Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
117° (1999), Vol. XXIII, fasc. 1, pagg. 125-166

B.-W. SCHULZE - M. I. VISHIK - I. WITT - S. V. ZELIK (*)

The Trajectory Attractor for a Nonlinear Elliptic System in a Cylindrical Domain with Piecewise Smooth Boundary

SUMMARY. — In the half-cylinder $\Omega_+ = \mathbb{R}_+ \times \omega$, where ω is a bounded polyhedral domain in $\mathbb{R}^n$, the quasilinear elliptic second-order system

$$(1) \qquad \begin{cases} a(\partial_t^2 u + \Delta u) + \gamma \partial_t u - f(u) = g(t), \\ u|_{t=0} = u_0, \quad \partial_\nu u|_{\partial\omega} = 0 \end{cases}$$

is considered. Here $(t, x) \in \Omega_+$, $\Delta = \Delta_x$, $u = u(t, x) = (u^1, \ldots, u^k)$ is the unknown vector-function, f, g are given functions, and $a = a^* > 0$, γ are given $k \times k$ matrices. Under certain natural assumptions on the nonlinear part $f(u)$ and the right-hand side g the trajectory attractor for the problem (1) that describes the long-term behaviour of the solutions as $t \to +\infty$ is constructed. In the three-dimensional case ($\dim \omega = 2$), the behaviour of the solutions to the problem (1) near the edges of $\mathbb{R}_+ \times \omega$ is also investigated as $t \to +\infty$.

Studio dell'attrattore per un sistema ellittico non lineare in un dominio cilindrico con frontiera regolare a tratti

SUNTO. — Si considera nel semicilindro $\Omega_+ = \mathbb{R}_+ \times \omega$ (dove ω è un dominio poliedrico in $\mathbb{R}^n$) il sistema del secondo ordine, ellittico, quasi lineare

$$(1) \qquad \begin{cases} a(\partial_t^2 u + \Delta u) + \gamma \partial_t u - f(u) = g(t), \\ u|_{t=0} = u_0, \quad \partial_\nu u|_{\partial\omega} = 0 \end{cases}$$

(*) Indirizzi degli Autori: B.-W. Schulze: University of Potsdam, Institute for Mathematics, PF 601553, D-14415 Potsdam, Germany; E-mail: schulze@math.uni-potsdam.de); M. I. Vishik: Institute for Problems of Transmission Information, Russian Academy of Sciences, Bolshoi Karetnii 19, Moscow 101447, GSP-4, Russia; E-mail: vishik@ippi.ac.msk.su); I. Witt: University of Potsdam, Institute for Mathematics, PF 601553, D-14415 Potsdam, Germany; E-mail: ingo@math.uni-potsdam.de); S. V. Zelik, Institute for Problems of Transmission Information, Russian Academy of Sciences, Bolshoi Karetnii 19, Moscow 101447, GSP-4, Russia; E-mail: zelik@iitp.ru)

(**) Memoria presentata l'8 giugno 1999 da Mark Višik, uno dei XL.

con $(t, x) \in \Omega_+$, $\Delta = \Delta_x$, dove $u = u(t, x) = (u^1, \ldots, u^k)$ è la funzione vettoriale incognita, e f, g sono funzioni assegnate, mentre $a = a^* > 0$ e γ sono assegnate matrici $k \times k$. Sotto condizioni abbastanza naturali imposte alla parte non lineare $f(u)$ e al secondo membro g, si costruisce, per il problema (1), l'attrattore che descrive il comportamento asintotico delle soluzioni al tendere di t all'infinito. Nel caso tridimensionale ($\dim \omega = 2$), si studia anche il comportamento asintotico delle soluzioni del problema (1) in prossimità dei lati di $\mathbb{R}_+ \times \omega$.

Introduction

In the half-cylinder $\Omega_+ = \mathbb{R}_+ \times \omega$, where ω is a bounded polyhedral domain in $\mathbb{R}^n$, we consider the following elliptic system:

(0.1)
$$\begin{cases} a(\partial_t^2 u + \Delta u) + \gamma \partial_t u - f(u) = g(t), \\ u|_{t=0} = u_0, \qquad \partial_\nu u|_{\partial\omega} = 0. \end{cases}$$

Here (t, x) are the variables in Ω_+, $u = u(t, x) = (u^1, \ldots, u^k)$, $g = (g^1, \ldots, g^k)$, and $f(u)$ are vector-valued functions, Δ is the Laplacian with respect to the variable $x = (x^1, \ldots, x^n)$, and γ and a are constant $k \times k$ matrices with $a = a^* > 0$.

Notice that domains ω with non-degenerate edges and corners on the boundary are admitted. To be more rigorous, the domain ω is said to be polyhedral if any of its boundary points b is either regular or there are a polyhedron $P \subset \mathbb{R}^n$, a non-regular boundary point b_1 of P, open subsets U, V of $\mathbb{R}^n$ with $b \in U$, $b_1 \in V$, and a C^∞-diffeomorphism $\chi: U \to V$ such that $\chi(b) = b_1$ and $\chi(\overline{\omega} \cap U) = \overline{P} \cap V$.

On the nonlinear term $f(u)$ we impose the following conditions:

(0.2)
$$\begin{cases} (1) \quad f \in C(\mathbb{R}^k, \mathbb{R}^k); \\ (2) \quad f(u) \cdot u \geq -C_1 + C_2 |u|^p, \quad 2 < p < 2 + 4/(n-3); \\ (3) \quad |f(u)| \leq C(1 + |u|^{p-1}). \end{cases}$$

Here and below $u \cdot v$ denotes the inner product in $\mathbb{R}^k$.

We suppose that the right-hand side g belongs to the space $[L^2_{loc}(\Omega_+)]^k$ and has a finite norm

(0.3)
$$|g|_b = \sup_{T \geq 0} \|g, \Omega_T\|_{0,2} < \infty,$$

where $\Omega_T = (T, T+1) \times \omega$. In the sequel we denote by $H^{l,p}(U)$ the Sobolev space of functions on U whose generalized derivatives up to the order l belong to the space $L^p(U)$ (see [18]). Further we write $H^l(U)$ instead of $H^{l,2}(U)$ and $\|\cdot, U\|_{l,p}$ instead of $\|\cdot\|_{H^{l,p}(U)}$.

A solution $u(t, x)$ to the problem (0.1) is defined as a function that belongs to the space

$$(0.4) \qquad [H^2_Q(\Omega_T)]^k = [H^2((T, T+1), L^2(\omega)) \cap L^2((T, T+1), H^2_Q(\omega))]^k$$

for each $T \geq 0$ and satisfies Eq. (0.1) in the sense of distributions. Here we denote by $H^2_Q(\omega)$ the domain of the Laplace operator $-\Delta$ in $L^2(\omega)$ with the homogeneous Neumann boundary condition. Appendix A is devoted to a comprehensive study of $H^2_Q(\Omega_T)$. Let us notice already here that for polyhedral domains in general $H^2_Q(\omega) \neq \{u \in H^2(\omega), \partial_n u|_{\partial\omega} = 0\}$, in contrast to the case of smooth $\partial\omega$ (see [9]).

It is proved in the appendix that the third assumption of (0.2) implies that, for $u \in [H^2_Q(\Omega_T)]^k$, the function $f(u)$ belongs to $[L^2(\Omega_T)]^k$. Hence Eq. (0.1) can be considered as an equality in the space $[L^2_{loc}(\Omega_+)]^k$.

The initial data u_0 are assumed to belong to the trace space V_0 on $\{t = 0\}$ of functions in $[H^2_{Q,\,loc}(\Omega_+)]^k$.

For domains ω with smooth boundary, the problem (0.1) has been investigated under different assumptions on the nonlinear part f and the right-hand side g in [1], [3], [4], [20].

The main objective of this paper is to study the behaviour of the solutions to Eq. (0.1) as $t \to +\infty$ for polyhedral cross-sections ω. The following estimate is of fundamental significance in that connection:

$$(0.5) \qquad \|u, \Omega_T\|_{2,\,Q} \leq C(1 + \chi(1-T)\|u_0\|^p_{V_0}{}^{-1} + \|g, \tilde{\Omega}_T\|_{0,\,2}).$$

Here $\tilde{\Omega}_T = (\max\{0, T-1\}, T+2) \times \omega$, $\chi(z)$ is the Heaviside function, i.e., $\chi(z) = 1$ for $z \geq 0$ and $\chi(z) = 0$ for $z < 0$, and the constant C is independent of u_0. This estimate makes it possible to apply the methods of the theory of attractors ([2], [5-8]) to the problem (0.1).

Furthermore, estimate (0.5) implies that every solution $u(t, x)$ to the problem (0.1) is bounded as $t \to \infty$, i.e.,

$$(0.6) \qquad \|u\|_b = \sup_{T \geq 0} \|u, \Omega_T\|_{2,\,Q} < \infty.$$

The subspace of functions $u \in [\mathcal{D}'(\Omega_+)]^k$ which have finite norm (0.6) is denoted by F_0^+.

Since the conditions that we impose on the nonlinear function f (see (0.2)) guarantee in general only the existence (but not the uniqueness) of a solution to the problem (0.1), in order to describe the behaviour of solutions as $t \to +\infty$ we construct a trajectory attractor for the dynamical system generated by the semigroup $\{T_s, s \geq 0\}$ of positive shifts of the solutions to (0.1) along the t-axis (see [7], [8], [20]). Here, since g explicitly depends on t, it is natural to study the family of equations of the form (0.1) generated by all positive shifts of this equation with respect to t and their limits in a suitable topology (see § 3). The trajectory attractor A attracts the set K^+ of all trajectories of the above family as $t \to +\infty$. Recall that the attracting property is usually required only for those subsets of the phase space K^+ which are, in a certain sense, *bounded*. But in our case estimate (0.5) allows us to verify the following improved version (see also [22]): for any neighbourhood $\mathcal{O}(\mathrm{A})$ of the attractor A in the space K^+, there exists a number $T = T(\mathcal{O})$ such that

$$T_s K^+ \subset \mathcal{O}(\mathrm{A}) \qquad \text{for all } s \geq T. \tag{0.7}$$

Moreover, like an ordinary attractor, the trajectory attractor A is strictly invariant with respect to the semigroup $\{T_s, s \geq 0\}$ and is generated by all the trajectories of this semigroup that are defined and bounded for $t \in \mathbb{R}$.

We also study the problem of stabilization of the solutions to Eq. (0.1) as $t \to +\infty$ in the case that the nonlinear part f has a potential ($f = \nabla F$, F: $\mathbb{R}^k \to \mathbb{R}$). Especially, it is shown that in the autonomous case ($g(t, x) \equiv g(x)$) every solution to the problem (0.1) in the whole cylinder $\Omega = \mathbb{R} \times \omega$ is a heteroclinic orbit connecting two stationary solutions (see § 4).

