independent on each $X_n$, $Y_n$. It follows

$$|\mathbf{Cov}\,(I_A(U_p),\, I_{]-\infty,\,x]}(U'_q))| \leqslant \frac{P(H)}{\pi}\int e^{-\frac{t^2(q-p)}{2q}}\,\boldsymbol{E}^{Q}[\,|U'_q - V'_q\,|\,]\,dt$$

$$\leqslant \frac{1}{\pi}\int e^{-\frac{t^2(q-p)}{2q}}\,\boldsymbol{E}^{P}[\,|U'_q - V'_q\,|^2\,]^{1/2}\,dt$$

$$\leqslant \frac{1}{\pi}\int e^{-\frac{t^2}{2}}\,\boldsymbol{E}^{P}[\,|U'_q - V'_q\,|^2\,]^{1/2}\,dt$$

$$\leqslant \boldsymbol{E}^{P}[\,|U'_q - V'_q\,|^2\,]^{1/2}$$

$$\leqslant 2\sqrt{\frac{p}{q}}\,.$$

Let $a_{p,\,q}$ be the second term $\mathbf{Cov}\,(I_A(U_p),\, I_{]-\infty,\,x]}(U_q) - I_{]-\infty,\,x]}(U'_q))$; then

$$|a_{p,\,q}| = |b_{p,\,q} - P\{U_p \in A\}(P\{U_q \leqslant x\} - P\{U'_q \leqslant x\})\,|\,,$$

where $b_{p,\,q}$ denotes $P\{U_p \in A,\, U_q \leqslant x\} - P\{U_p \in A,\, U'_q \leqslant x\}$. By the Berry-Esseen inequality, we get

$$|b_{p,\,q}| = \int_A |F_{p,\,q}(g(x_1,\,\ldots,\,x_p)) - N(g(x_1,\,\ldots,\,x_p))\,|\,d\mu(x_1,\,\ldots,\,x_p)$$

$$\leqslant \frac{E[\,|X_1|^3\,]}{\sqrt{q-p}}\,,$$

where $\mu$ is the law of $U_p$ under $P$, $N$ the distribution function of the standard gaussian law and $g(x_1,\,\ldots,\,x_p)$ the real number $\dfrac{x\sqrt{q} - (x_1 + \ldots + x_p)}{\sqrt{q-p}}$. By arguing analogously, we get also

$$|P\{U_p \in A\}(P\{U_q \leqslant x\} - P\{U'_q \leqslant x\})\,| \leqslant \frac{E[\,|X_1|^3\,]}{\sqrt{q-p}}\,.$$

From the two above relations and the inequality $p \leqslant q/2$ it follows that

$$|a_{p,q}| \leqslant 2\,\frac{\boldsymbol{E}[\,|X_1|^3\,]}{\sqrt{q-p}} \leqslant 4\boldsymbol{E}[\,|X_1|^3\,]\sqrt{\frac{p}{q}}\,.$$

This concludes the proof.

REMARK: It is for the moment an open problem whether the relation in theorems 2 e 3 holds good even without any assumption on the third moment or the characteristic function of $X_1$.

*Acknowledgements.* I am greatly indebted to Prof. M. Weber for stimulating my interest on this topic; I am also grateful to Prof. Luca Pratelli: I would have not brought this paper to an end without his ideas and suggestions.

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