The second part is devoted to the detailed investigation of the three-dimensional case ($\dim \omega = 2$). In § 5, the singular behaviour of functions $u \in F_0^+$ close to an edge in $\mathbb{R}_+ \times \partial\omega$ is described. This employs an approach for deriving edge asymptotics for solutions to elliptic differential equations developed by B.-W. Schulze and others (see [14-17]).

Furthermore, there is a non-canonical splitting of K^+ into a regular part and the space K^+_{sing} which contains the edge asymptotics of solutions belonging to K^+. This splitting is chosen in a way such that K^+_{sing} becomes invariant with respect to the semigroup $\{T_s, s \geq 0\}$ of positive shifts along the t-axis:

$$T_s\colon K^+_{\mathrm{sing}} \to K^+_{\mathrm{sing}} \qquad \text{for } s \geq 0\,. \tag{0.8}$$

We then show in § 6 that the semigroup $\{T_s, s \geq 0\}$ restricted to K^+_{sing} possesses an attractor $\mathrm{A}_{\mathrm{sing}}$ which in turn is interpreted as the singular part of the trajectory attractor A of Eq. (0.1).

1. - A PRIORI ESTIMATES

In this section we obtain *a priori* estimates for solutions to the problem (0.1). In the sequel these estimates will be used to prove existence of solutions and to construct the trajectory attractor.

THEOREM 1.1: *Let u be a solution to* (0.1). *Then*

$$\|u, \Omega_T\|^2_{1,2} \leq C(1 + \chi(1-T)\|u_0\|^2_{V_0} + \|g, \tilde{\Omega}_T\|^2_{0,2})\,, \tag{1.1}$$

where C is independent of u.

PROOF: By definition of V_0, there exists a function $v \in [H^2_{0,b}(\Omega_+)]^k$ (see Definition A.2) such that $\operatorname{supp} v \subset \Omega_0$, $v|_{t=0} = u_0$, and

$$\|v, \Omega_0\|_{2,Q} \leqslant C\|u_0\|_{V_0}, \tag{1.2}$$

where the constant C is independent of u_0.

Let us rewrite Eq. (0.1) for the function $w = u - v$,

$$\begin{cases} a(\partial_t w + \Delta w) + \gamma\partial_t w - f(w + v) = g(t) - a(\partial_t v + \Delta v) - \gamma\partial_t v \equiv h(t), \\ w|_{t=0} = 0. \end{cases} \tag{1.3}$$

From the choice of v it follows that

$$\|h, \Omega_T\|_{b,2} \leqslant C(\|g, \Omega_T\|_{b,2} + \chi(1-T)\|u_0\|_{V_0}). \tag{1.4}$$

Let $\phi(t) = \phi_T(t)$ be the following cut-off function:

$$\phi(t) = \begin{cases} (1 - |t - T - 1/2|)^{2p/(p-2)}, & \text{for } t \in (T - 1/2, T + 3/2), \\ 0, & \text{for } t \notin (T - 1/2, T + 3/2). \end{cases}$$

It is readily seen that $\phi' \in L^\infty(\mathrm{R})$. Moreover, the following estimate is valid:

$$|\phi'(t)| \leqslant C\phi(t)^{1/2 + 1/p} \quad \text{for } t \in \mathrm{R}. \tag{1.5}$$

Multiplying Eq. (1.3) by ϕw in R^k and integrating over Ω_+ gives us

$$\langle a\partial_t^2 w, \phi w\rangle + \langle a\Delta w, \phi w\rangle + \langle\gamma\partial_t w, \phi w\rangle - \langle f(v + w), \phi w\rangle = \langle h, \phi w\rangle. \tag{1.6}$$

From the positivity of a and (1.5), it follows that

$$-\langle a\partial_t^2 w, \phi w\rangle \geqslant C_1\langle\phi|\partial_t w|^2, 1\rangle - \langle|\phi'||\partial_t w|, |w|\rangle \geqslant C_1\langle\phi|\partial_t w|^2, 1\rangle - \tag{1.7}$$
$$-\frac{1}{2}C_1\langle\phi|\partial_t w|^2, 1\rangle - C\langle\phi^{2/p}|w|^2, 1\rangle \geqslant C_2\langle\phi|\partial_t w|^2, 1\rangle - C\langle\phi^{2/p}|w|^2, 1\rangle.$$

Applying Hölder's inequality to the third term in (1.6), we obtain the estimate

$$|\langle\gamma\partial_t w, \phi w\rangle| \leqslant \mu\langle\phi|\partial_t w|^2, 1\rangle + C_\mu\langle\phi|w|^2, 1\rangle \leqslant \tag{1.8}$$
$$\leqslant \mu\langle\phi|\partial_t w|^2, 1\rangle + C_\mu\langle\phi^{2/p}|w|^2, 1\rangle$$

for any $\mu > 0$ with some constant $C_\mu > 0$.

In view of assumption (0.2) on the nonlinear term $f(u)$, we further have

$$\langle f(w + v), \phi w\rangle = \langle f(w + v)\cdot(w + v), \phi\rangle - \langle f(w + v), v\phi\rangle \geqslant \tag{1.9}$$
$$\geqslant -C + C_1\langle\phi|w + v|^p, 1\rangle - C\langle 1 + |w + v|^{p-1}, \phi|v|\rangle \geqslant$$
$$\geqslant -C_2(1 + \langle\phi|v|^p, 1\rangle) + C_3\langle\phi|w|^p, 1\rangle \geqslant -C_4(1 + \chi(1-T)\|u_0\|_{V_0}^p) + C_5\langle\phi|w|^p, 1\rangle.$$

Here we have employed (1.2) and the embedding (A.11).

Using the positivity of a again, after integration by parts we find

$$-\langle a\Delta w, \phi w\rangle \geq C\langle \phi|\nabla w|^2, 1\rangle. \tag{1.10}$$

Finally, from (1.4) and Hölder's inequality we conclude

$$\begin{aligned} |\langle b, \phi w\rangle| &\leq \langle \phi|b|^2, 1\rangle + \langle \phi|w|^2, 1\rangle \leq \\ &\leq C(\langle \phi|g|^2, 1\rangle + \chi(1-T)\|u_0\|_{V_0}^p) + C_1\langle \phi^{2/p}|w|^2, 1\rangle. \end{aligned} \tag{1.11}$$

By inserting all the estimates (1.7)-(1.11) into (1.6), a short calculation yields

$$\begin{aligned} &\langle \phi|\partial_t w|^2, 1\rangle + \langle \phi|\nabla w|^2, 1\rangle + \langle \phi|w|^p, 1\rangle - C\langle \phi^{2/p}|w|^2, 1\rangle \leq \\ &\leq C_1(1 + \langle \phi|g|^2, 1\rangle + \chi(1-T)\|u_0\|_{V_0}^p). \end{aligned} \tag{1.12}$$

We estimate the last term of the left-hand side in (1.12) using Hölder's inequality,

$$\langle \phi^{2/p}|w|^2, 1\rangle = \langle |\phi^{1/p} w|^2, 1\rangle \leq C\langle \phi|w|^p, 1\rangle^{2/p} \leq \mu\langle \phi|w|^p, 1\rangle + C_\mu,$$

which holds for any $\mu > 0$. Choosing $\mu > 0$ sufficiently small, this estimate inserted into (1.12) yields

$$\langle \phi|\partial_t w|^2, 1\rangle + \langle \phi|\nabla w|^2, 1\rangle + \langle \phi|w|^p, 1\rangle \leq C_2(1 + \langle \phi|g|^2, 1\rangle + \chi(1-T)\|u_0\|_{V_0}^p). \tag{1.13}$$

Recall that $\phi(t) > C_0 > 0$ for $t \in (T, T+1)$. Hence from (1.13) we infer that

$$\|w, \Omega_T\|_{1,2}^2 \leq C(1 + \chi(1-T)\|u_0\|_{V_0}^p + \|g, \tilde{\Omega}_T\|_{0,2}^2). \tag{1.14}$$

Theorem 1.1 is proved. ∎

REMARK 1.2: In a similar manner it follows from (1.13) that

$$\|u, \Omega_T\|_{0,p}^p \leq C(1 + \chi(1-T)\|u_0\|_{V_0}^p + \|g, \tilde{\Omega}_T\|_{0,2}^2) \tag{1.15}$$

THEOREM 1.3: *Let u be a solution to (0.1). Then, for each $T \geq 0$, we have*

$$\|u, \Omega_T\|_{0,2(p-1)}^{2(p-1)} \leq C(1 + \chi(1-T)\|u_0\|_{V_0}^{2(p-1)} + \|g, \tilde{\Omega}_T\|_{0,2}^2 + \|u, \tilde{\Omega}_T\|_{0,p}^p). \tag{1.16}$$

The exponent p is defined in (0.2).

PROOF: We fix some $T \geq 0$ and take another cut-off function $\varphi(t) \in C_0^\infty(\mathrm{R})$ such that $\varphi(t) = 1$ for $t \in (T, T+1)$, $\varphi(t) = 0$ for $t \notin (T-1, T+2)$, and $0 \leq \varphi(t) \leq 1$.
Multiplying Eq. (1.3) by $\varphi w|w|_a^{p-2}$, where $|w|_a \equiv (aw \cdot w)^{1/2}$, and afterwards inte-

grating over Ω_+, we obtain the following equality:

$$(1.17)\quad \langle a(\partial_t^2 w + \Delta w), \varphi w|w|_a^{p-2}\rangle = \\ = -\langle \varphi\gamma\partial_t w, w|w|_a^{p-2}\rangle + \langle \varphi f(w+v)\cdot w, |w|_a^{p-2}\rangle + \langle \varphi b, w|w|_a^{p-2}\rangle.$$

By definition of $[H_0^2(\bar{\Omega}_T)]^k$, $\partial_t^2 w + \Delta w \in [L_b^2(\bar{\Omega}_T)]^k$. By (A.11), the functions $w|w|_a^{p-2}$ and $f(w+v)$ also belong to $[L_b^2(\bar{\Omega}_T)]^k$. Hence, all the integrals in (1.17) are correctly defined. Moreover, by virtue of Theorem A.7, we have $w|w|_a^{p-2} \in H^1(\bar{\Omega}_T)$. Thus, on the left-hand side of (1.17), we can integrate by parts and get

$$(1.18)\quad \langle a\partial_t^2 w, \varphi w|w|_a^{p-2}\rangle = -\langle a\partial_t w, \partial_t(\varphi w|w|_a^{p-2})\rangle = \\ = -\frac{1}{p}\langle \varphi', \partial_t(|w|_a^p)\rangle - \langle \varphi|\partial_t w|_a^2, |w|_a^{p-2}\rangle - (p-2)\langle \varphi(a\partial_t w.\, w)^2, |w|_a^{p-4}\rangle = \\ = \frac{1}{p}\langle \varphi'', |w|_a^p\rangle - \langle \varphi|\partial_t w|_a^2, |w|_a^{p-2}\rangle - \frac{4(p-2)}{p^2}\langle \varphi\partial_t(|w|_a^{p/2}), \partial_t(|w|_a^{p/2})\rangle \le \\ \le C_1\|w, \bar{\Omega}_T\|_{0,p}^p - C_2\langle \varphi\partial_t(|w|_a^{p/2}), \partial_t(|w|_a^{p/2})\rangle.$$

Analogously,

$$\langle a\Delta w, \varphi w|w|_a^{p-2}\rangle \le -C_2\langle \varphi\nabla(|w|_a^{p/2}), \nabla(|w|_a^{p/2})\rangle.$$

Hence we obtain

$$(1.19)\quad -\langle a(\partial_t^2 w + \Delta w), \varphi w|w|_a^{p-2}\rangle \ge -C_1\|w, \bar{\Omega}_T\|_{0,p}^p + \\ + C_2(\langle \varphi\partial_t(|w|_a^{p/2}), \partial_t(|w|_a^{p/2})\rangle + \langle \varphi\nabla(|w|_a^{p/2}), \nabla(|w|_a^{p/2})\rangle).$$

It follows from Hölder's inequality that

$$(1.20)\quad |\langle \gamma\partial_t w, \varphi w|w|_a^{p-2}\rangle| \le \mu\langle \varphi\partial_t(|w|_a^{p/2}), \partial_t(|w|_a^{p/2})\rangle + C_\mu\langle \varphi|w|^p, 1\rangle$$

and

$$(1.21)\quad |\langle b, \varphi w|w|^{p-2}\rangle| \le \mu\langle \varphi|w|^{2(p-1)}, 1\rangle + C_\mu\langle \varphi|b|^2, 1\rangle \le \\ \le \mu\langle \varphi|w|^{2(p-1)}, 1\rangle + C_\mu(\|g, \bar{\Omega}_T\|_{0,2}^2 + \chi(1-T)\|u_0\|_{V_0}^2).$$

Here $\mu > 0$ is an arbitrary number.

Argueing as for (1.9) above we obtain

$$(1.22)\quad \langle f(w+v), \varphi w|w|_a^{p-2}\rangle \ge -C_1(1 + \langle \varphi|v|^{2(p-1)}, 1\rangle) + C_2\langle \varphi|w|^{2(p-1)}, 1\rangle \ge \\ \ge -C_3(1 + \chi(1-T)\|u_0\|_{V_0}^{2(p-1)}) + C_2\langle \varphi|w|^{2(p-1)}, 1\rangle.$$

Now replacing all terms in equality (1.17) by their corresponding bounds in (1.19) to

(1.22) and taking $\mu > 0$ sufficiently small, after a short calculation we get

$$(1.23) \qquad \langle \varphi |w|^{2(p-1)}, 1 \rangle \leqslant C(1 + \chi(1-T)\|u_0\|_{V_0}^{p(p-1)} + \|g, \tilde{\Omega}_T\|_{0,2}^p + \|w, \tilde{\Omega}_T\|_{0,p}^p).$$

This proves Theorem 1.3. ∎

COROLLARY 1.4: *Let u be a solution to* (0.1). *Then, for each $T \geqslant 0$, we have*

$$(1.24) \qquad \|f(u), \Omega_T\|_{0,2} \leqslant C(1 + \chi(1-T)\|u_0\|_{V_0}^{p-1} + \|g, \tilde{\Omega}_T\|_{0,2}).$$

This follows from the estimates (1.15), (1.16).

THEOREM 1.5 (The main estimate): *Let u be a solution to the problem* (0.1). *Then the following estimate holds:*

$$(1.25) \qquad \|u, \Omega_T\|_{2,Q} \leqslant C(1 + \chi(1-T)\|u_0\|_{V_0}^{p-1} + \|g, \tilde{\Omega}_T\|_{0,2}).$$

PROOF: Rewrite Eq. (1.3) in the following form:

$$(1.26) \qquad \begin{cases} \partial_t^2(\varphi w) + \Delta(\varphi w) = h_w(t), \\ \varphi w|_{t=\max\{T-1,0\}} = 0, \quad \varphi w|_{t=T+2} = 0, \quad \partial_n(\varphi w)|_{\partial\omega} = 0. \end{cases}$$

Here φ is a cut-off function as in the proof of Theorem 1.3 and

$$(1.27) \qquad h_w(t) = \varphi'' w + 2\varphi' \partial_t w - a^{-1}(\varphi h(t) + \varphi f(u) - \gamma \partial_t w).$$

By (1.1) and (1.24) we have the estimate

$$(1.28) \qquad \|h_w, \hat{\Omega}_T\|_{0,2} \leqslant C(1 + \chi(2-T)\|u_0\|_{V_0}^{p-1} + \|g, \hat{\Omega}_T\|_{0,2}),$$

where $\hat{\Omega}_T = (\max\{0, T-2\}, T+3) \times \omega$. By the L^2-regularity theorem (see Appendix A), we obtain

$$(1.29) \qquad \|w, \Omega_T\|_{2,Q} \leqslant C_1 \|\varphi w, \tilde{\Omega}_T\|_{2,Q} \leqslant$$
$$\leqslant C\|h_w, \hat{\Omega}_T\|_{0,2} \leqslant C_2(1 + \chi(2-T)\|u_0\|_{V_0}^{p-1} + \|g, \hat{\Omega}_T\|_{0,2}).$$

This completes the proof of Theorem 1.5. ∎

REMARK 1.6: Let the condition (0.3) be satisfied. Then each solution u to (0.1) that is in $[H^2_{Q,\mathrm{loc}}(\Omega_+)]^k$ belongs automatically to the space $[H^2_{Q,b}(\Omega_+)]^k$. More precisely, we have the estimate

$$(1.30) \qquad \|u\|_b \equiv \sup_{T \geqslant 0} \|u, \Omega_T\|_{2,Q} \leqslant C(1 + \|u_0\|_{V_0}^{p-1} + |g|_b).$$

In fact, (1.31) is a consequence of (1.25).

2. - The existence of solutions

In this section we shall prove solvability for the problem (0.1). We first solve the following auxiliary problem in a finite cylinder:

$$(2.1)\qquad \begin{cases} a(\partial_t^2 u + \Delta u) + \gamma \partial_t u - f(u) = g(t)\,, \\ u|_{t=0} = u_0, \quad u|_{t=M} = u_1, \quad \partial_n w|_{\partial\omega} = 0\,. \end{cases}$$

Here $u_0, u_1 \in V_0$ and $u \in [H^2_Q(\Omega_{0,M})]^k$. Then we shall obtain a solution u to the main problem (0.1) as the limit as $M \to \infty$ of solutions u_M to the corresponding auxiliary problems (2.1).

Theorem 2.1: *Let u be the solution to the problem (2.1). Then the following estimate holds uniformly with respect to $M \to \infty$:*

$$(2.2)\qquad \|u, \Omega_T\|_{2,Q} \le C(1 + \chi(1-T)\|u_0\|_{V_0}^{p-1} + \\ + \chi(T - M + 1)\|u_1\|_{V_0}^{p-1} + \|g, \bar{\Omega}_T \cap \Omega_{0,M}\|_{0,2})\,.$$

The proof of (2.2) is analogous to that of (1.25) given in § 1 in the case of the half-cylinder.

Theorem 2.2: *For every $u_0, u_1 \in V_0$, the problem (2.1) has at least one solution.*

Proof: Introduce the space

$$(2.3)\qquad \mathfrak{W}_M = \{w \in [H^2_Q(\Omega_{0,M})]^k: \quad w|_{t=0} = w|_{t=M} = 0\}\,.$$

and reformulate problem (2.1) with respect to the new function $w = u - v$, where $w \in \mathfrak{W}_M$, $v \in [H^2_Q(\Omega_{0,M})]^k$:

$$(2.4)\qquad \begin{cases} \partial_t^2 w + \Delta w = a^{-1}(-\gamma \partial_t w + f(v + w) + g_1(t))\,, \\ w|_{t=0} = 0\,, \quad w|_{t=M} = 0\,, \quad \partial_n w|_{\partial\omega} = 0\,. \end{cases}$$

Here $g_1 = -a(\partial_t^2 v + \Delta v) - \gamma \partial_t v + g$.

Let A denote the inverse to the Laplace operator with respect to the variables $(t, x) \in \Omega_{0,M}$ and the boundary conditions $w|_{t=0} = 0$, $w|_{t=M} = 0$, $\partial_n w|_{\partial\omega} = 0$. Then from Appendix A we get

$$(2.5)\qquad A: [L_2(\Omega_{0,M})]^k \to \mathfrak{W}_M\,.$$

Applying the operator A to both sides of Eq. (2.4) we obtain

$$(2.6)\qquad w + F(w) = h \equiv -A(\partial_t^2 v + \Delta v)\,,$$

where

$$F(w) = -Aa^{-1}(-\gamma\partial_t w + f(v+w) + g - \gamma\partial_t v)\,.$$

Now we use the Leray-Schauder principle in the following form (see [10]):

LERAY-SCHAUDER PRINCIPLE: *Let D be a bounded open set in a Banach space W and let F: $\overline{D} \to W$ be a compact and continuous operator. Further let the point $b \in D$ be such that*

$$w + sF(w) \neq b \quad \textit{for all } w \in \partial D,\ s \in [0, 1]. \tag{2.7}$$

Then the equation

$$w + F(w) = b$$

has at least one solution in D.

Let B_R be an open ball in $\mathcal{W}_M$ of sufficiently large radius and suppose that

$$w_s + sF(w_s) = b \quad \text{for some } w_s \in \partial D,\ s \in [0, 1]. \tag{2.8}$$

Eq. (2.8) can be rewritten in the form

$$\begin{cases} a(\partial_t^2 u_s + \Delta u_s) + s\gamma\partial_t u_s - sf(u_s) = sg(t)\,, \\ u_s|_{t=0} = u_0, \quad u_s|_{t=M} = u_1, \quad \partial_n u_s|_{\partial\omega} = 0\,, \end{cases} \tag{2.9}$$

where $u_s = w_s + v$.

LEMMA 2.3: *Let u_s be a solution to* Eq. (2.9), $s \in [0, 1]$. *Then the following estimate holds:*

$$\|u_s, \Omega_{0,M}\|_{2,Q} \leq C(\|u_0\|_{V_0}^{p-1} + \|u_1\|_{V_0}^{p-1} + \|g, \Omega_{0,M}\|_{0,2})\,. \tag{2.10}$$

Moreover, the constant C in (2.10) is independent of $s \in [0, 1]$.

The proof of this lemma is analogous to the proof of Theorem 1.5 and even simpler than it, since we need not use cut-off functions $\phi(t)$ and $\varphi(t)$.

REMARK 2.4: Eq. (2.9) has the form (2.1), hence for any $s > 0$ inequality (2.10) is a consequence of Theorem 2.1. But the estimates thus obtained do not hold uniformly with respect to $s \in (0, 1]$. Indeed, for $s = 0$, (2.9) is a linear equation, and then the estimates (1.1) and (1.25) are evidently impossible. Actually, the proof of Theorem 1.1 which is based on the special choice of the cut-off function ϕ does not go through if $s = 0$. However, in the estimate (2.10) no uniformity with respect to $M \to \infty$ is included, hence in its proof there is no need for any cut-off function.

Since v is independent of s, it follows from Lemma 2.3 that

$$\|w_s\|_{\mathcal{W}_M} \leq K$$

for all solutions u_s to (2.9) uniformly in $s \in [0, 1]$. Therefore, condition (2.7) is fulfilled if the radius of B_R is chosen larger than K.

We prove compactness for the operator F. It is sufficient to prove compactness for the nonlinear part $Aa^{-1}f(w+v)$. To do this decompose the nonlinear part as a composition of three continuous operators $A \circ F_2 \circ F_1$, with one of them being compact: $F_1 : \mathcal{W}_M \to [L^{2(p-1)}(\Omega_{0,M})]^k$ is the embedding which is compact because of $2(p-1) < q_0$ (see Theorem A.7) and $F_2 w = a^{-1} f(v+w)$. The operator F_2 is continuous from $[L^{2(p-1)}(\Omega_{0,M})]^k$ to $[L^2(\Omega_{0,M})]^k$ in view of condition (0.2) and Krasnoselski's theorem (see [11])). Hence the operator F is compact and according to the Leray-Schauder principle the problem (2.1) has at least one solution. ■

THEOREM 2.5: *The problem* (0.1) *has at least one solution* $u \in [H^2_{Q,b}(\Omega_+)]^k$.

PROOF: Consider a sequence u_M, $M = 1, 2, \ldots$, of solutions to the auxiliary problems (2.1) with $u_1|_{r=M} = 0$. It follows from Theorem 2.1 that, for every fixed N,

$$\|u_M, \Omega_{0,N}\|_{2,Q} \leq C(u_0, N, g)$$

holds uniformly with respect to $M \geq N$. Using Cantor's diagonalization procedure we extract a subsequence from u_M, again denoted by u_M, obeying the following property:

$$u_M|_{\Omega_{0,N}} \to u|_{\Omega_{0,N}} \quad \text{weakly in } [H^2_Q(\Omega_{0,N})]^k$$

for a certain $u \in [H^2_{Q,b}(\Omega_+)]^k$. We finally show that u is a solution to (0.1). It is sufficient to prove that, for every $\Phi \in [C_0^\infty(\Omega_+)]^k$, the following equality holds:

$$-\langle a\partial_t u, \partial_t \Phi\rangle - \langle a\nabla u, \nabla\Phi\rangle + \langle \gamma\partial_t u, \Phi\rangle - \langle f(u), \Phi\rangle = \langle g, \Phi\rangle. \tag{2.11}$$

From the definition of u_M we conclude that

$$-\langle a\partial_t u_M, \partial_t \Phi\rangle - \langle a\nabla u_M, \nabla\Phi\rangle + \langle \gamma\partial_t u_M, \Phi\rangle - \langle f(u_M), \Phi\rangle = \langle g, \Phi\rangle \tag{2.12}$$

when M is sufficiently large. Taking the limit $M \to \infty$ in (2.12) we obtain (2.11). In fact, the only non-trivial part in its proof is to show that

$$\langle f(u_M), \Phi\rangle \to \langle f(u), \Phi\rangle \tag{2.13}$$

holds. Suppose that $\operatorname{supp}\Phi \subset \Omega_{0,N}$. By Theorem A.7, the embedding $[H^2_Q(\Omega_{0,N})]^k \subset [L^{2(p-1)}(\Omega_{0,N})]^k$ is compact. Hence $u_M \to u$ in $[L^{2(p-1)}(\Omega_{0,N})]^k$ and, by condition (0.2), $f(u_M) \to f(u)$ in $[L^2(\Omega_{0,N})]^k$. Theorem 2.5 is proved. ■

3. - The trajectory attractor for the nonlinear elliptic system

Now we are going to construct the trajectory attractor for the problem (0.1). First we briefly recall the main concepts and definitions from the abstract theory of trajectory attractors for dynamical systems (see [6], [7] for more details).

Definition 3.1: The right-hand side g of (0.1) is said to be translation-compact in

$$\Xi^+ = [L^2_{loc}(\mathbb{R}_+, L_2(\omega))]^k$$

if its hull

$$\mathcal{H}^+(g) = [T_s g, s \geq 0]_{\Xi^+}, \qquad (T_s g)(t) = g(t+s)$$

is compact in Ξ^+. Here $[\cdot]_{\Xi^+}$ means the closure in the space Ξ^+.

The right-hand side g of (0.1) is said to be weakly translation-compact in the space Ξ^+ if its weak hull

$$\mathcal{H}^+_w(g) = [T_s g, s \geq 0]_{\Xi^+_w}$$

is compact in Ξ^+_w. Here Ξ^+_w denotes the space Ξ^+ equipped with the weak topology.

Remark 3.2: If the function g is translation-compact for the strong topology, then it is weakly translation-compact and

(3.1) $$\mathcal{H}^+(g) = \mathcal{H}^+_w(g)$$

(see [20]).

Remark 3.3: A functions g that is almost-periodic in t with values in $L^2(\omega)$ in the sense of Bochner, in particular, a periodic or a quasi-periodic function, is evidently translation-compact in the space Ξ^+ (for its strong topology). Hence translation-compactness is a generalization of the concept of almost-periodicity.

Remark 3.4: It follows from the definition of the hull that

(3.2) $$T_s\mathcal{H}^+(g) \subseteq \mathcal{H}^+(g), \qquad T_s\mathcal{H}^+_w(g) \subseteq \mathcal{H}^+_w(g) \qquad \text{for } s \geq 0,$$

i.e., the semigroup $\{T_s, s \geq 0\}$ of shifts acts on $\mathcal{H}^+(g)$ and $\mathcal{H}^+_w(g)$, respectively.

Next we formulate necessary and sufficient conditions for translation-compactness and weak translation-compactness in the space Ξ^+.

Theorem 3.5 [8]: (1) *A function g is weakly translation-compact in Ξ^+ if and only if it is bounded with respect to $t \to \infty$, i.e., $|g|_b < \infty$.*

(2) *A function g is translation-compact in Ξ^+ if and only if the following conditions hold:*

(a) *for any fixed* $t > 0$, *the set* $\left\{ \int_s^{s+t} g(z)\,dz,\ s \in \mathbb{R}_+ \right\}$ *is precompact in the space* $[L^2(\omega)]^k$;

(b) *there exists a function* $\beta(s)$, $s \geq 0$, $\beta(s) \to 0$ *as* $s \to +0$, *such that*

$$\int_t^{t+1} \|g(z) - g(z+l)\|_{L_2(\omega)}\,dz \leq \beta(|l|) \quad \text{for all } t \in \mathbb{R}_+ \text{ with } t + l \in \mathbb{R}_+. \tag{3.3}$$

REMARK 3.6: Condition (3.3) is fulfilled, e.g., if

$$\|T_s g, \Omega\|_{\delta, 2} \leq C, \qquad s \geq 0,$$

for a suitable $\delta > 0$.

To construct the trajectory attractor for the problem (0.1), we consider the family of problems of the form (0.1) obtained from all positive shifts of the initial problem (0.1) together with all limits in the appropriate topology:

$$\begin{cases} a(\partial_t^2 u + \Delta u) + \gamma \partial_t u - f(u) = \sigma(t), & \sigma \in \Sigma, \\ u|_{t=0} = u_0, \quad \partial_n u|_{\partial\omega} = 0. \end{cases} \tag{3.4}$$

Here we take $\Sigma = \mathcal{K}^+(g)$, if g is translation-compact for the strong topology, and $\Sigma = \mathcal{K}_w^+(g)$ otherwise.

DEFINITION 3.7: For each $\sigma \in \Sigma$, K_σ^+ denotes the space of all solutions to (3.4) with an arbitrary $u_0 \in V_0$. Further define K_Σ^+ as the union of all K_σ^+:

$$K_\Sigma^+ = \bigcup_{\sigma \in \Sigma} K_\sigma^+ .$$

It follows from (3.2) that the semigroup $\{T_s, s \geq 0\}$ of non-negative shifts along the t-axis ($(T_s v)(t) \equiv v(t+s)$) acts on the space K_Σ^+, i.e.,

$$T_s K_\Sigma^+ \subseteq K_\Sigma^+ \qquad \text{for } s \geq 0. \tag{3.5}$$

The set K_Σ^+ is endowed with the relative topology induced from the embedding $K_\Sigma^+ \subset \Theta_0^+$ if $\Sigma = \mathcal{K}^+(g)$ (in case of the strong topology) and induced from the embedding $K_\Sigma^+ \subset (\Theta_0^+)^w$ if $\Sigma = \mathcal{K}_w^+(g)$ (in case of the weak topology), respectively. (For the definition of Θ_0^+, see Appendix 1.)

DEFINITION 3.8: The (global) attractor of the semigroup $\{T_s, s \geq 0\}$ acting on the topological space K_Σ^+ is called the trajectory attractor of the family (3.4). That means that a set $\mathfrak{A}_\Sigma \subset K_\Sigma^+$ is the trajectory attractor of the family (3.4) if the following conditions hold:

(1) $\mathfrak{A}_\Sigma$ is compact in K_Σ^+;

(2) $\mathfrak{A}_\Sigma$ is strongly invariant with respect to $\{T_s, s \geq 0\}$, i.e.,

$$T_s \mathfrak{A}_\Sigma = A_\Sigma \quad \text{for all } s \geq 0;$$

(3) $\mathfrak{A}_\Sigma$ is attracting for $\{T_s, s \geq 0\}$, i.e., for any neighbourhood $\mathcal{O} = \mathcal{O}(\mathfrak{A}_\Sigma)$ in the topology of K_Σ^+ there is an $s_0 > 0$ such that

$$T_s \mathcal{K}_\Sigma^+ \subset \mathcal{O} \quad \text{for all } s \geq s_0. \tag{3.6}$$

Remark 3.9: The attracting property is usually required only for (in some sense) bounded subsets of K_Σ^+. In view of estimate (1.26), however, the set $T_1 K_\Sigma^+$ is already bounded both in F_0^+ and Θ_0^+. Hence the attracting property (3.6) is automatically implied for all subsets of K_Σ^+, with the same constant s_0 (see also [22]).

Theorem 3.10 [8]: *Let the following conditions be satisfied:*

(1) *There exists a compact attracting set $P \subset K_\Sigma^+$ for the semigroup $\{T_s, s \geq 0\}$;*

(2) *the set K_Σ^+ is closed in the space Θ_0^+ in case of the strong topology and sequentially closed in the space $(\Theta_0^+)^w$ in case of the weak topology, respectively.*

Then the family (3.6) possesses a trajectory attractor $A = \mathfrak{A}_\Sigma$ in K_Σ^+.

Definition 3.11: The trajectory attractor A^w of the family (3.6) with $\Sigma = \mathcal{K}_w^+(g)$ (the case of the weak topology) is called the weak trajectory attractor of the initial problem (0.1).

Analogously the trajectory attractor $A = A^s$ of the family (3.6) with $\Sigma = \mathcal{K}^+(g)$ (the case of the strong topology) is called the (strong) trajectory attractor of the initial problem (0.1).

Theorem 3.12 (1): *Let condition (0.3) hold. Then the problem (0.1) possesses a weak trajectory attractor A^w.*

(2) *Let the right-hand side g be translation-compact in Ξ^+ (endowed with the strong topology). Then the problem (0.1) possesses a strong trajectory attractor $A = A^s$.*

We check the conditions of Theorem 3.10.

Lemma 3.13: *The set K_Σ^+ is sequentially closed in the space $(\Theta_0^+)^w$.*

Proof: Let $u_m \in K_{\sigma_m}^+$, $u_m \to u$ in $(\Theta_0^+)^w$. Without loss of generality we may suppose that $\sigma_m \to \sigma$ weakly in Ξ^+, since Σ is compact in Ξ_w^+. We have to prove that $u \in K_\sigma^+$. By definition, the functions $u_m(t)$ are bounded solutions to the following problems:

$$(3.7)\qquad \begin{cases} a(\partial_t^2 u_m + \Delta u_m) + \gamma \partial_t u_m - f(u_m) = \sigma_m(t), \\ u_m|_{t=0} = u_m^0, \quad u_m^0 \in V_0. \end{cases}$$

As in the proof of Theorem 2.5, taking the limit $m \to \infty$ in (3.7) we obtain that $u \in K_\sigma^+$. ■

Hence the second condition of Theorem 3.10 holds. Let us check the first condition. From the estimate (1.25) we infer that the set

$$P = B_R \cap K_\Sigma^+, \quad s > 0,$$

where B_R is a sufficiently large ball in the space F_0^+, is an absorbing set for the semigroup $\{T_s, s \geq 0\}$. First we consider the case of the weak topology. The set B_R is a compact and metrizable subset of $(\Theta_0^+)^w$. In fact, the ball B_R is bounded in Θ_0^+ and Θ_0^+ is a reflexive and separable Fréchet space, hence B_R is semi-compact and metrizable for the weak topology. Due to convexity, B_R is a metrizable compact. From Lemma 3.13 we finally conclude that the set P is compact in $(\Theta_0^+)^w$.

Now let us suppose that the right-hand side g of the problem (0.1) is translation-compact for the strong topology.

Lemma 3.14: *Let the previous condition hold. Then the set $T_s P$ is compact in Θ_0^+ for all $s \geq 0$.*

Proof: Without loss of generality we suppose that $s = 1$. Let $\{u_m\}$, $u_m \in B_R \cap K_{\sigma_m}^+$, be an arbitrary sequence. We can suppose that $\sigma_m \to \sigma$ in Ξ^+, since the hull Σ is compact in Ξ^+ (for the strong topology). Further we can suppose that $\{u_m\}$ is is weakly convergent to some $u \in K_\sigma^+$, since P is weakly compact.

To prove the lemma it suffices to show that $u_m \to u$ in $[H_Q^2(\Omega_T)]^k$ for an arbitrary $T \geq 1$. The functions u_m satisfy Eq. (3.7). Multiplying (3.7) by a cut-off function $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi(t) = 1$ for $t \in (T, T+1)$ and $\phi(t) = 0$ for $t \notin (T-1, T+2)$, we obtain

$$(3.8)\qquad \begin{cases} \partial_t^2(\phi u_m) + \Delta(\phi u_m) = a^{-1}(\phi\sigma_m + \phi f(u_m) - \gamma\phi\partial_t u_m) + \\ \qquad\qquad + 2\partial_t\phi\partial_t u_m + (\partial_t^2\phi)\, u_m \equiv b_m(t), \\ \phi u_m|_{t=T-1} = 0, \quad \phi u_m|_{t=T+2} = 0, \quad \partial_\nu(\phi u_m)|_{\partial\omega} = 0. \end{cases}$$

Then arguing as in the proof of Theorem 2.5 we can show that $f(u_m) \to f(u)$ and $\partial_t u_m \to \partial_t u$ in Ξ^+. So $b_m \to b$ in $[L_2(\Omega_{T-1,T+2})]^k$ and

$$\phi u_m \to \phi u \quad \text{in } \mathfrak{G}_0$$

(see Appendix A). Hence,

$$u_m \to u \quad \text{in } [H_Q^2(\Omega_T)]^k.$$

The proof is finished. ■

Therefore, all conditions of Theorem 3.10 hold; so the proof of Theorem 3.12 is complete.

COROLLARY 3.15: *Let the right-hand side g be translation-compact in* Ξ^+. *Then*

$$\operatorname{dist}_{2,Q}(\Pi_{T_1,T_2} T_s K_\Sigma^+, \Pi_{T_1,T_2} \mathrm{A}) \to 0 \quad \text{as } s \to +\infty,$$

where

$$\operatorname{dist}_{2,Q}(M, N) = \sup_{x \in M} \inf_{y \in N} \|x - y, \Omega_{T_1,T_2}\|_{2,Q}.$$

Here Π_{T_1,T_2} *denotes the restriction in t to the interval* (T_1, T_2).

This corollary is immediate from the definition of trajectory attractor.

COROLLARY 3.16: *Let the right-hand side g be weakly translation-compact in* Ξ^+. *Then*

$$\operatorname{dist}_{3/2+\varepsilon,2}(\Pi_{T_1,T_2} T_s K_\Sigma^+, \Pi_{T_1,T_2} \mathrm{A}) \to 0 \quad \text{as } s \to +\infty,$$

and

$$\operatorname{dist}_{0,q}(\Pi_{T_1,T_2} T_s K_\Sigma^+, \Pi_{T_1,T_2} \mathrm{A}) \to 0 \quad \text{as } s \to +\infty,$$

where $\varepsilon > 0$ *is sufficiently small and* $q < 2(n+1)/(n-3)$.

This corollary follows from the compactness of the embeddings $[H_Q^2(\Omega_{T_1,T_2})]^k \subset [H^{3/2+\varepsilon,2}(\Omega_{T_1,T_2})]^k$ and $[H_Q^2(\Omega_{T_1,T_2})]^k \subset [L^q(\Omega_{T_1,T_2})]^k$ proved in Appendix A.

Next we investigate the structure of the trajectory attractor A.

Let $\omega(\Sigma)$ be the ω-limit set (the attractor) of the semigroup $\{T_s, s \geq 0\}$ acting on the compact space Σ. It is non-empty and can be represented in the form

$$\omega(\Sigma) = \bigcap_{t \geq 0} \left[\bigcup_{s \geq t} T_s \Sigma \right]_\Sigma$$

(see [2]). Here $[\cdot]_\Sigma$ denotes the closure in the space Σ.

DEFINITION 3.17: A function $\xi(t)$, $t \in \mathbb{R}$, is called a complete symbol of (3.4) if

$$\Pi_+ \xi_s(\cdot) \in \omega(\Sigma) \quad \text{for all } s \in \mathbb{R}\,. \tag{3.8}$$

Here $\xi_s(t) = \xi(t+s)$. The operator Π_+ is the restriction to the half-axis $\mathbb{R}_+$.
The set of all complete symbols of (3.4) is denoted by $Z(\Sigma)$.

LEMMA 3.18 [8]: *For each $\sigma \in \omega(\Sigma)$, there exists a complete symbol $\xi \in Z(\Sigma)$ such that $\Pi_+ \xi = \sigma$.*

DEFINITION 3.19: For $\xi \in Z(\Sigma)$, K_ξ is the set of all bounded solutions to Eq. (3.4) on the whole axis $t \in \mathbb{R}$, where $\sigma(t)$ is replaced by $\xi(t)$.

THEOREM 3.20 [8]: *The attractor* A *has the following structure:*

$$\mathrm{A} = \Pi_+ \bigcup_{\xi \in Z(\Sigma)} K_\xi\,. \tag{3.9}$$

COROLLARY 3.21 [20]: *Let the right-hand side g be translation-compact in Ξ^+. Then the weak trajectory attractor of the problem* (0.1) *coincides with the strong trajectory attractor:*

$$\mathrm{A}^s = \mathrm{A}^w.$$

4. - STABILIZATION OF SOLUTIONS IN THE POTENTIAL CASE

In this section we shall study the long-term behaviour of solutions when the right-hand side $g(t)$ of (0.1) has the form

$$g(t,x) = g_+(x) + g_1(t,x)\,, \tag{4.1}$$

where $g_+ \in L^2(\omega)$ is independent of t and g_1 satisfies the condition

$$T_s\, g_1 \to 0 \quad \text{as } s \to +\infty\,, \tag{4.2}$$

in Ξ^+ and $(\Xi^+)^w$, respectively. It is not difficult to see that in the first case the function g is strongly translation-compact in Ξ^+, while in the second case it is weakly translation-compact in Ξ^+.

THEOREM 4.1: *Suppose that condition* (4.2) *holds. Then the problem* (0.1) *with right-hand side* (4.1) *possesses a strong and weak trajectory attractor* $\mathrm{A} = \mathrm{A}_\xi$*, respectively. It coincides with the attractor of the limit autonomous equation*

$$a(\partial_t^2 u + \Delta u) - \gamma \partial_t u - f(u) = g_+\,, \tag{4.3}$$

i.e.,

$$\mathcal{A}_{\Sigma} = \mathcal{A}_{g_+}. \tag{4.4}$$

PROOF: The existence of the trajectory attractor follows immediately from Theorem 3.12. Thus we show (4.4).

From condition (4.2) we get that

$$Z(g) \equiv Z(\Sigma) = \omega(\Sigma) = g_+.$$

Here Σ is the respectively strong and weak hull of the right-hand side g in the space Ξ^+ (see § 3). Hence formula (4.4) holds in view of Theorem 3.20. Theorem 4.1 is proved. ■

Now we assume that the nonlinear term $f(u)$ on the left-hand side of Eq. (0.1) is gradient-like, i.e.,

$$f(u) = -\nabla F(u), \qquad F \in C(\mathbb{R}^k, \mathbb{R}). \tag{4.5}$$

For $u \in [H^2_{Q,b}(\Omega_+)]^k$, we introduce the function $\mathcal{F}_u(t)$ by

$$\mathcal{F}_u(t) = \frac{1}{2}(a\partial_t u(t), \partial_t u(t)) - \frac{1}{2}(a\nabla u(t), \nabla u(t)) + (F(u(t)), 1) - (g_+, u(t)), \tag{4.6}$$

where $(\cdot, \cdot)$ is the L^2-scalar product in the cross-section.

THEOREM 4.2: (1) *For every* $u \in [H^2_{Q,b}(\Omega_+)]^k$, *the function* $\mathcal{F}_u$ *is well-defined and belongs to the space* $H^{1,1}_b(\mathbb{R}_+)$.

(2) *If u is a solution to the problem* (0.1), *then*

$$\frac{d\mathcal{F}_u(t)}{dt} = -(\gamma\partial_t u(t), \partial_t u(t)) + (g_1(t), \partial_t u(t)). \tag{4.7}$$

PROOF: Let $u \in [H^2_{Q,b}(\Omega_+)]^k$. Then, according to the embedding (A.19), the first, the second, and the fourth term on the right-hand side of (4.6) are well-defined. It remains to consider the third term. From (0.2) and (4.5) we infer that

$$|F(u)| \leq C(1 + |u|^p). \tag{4.8}$$

Then (A.15) and Krasnoselski's theorem yield that

$$(F(u(t)), 1) \in C_b(\overline{\mathbb{R}}_+).$$

Hence $\mathcal{F}_u(t)$ is well-defined.

When calculating the derivative, using standard methods of distribution theory, we obtain that $\mathcal{F}_s \in H^{1,1}_{loc}(\mathbf{R}_+)$,

$$\frac{d\mathcal{F}_s(t)}{dt} = (a(\partial_t^2 u + \Delta u) - f(u) - g_*, \partial_t u). \tag{4.9}$$

Hence the first part of the proof is finished.

Now suppose that u is a solution to the problem (0.1). Then (4.5) follows immediately from (4.9). The second part of the proof is also finished. ∎

Theorem 4.3: *Suppose that conditions* (4.2) *and* (4.5) *hold. Further suppose that the matrix* γ *on the left-hand side of* (0.1) *is sign-definite, i.e.,*

$$\text{either } \gamma + \gamma^* > 0 \text{ or } \gamma + \gamma^* < 0,$$

and the function $g_1(t) = g_1(t, x)$ *satisfies at least one of the following conditions:*

$$\left\{\begin{array}{ll} \text{(i)} & \int_0^\infty \|g_1(t)\|_{0,2}\, dt < \infty; \\ \text{(ii)} & \partial_t g_1 \in L_1^{loc}(\mathbf{R}_+, L_2(\omega)) \text{ and } \int_0^\infty \|\partial_t g_1(t)\|_{0,2}\, dt < \infty; \\ \text{(iii)} & \sum_{N=0}^\infty \|G_1, \Omega_N\|_{0,2} < \infty \text{ for some } G_1 \text{ such that } \partial_t G_1 = g_1. \end{array}\right. \tag{4.10}$$

Then every solution u *to the problem* (0.1) *possesses the finite dissipative integral*

$$\int_0^\infty \|\partial_t u(t)\|_{0,2}^2\, dt < \infty. \tag{4.11}$$

Proof: We integrate (4.6) over $t \in [0, T]$ and obtain

$$\int_0^T (\gamma \partial_t u, \partial_t u)\, dt = \mathcal{F}_s(0) - \mathcal{F}_s(T) + \int_0^T (g_1, \partial_t u)\, dt.$$

Now it follows from the sign-definiteness of the matrix γ that

$$\int_0^T \|\partial_t u(t)\|_{0,2}^2\, dt \le C|\mathcal{F}_s(T) - \mathcal{F}_s(0)| + C\left|\int_0^T (g_1, \partial_t u)\, dt\right|. \tag{4.12}$$

Theorem 4.2 implies that function $\mathcal{F}_s(T)$ is bounded as $T \to \infty$. Hence it suffices to show the boundedness of the integral on the right-hand side of (4.12).

Suppose that condition (i) of (4.10) holds. Then

$$(4.13)\qquad \left|\int_0^T (g_1, \partial_t u)\, dt\right| \le \int_0^T \|g_1(t)\|_{0,2} \|\partial_t u(t)\|_{0,2}\, dt \le$$

$$\le \sup_{t\in[0,T]} \|\partial_t u(t)\|_{0,2} \int_0^T \|g_1(t)\|_{0,2}\, dt \le \|u\|_b \int_0^\infty \|g_1(t)\|_{0,2}\, dt .$$

Thus $\left|\int_0^T (g_1, \partial_t u)\, dt\right|$ is bounded as $T \to \infty$.

Now suppose that condition (ii) of (4.10) holds. Then we obtain by integration by parts

$$(4.14)\qquad \left|\int_0^T (g_1, \partial_t u)\, dt\right| \le |(g_1(T), u(T))| + |(g_1(0), u(0))| + \left|\int_0^T (\partial_t g_1(t), u(t))\, dt\right|.$$

The integral on the right-hand side of (4.14) is estimated in the same manner as the integral in (4.13). To estimate the first two terms on the right-hand side it suffices to prove that under the above assumptions $g_1 \in C_b(\overline{\mathbb{R}}_+, L^2(\omega))$. Let $[N, N+1] \subset \overline{\mathbb{R}}_+$ be an arbitrary interval and let $[t, T]$ be in that interval. Then

$$(4.15)\qquad \|g_1(T)\|_{0,2} \le \|g_1(t)\|_{0,2} + \|g_1(T) - g_1(t)\|_{0,2} \le$$

$$\le \|g_1(t)\|_{0,2} + \int_t^T \|\partial_t g_1(t)\|_{0,2}\, dt \le \|g_1(t)\|_{0,2} + \int_0^\infty \|\partial_t g_1(t)\|_{0,2}\, dt .$$

Integrating (4.15) over $t \in [N, N+1]$, we get

$$\|g_1(T)\|_{0,2} \le C\|g_1, \Omega_N\|_{0,2} + \int_0^\infty \|\partial_t g_1(t)\|_{0,2}\, dt \le |g_1|_b + \|\partial_t g_1\|_{L_1(\mathbb{R}_+, L^2(\omega))}.$$

Since the constant N was chosen arbitrarily, we find $g_1 \in C_b(\mathbb{R}_+, L^2(\omega))$.

Now suppose that condition (iii) of (4.10) holds. Again integrating by parts we obtain that

$$\left|\int_0^T (g_1, \partial_t u)\, dt\right| \le |(G_1(T), \partial_t u(T))| + |(G_1(0), \partial_t u(0))| + \left|\int_0^T (G_1(t), \partial_t^2 u(t))\, dt\right|.$$

The first two terms on the right-hand side can be estimated as before. The third term is estimated as follows:

$$\left| \int_0^T (G_1(t), \partial_t^2 u(t))\, dt \right| \leqslant \int_0^T \|G_1(t)\|_{0,2} \|\partial_t^2 u(t)\|_{0,2}\, dt \leqslant$$

$$\leqslant \sum_{N=0}^{[T]} \|G_1, \Omega_N\|_{0,2} \|\partial_t^2 u, \Omega_N\|_{0,2} \leqslant C \|u\|_b \sum_{N=0}^{\infty} \|G_1, \Omega_N\|_{0,2}.$$

Theorem 4.3 is proved. ∎

THEOREM 4.4: *Suppose that all the assumptions of the previous theorem hold. Further suppose that the limit problem in the cross section* ω

$$\text{(4.16)} \qquad \begin{cases} a \Delta v_+ - f(v_+(x)) = g_+(x), \\ \partial_n v_+ |_{\partial\omega} = 0, \end{cases}$$

has only a finite number of solutions

$$\text{(4.17)} \qquad v_+ \in \nabla_+ = \{ v_+^1(x), \ldots, v_+^l(x) \}.$$

Then, for every solution u *to the problem* (0.1), *there exists an equilibrium* $v_+^N(x) \in \nabla_+$ *such that*

$$\text{(4.18)} \qquad (T_s u)(t, x) \to v_+^N(x) \quad \text{in } \Theta^+ \text{ as } s \to +\infty.$$

Here Θ^+ *denotes the space* Θ_0^+ *if* g *is strongly translation-compact in* Ξ *and the space* $(\Theta_0^+)^w$ *if* g *is weakly translation-compact in* Ξ, *respectively.*

REMARK: As is known (see, for instance, [2]), there exists an open dense subset in $L^2(\omega)$ such that ∇_+ is finite for every g_+ belonging to this set.

PROOF: Let u be a solution to the problem (0.1). Consider the ω-limit set $\omega(u)$ of $u \in \Theta^+$ under the action of the semigroup $\{T_s, s \geqslant 0\}$. Recall that $u_+ \in \omega(u)$ if and only if there exists the sequence $\{s_j\}_{j \in \mathbb{N}}$, $s_j \to \infty$, such that

$$\text{(4.19)} \qquad T_{s_j} u \to u_+ \quad \text{in } \Theta^+.$$

By Theorem 4.1, $\{T_s, s \geqslant 0\}$ possesses an attractor A in $K_\Xi^+ \subset \Theta^+$, hence $\omega(u)$ is a nonempty, compact, and connected subset of Θ^+ (see [2]).

Let u_+ be in $\omega(u)$. Let $\{s_j\}_{j \in \mathbb{N}}$ be a sequence as in (4.19). Then, for every $T > 0$,

$$T_{s_j} u \to u_+ \quad \text{weakly in } H_0^2(\Omega_T) \text{ as } s_j \to \infty.$$

In particular,

$$\|T_{s_j} \partial_t u - \partial_t u_+, \Omega_T\|_{0,2} \to 0 \quad \text{as } s_j \to \infty.$$

Now the finiteness of the dissipative integral (4.11) implies that

$$\|T_{s_j}\partial_t u,\ \Omega_T\|_{0,2} = \|\partial_t u,\ T_{s_j}\Omega_T\|_{0,2} \to 0 \quad \text{as } s_j \to \infty .$$

Therefore, $\|\partial_t u_+,\ \Omega_T\|_{0,2} = 0$ and $u_+(t, x) \equiv u_+(x)$.

From condition (4.2) and Lemma 3.3, however, we conclude that $u_+(x)$ is a solution to the limit problem (4.16). Thus

$$(4.20) \qquad \omega(u) \subset \mathfrak{V}_+ .$$

Since $\omega(u)$ is connected and $\mathfrak{V}_+$ is discrete, we eventually get

$$(4.21) \qquad \omega(u) = \{v_+^N\} \quad \text{for some } N \in \{1, \ldots, l\}.$$

Finally (4.18) is a consequence of the attracting property for $\{T_s, s \geq 0\}$ (see § 3). Theorem 4.4 proved. ■

COROLLARY 4.5: *Both in the case of strong translation-compactness and of weak translation-compactness of g, (4.20) implies that*

$$(4.22) \qquad \begin{cases} \lim\limits_{t\to+\infty} \|u(t,\cdot) - v_+^N(\cdot)\|_{0,p_0} = 0 , \\ \lim\limits_{t\to+\infty} \|\partial_t u(t,\cdot)\|_{\varepsilon,2} = 0 , \end{cases}$$

where the exponent p_0 is defined in Corollary A.8 *and $\varepsilon < 1/2$.*

This is obtained similarly to the proof of Corollary 3.16.

COROLLARY 4.6: *Suppose that the function g_+ satisfies the conditions of Theorem 4.4. Then, any solution $u(t)$, $t \in \mathbb{R}$, to Eq. (4.4) in the full cylinder $\Omega = \mathbb{R} \times \omega$, which is not an equilibrium itself, is a heteroclinic orbit, i.e., there exist two different equilibria w_u^+ and w_u^- belonging to $\mathfrak{V}_+$ such that*

$$(4.23) \qquad T_s u \to w_u^+ \quad \textit{as } s \to +\infty, \qquad T_s u \to w_u^- \quad \textit{as } s \to -\infty .$$

In fact, in view of estimate (1.25), see Remark 1.7, any solution $u(t)$ to the problem (3.4) is bounded with respect to both $t \to \infty$ and $t \to -\infty$. Thus the convergence (4.23) follows from Theorem 4.4. Hence it remains to prove that $w_u^+ \neq w_u^-$. Integrating (4.7) over $\mathbb{R}$, where $g_1 \equiv 0$, we get

$$(4.24) \qquad \mathcal{F}_u(+\infty) - \mathcal{F}_u(-\infty) = \mathcal{F}_{w^+} - \mathcal{F}_{w^-} = -\int_{\mathbb{R}} (\gamma \partial_t u,\ \partial_t u)\, dt \neq 0 .$$

Thus $w^+ \neq w^-$.

We now give examples for the perturbation term $g_1(t, x)$ satisfying the conditions of Theorem 4.4.

EXAMPLE 4.7: Let

$$g_1(t, x) = \varphi(t)\, g_0(x), \tag{4.25}$$

where $g_0 \in L^2(\omega)$ and

$$\varphi(t) = \frac{|\sin(t^2)|}{1+t^2} \tag{4.26}$$

Then condition (i) of (4.10) is fulfilled. (4.2) holds for the strong topology.

EXAMPLE 4.8: Let $g_1(t, x)$ be as in (4.25), where

$$\varphi(t) = \frac{t}{1+t^2}. \tag{4.27}$$

Then condition (ii) of (4.10) is fulfilled. (4.2) holds for the strong topology.

EXAMPLE 4.9: Let $g_1(t, x)$ be as in (4.25), where

$$\varphi(t) = \sin(t^3). \tag{4.28}$$

Then condition (iii) of (4.10) is fulfilled. (4.2) holds for the weak topology.

Part 2. Asymptotics in the three-dimensional case

In this second part, we describe the asymptotics of solutions to the linear system corresponding to (0.1) when the half-cylinder $\Omega_+ = \mathrm{R}_+ \times \omega$ is three-dimensional and conclude from that the existence of the trajectory attractor for the singular part of the solutions to the nonlinear elliptic system (0.1).

5. - EDGE ASYMPTOTICS IN THE THREE-DIMENSIONAL CASE

To start with, we discuss elliptic regularity for the Neumann problem for the Laplace operator on an infinite cone $\Gamma \subset \mathrm{R}^2$ and the infinite wedge $\mathrm{R} \times \Gamma \subset \mathrm{R}^3$. Let $\Gamma \subset \mathrm{R}^2$ be an open cone with angle α. Throughout we shall suppose that $\Gamma = \{(r, \theta);\ 0 < \theta < \alpha\}$. Here (r, θ) denote polar coordinates in R^2. We further suppose that $\alpha > \pi$ (see Remark 5.3 (a)).

Since the model cone Γ arises from flatting out the boundary of ω near a fixed conical point of $\partial\omega$, we shall consider operators $1 - \Delta_y - M(y, \partial_y)$ on Γ, where $y = (y_1, y_2)$ are Euclidian coordinates in R^2 and $M(y, \partial_y) = \sum_{|\gamma| \le 2} b_\gamma(y)\, \partial_y^\gamma$ is a second-order partial differential operator subject to the following conditions: For $\gamma \in \mathrm{N}^2$, $|\gamma| = 2$, $b_\gamma \in C^\infty(\bar{\Gamma})$ and

(a1) $\|b_\gamma\|_{L^\infty(\Gamma)} \le \delta$, $b_\gamma(0) = 0$;

(a2) $\|\nabla_y b_\gamma\|_{L^\infty(B_{0,R} \cap \Gamma)} \le K$, where $B_\varrho = \{y \in \mathbb{R}^2;\ |y| \leq \varrho\}$. For $\gamma \in \mathbb{N}^2$, $|\gamma| \leq 1$, $b_\gamma = b_{\gamma 1} + b_{\gamma 2}$, where $b_{\gamma 1}, b_{\gamma 2} \in C^\infty(\bar{\Gamma})$ and

(b1) $\operatorname{supp} b_{\gamma 1} \subseteq B_\delta \cap \bar{\Gamma}$, $\|b_{\gamma 1}\|_{L^\infty(\Gamma)} \leq K$;

(b2) $\|b_{\gamma 2}\|_{L^\infty(\Gamma)} \leq \delta$

for some constant $K > 1$ and a certain $\delta = \delta(K)$ sufficiently small, $0 < \delta < 1$.

Let $H^2_\Diamond(\Gamma)$ denote the space of all variational solutions v to the problem

$$(5.1) \qquad (1 - \Delta_y - M(y, \partial_y))\, v = g\,, \qquad \partial_n v|_{\partial\Gamma} = 0$$

with right-hand side $g \in L^2(\Gamma)$ (see Appendix A).

Remark 5.1: It is a well-known fact that the space $H^2_\Diamond(\Gamma)$ is actually independent of the choice of the operator $M(y, \partial_y)$ satisfying (a), (b) provided that $\delta > 0$ is small enough (see [13], [17]).

Moreover, from Theorem A.13 and its corresponding version for a model cone it follows that a solution to (5.1) belongs to $H^{3/2+\varepsilon}(\Gamma)$ for a certain $\varepsilon > 0$.

For the special case $M(y, \partial_y) \equiv 0$, it is readily seen that

$$(5.2) \qquad H^2_\Diamond(\Gamma) = H^2_N(\Gamma) \oplus \operatorname{span}\{S\}, \qquad S(y) = \psi(r)\, r^{\pi/\alpha} \cos(\pi\theta/\alpha),$$

where $H^2_N(\Gamma) = \{v \in H^2(\Gamma);\ \partial_n v|_{\partial\Gamma} = 0\}$ (see [9], [13]). Here $\psi \in C_0^\infty(\bar{\Gamma})$ is some fixed cut-off function, depending only on the radial coordinate r, such that $\psi(r) = 1$ in a neighbourhood of 0 and ψ is supported sufficiently close to 0. Notice that $S \in H^{1+\pi/\alpha-\varepsilon}(\Gamma)$ for any $\varepsilon > 0$, but $S \notin H^{1+\pi/\alpha}(\Gamma)$.

Lemma 5.2: *For $\delta > 0$ sufficiently small (depending on K), the differential operator*

$$(5.3) \qquad 1 - \Delta_y - M(y, \partial_y)\colon H^2_\Diamond(\Gamma) \to L^2(\Gamma)$$

induces an isomorphism, where $H^2_\Diamond(\Gamma)$ is the space given in (5.2). Moreover, we have the estimate

$$(5.4) \qquad \|v\|_{H^2_\Diamond(\Gamma)} \leq C \|(1 - \Delta_y - M(y, \partial_y))\, v\|_{L^2(\Gamma)}$$

for $v \in H^2_\Diamond(\Gamma)$, where the constant $C > 0$ only depends on δ, K.

Proof: It is known that $1 - \Delta$ is an isomorphism from $H^2_\Diamond(\Gamma)$ onto $L^2(\Gamma)$. Furthermore, it is seen that $M(y, \partial_y)$ maps $H^2_\Diamond(\Gamma)$ into $L^2(\Gamma)$, and it can be shown that

$$(5.5) \qquad \|M(y, \partial_y)\|_{H^2_\Diamond(\Gamma) \to L^2(\Gamma)} \leq C(K)\, \delta^{1/2}$$

with some constant $C(K) > 0$. To prove (5.5) it suffices to observe that

$$\Big\| \sum_{|\gamma|=2} b_\gamma \partial_y^\gamma S \Big\|_{L^2(\Gamma)} \leq CK^{1-\pi/\alpha} \delta^{\pi/\alpha}$$

and

$$\Big\| \sum_{|\gamma|\leq 1} b_{\gamma 1} \partial_y^\gamma v \Big\|_{L^2(\Gamma)} \leq \sum_{|\gamma|\leq 1} \|b_{\gamma 1}\|_{L^4(\Gamma)} \|v\|_{H^{1,4}(\Gamma)} \leq CK\delta^{1/2} \|v\|_{H^2_Q(\Gamma)}.$$

In fact, for $|\gamma| = 2$, we have $b_\gamma(y) = r\tilde{b}_\gamma(y)$ with $\tilde{b}_\gamma(y) = \int_0^1 (y/r) \cdot \nabla_y b_\gamma(sy)\, ds$ such that $\|\tilde{b}_\gamma\|_{L^\infty(B_{\delta/K} \cap \Gamma)} \leq K$ and

$$\int_{B_1 \cap \Gamma} |b_\gamma(y)\, \partial_y^\gamma S(y)|^2 \, dy \leq$$

$$\leq C \int_0^{\delta/K} \int_0^\alpha |\tilde{b}_\gamma(y)|^2 r^{2\pi/\alpha - 1} \, d\theta \, dr + C \int_{\delta/K}^1 \int_0^\alpha |b_\gamma(y)|^2 r^{2\pi/\alpha - 3} \, d\theta \, dr \leq$$

$$\leq C(K^2 (\delta/K)^{2\pi/\alpha} + \delta^2 (\delta/K)^{2(\pi/\alpha - 1)}) = CK^{2(1-\pi/\alpha)} \delta^{2\pi/\alpha}.$$

Moreover, $H^2_Q(\Gamma) \subset H^{1,4}(\Gamma)$ follows from the explicit description of $H^2_Q(\Gamma)$ and Sobolev's embedding theorem. Now choose $\delta > 0$ dependent on K so small that

$$\|M(y, \partial_y)\|_{H^2_Q(\Gamma) \to L^2(\Gamma)} < \|(1 - \Delta)^{-1}\|_{L^2(\Gamma) \to H^2_Q(\Gamma)},$$

where $(1 - \Delta)^{-1}$ stands for the inverse to $1 - \Delta \colon H^2_Q(\Gamma) \to L^2(\Gamma)$. Then the differential expression $1 - \Delta - M(y, \partial_y)$ in (5.3) induces an isomorphism.

The estimate (5.4) immediately follows. ■

REMARK 5.3: (a) The same argumentation yields that $H^2_Q(\Gamma) = H^2_S(\Gamma)$ when $\alpha < \pi$. In subsequent discussion we always assume that $\alpha > \pi$.

(b) From (5.2) it follows that each $v \in H^2_Q(\Gamma)$ can uniquely be represented in the form

$$v = v_0 + dS, \tag{5.6}$$

where $v_0 \in H^2_S(\Gamma)$, $d \in \mathbb{C}$. It is important to observe that the coefficient d in (5.6) is independent of the particular cut-off function ψ, i.e., choosing another cut-off function possessing the same properties as ψ we obtain d as before.

Now we want to discuss the space $H^2_Q(\mathbb{R} \times \Gamma)$ of variational solutions v to

$$(1 - \partial_t^2 - \Delta_y - M(y, \partial_y))\, v = g, \qquad \partial_n v|_{\mathbb{R} \times \partial\Gamma} = 0 \tag{5.7}$$

with right-hand side $g \in L^2(\mathbb{R} \times \Gamma)$, where $M(y, \partial_y)$ is a second-order partial differential operator as above, but satisfying the additional conditions

$$\operatorname{supp} b_{\gamma 1} \subseteq B_{\delta^{1}/K} \cap \Gamma, \qquad b_{\gamma 2}(y) \equiv 0 \quad \text{for } |\gamma| \leq 1. \tag{5.8}$$

Again it turns out that the space $H^2_Q(\mathrm{R} \times \Gamma)$ is independent of the operator $M(y, \partial_y)$ provided that $\delta > 0$ is small enough.

We need the following result in the cases $s = 2$, $s = 0$. For a proof, see [9], [17].

Lemma 5.4: *Let $\Gamma \subset \mathrm{R}^2$ be an open cone, $s \in \mathrm{R}$. Then an equivalent norm on $H^s(\mathrm{R} \times \Gamma)$ is given by*

$$\|u\|_{H^s(\mathrm{R}\times\Gamma)} = \left\{ \int_{-\infty}^{\infty} \langle\tau\rangle^{2s} \|\kappa(\tau)^{-1}\hat{u}(\tau)\|^2_{H^s(\Gamma)} d\tau \right\}^{1/2}, \tag{5.9}$$

where $\hat{u}(\tau) = F_{t\to\tau} u(\tau)$, $\kappa(\tau) = \kappa_{\langle\tau\rangle}$, $\langle\tau\rangle = (1 + |\tau|^2)^{1/2}$, and

$$\kappa_\lambda u(y) = \lambda u(\lambda y), \qquad \lambda > 0, \; y \in \Gamma,$$

for $u \in H^s(\Gamma)$.

Notice that $\{\kappa_\lambda\}_{\lambda > 0}$ is a strongly continuous group on $H^s(\Gamma)$. It consists of isometries when $s = 0$.

Lemma 5.5: *Let $\Gamma \subset \mathrm{R}^2$ be an open cone as above. Then we have*

$$H^2_Q(\mathrm{R} \times \Gamma) = H^2_N(\mathrm{R} \times \Gamma) \oplus \tag{5.10}$$

$$\oplus (F^{-1}_{\tau\to t}\{\langle\tau\rangle \psi(r\langle\tau\rangle)(r\langle\tau\rangle)^{\pi/\alpha} \cos(\pi\theta/\alpha)\, \hat{d}(\tau)\};\; d \in H^2(\mathrm{R})),$$

where $H^2_N(\mathrm{R} \times \Gamma) = \{v \in H^2(\mathrm{R} \times \Gamma);\, \partial_n v|_{\mathrm{R}\times\partial\Gamma} = 0\}$.

Proof: Let v be a solution to (5.6) with right-hand side $g \in L^2(\mathrm{R} \times \Gamma)$. Upon applying the Fourier transformation $F_{t\to\tau}$ and afterwards the group action $\kappa(\tau)^{-1}$ we obtain the equation

$$\begin{cases} (1 - \Delta - M_\tau(y, \partial_y))\, \kappa(\tau)^{-1}\hat{v}(\tau) = \langle\tau\rangle^{-2} \kappa(\tau)^{-1}\hat{g}(\tau) & \text{in } \Gamma, \\ \partial_n (\kappa(\tau)^{-1}\hat{v}(\tau))|_{\partial\Gamma} = 0 \end{cases} \tag{5.11}$$

with parameter $\tau \in \mathrm{R}$, where $M_\tau(y, \partial_y) = \langle\tau\rangle^{-2} M(\langle\tau\rangle^{-1} y, \langle\tau\rangle\, \partial_y)$. Now it is seen that the operator $M_\tau(y, \partial_y) = \sum_{1\leq|\gamma|\leq 2} \langle\tau\rangle^{-2+|\gamma|} b_\gamma(\langle\tau\rangle^{-1} y)\, \partial^\gamma_y$ satisfies requirements (a), (b) with the same $\delta > 0$, $K > 1$ as before. Indeed, for $|\gamma| \leq 1$, we put $b_{\gamma 1,\tau}(y) = \langle\tau\rangle^{-1} b_\gamma(\langle\tau\rangle^{-1} y)$, $b_{\gamma 2,\tau}(y) = 0$ if $\langle\tau\rangle \leq K/\delta$ and $b_{\gamma 1,\tau}(y) = 0$, $b_{\gamma 2,\tau}(y) = \langle\tau\rangle^{-1} b_\gamma(\langle\tau\rangle^{-1} y)$ if $\langle\tau\rangle > K/\delta$.

Hence we conclude from Eq. (5.11) together with (5.1), (5.6) that

$$\kappa(\tau)^{-1}\hat{v}(\tau) = \kappa(\tau)^{-1}\hat{v}_0(\tau) + \hat{d}(\tau)\, S(y), \qquad S(y) = \psi(r)\, r^{\pi/\alpha} \cos(\pi\theta/\alpha).$$

Moreover, from (5.4) we derive the estimate

$$\|\kappa(\tau)^{-1}\hat{v}_0(\tau)\|^2_{H^2(\Gamma)} + |\hat{d}(\tau)|^2 \leq C\langle\tau\rangle^{-4}\|\kappa(\tau)^{-1}\hat{g}(\tau)\|^2_{L^2(\Gamma)}.$$

Therefore, we get

$$\int_{-\infty}^{\infty} \langle\tau\rangle^4 \|\kappa(\tau)^{-1}\hat{v}_0(\tau)\|^2_{H^2(\Gamma)}\, d\tau + \int_{-\infty}^{\infty} \langle\tau\rangle^4 |\hat{d}(\tau)|^2\, d\tau \leq$$

$$\leq C \int_{-\infty}^{\infty} \|\kappa(\tau)^{-1}\hat{g}(\tau)\|^2_{L^2(\Gamma)}\, d\tau = C\|g\|^2_{L^2(\mathrm{R}\times\Gamma)}$$

showing that $v_0 \in H^2(\mathrm{R}\times\Gamma)$, $d \in H^2(\mathrm{R})$ by Lemma 5.4. From (5.14) we finally get

$$v = v_0 + F^{-1}_{\tau\to t}(\hat{d}(\tau)(\kappa(\tau)\, S)(y)) \tag{5.15}$$

which gives us the decomposition (5.10), observing that the sum on the right-hand side of (5.10) is direct and is obviously contained in $H^2_Q(\mathrm{R}\times\Gamma)$. ■

Remark 5.6: The proof of Lemma 5.5 shows that

$$\|u\|_{H^2_Q(\mathrm{R}\times\Gamma)} = \left\{\int_{-\infty}^{\infty} \langle\tau\rangle^4 \|\kappa(\tau)^{-1}\hat{u}(\tau)\|^2_{H^2_Q(\Gamma)}\, d\tau\right\}^{1/2}$$

is an equivalent norm on $H^2_Q(\mathrm{R}\times\Gamma)$. Since $H^2_Q(\Gamma)$ is a cone Sobolev space of functions possessing asymptotics of a certain discrete asymptotic type near $y = 0$, $H^2_Q(\mathrm{R}\times\Gamma)$ is in fact a wedge Sobolev space in the sense of B.-W. Schulze (see [15-17]).

Next we discuss elliptic regularity and asymptotics for the cylinder $\Omega = \mathrm{R}\times\omega$ and the half-cylinder $\Omega_+ = \mathrm{R}_+\times\omega$. Let ω be a bounded, polyhedral domain in R^2. The boundary $\partial\omega$ is in particular smooth except for a finite number of conical points. Only the conical points with an obtuse angle deserve further interest, for H^2-regularity holds up to conical points with an acute angle (see Remark 5.3 (a)).

Let $\{b_1, \dots, b_\kappa\}$ denote the set of these conical points. Let α_j be the angle at b_j, $\alpha_j > \pi$. For every j, $1 \leq j \leq \kappa$, we choose an open cone $\Gamma_j \subset \mathrm{R}^2$, open subsets U_j, V_j in R^2 with $U_j \ni b_j$, $V_j \ni 0$, and a diffeomorphism $\chi_j\colon U_j \to V_j$ such that $\chi_j(b_j) = 0$ and $\chi_j(\overline{\omega} \cap U_j) = \overline{\Gamma}_j \cap V_j$. We assume that $\Gamma_j = \{(r, \theta);\, 0 < \theta < \alpha_j\}$. Furthermore, we suppose that the diffeomorphisms χ_j are chosen in such a manner that $\chi_j'(x)^t\chi_j'(x)$ is a positive scalar multiple of the identity for $x \in \partial\omega \cap U_j$. Moreover, $\chi_j'(b_j) \in \mathrm{SO}(2; \mathrm{R})$. It can be shown that such a choice is possible, if U_j is sufficiently small. Then the homogeneous Neumann boundary condition is preserved under the diffeomorphisms χ_j.

Notice that the assumption implies that

$$(5.14)\qquad (\chi_j)_* \Delta = \Delta + M_j(y, \partial_y)$$

close to $y = 0$, where $M_j(y, \partial_y)$ is a second-order differential operator (without zero-order terms). By shrinking U_j, if necessary, we may suppose that $M_j(y, \partial_y)$ satisfies, for $\Gamma = \Gamma_j$, $K > 1$ and $\delta > 0$ sufficiently small, the assumptions (a), (b) previous to Lemma 5.2 as well as condition (5.8).

Let $U_0 \subset \mathbb{R}^2$ be an open set not meeting $\{b_1, \ldots, b_\kappa\}$ such that $\{U_0\} \cup \{U_j\}_{j=1}^{\kappa}$ forms an open covering of $\overline{\omega}$. Let $\{\phi_0\} \cup \{\phi\}_{j=1}^{\kappa}$ be a subordinate partition of unity, $\phi_0 + \sum_{j=1}^{\kappa} \phi_j = 1$ on $\overline{\omega}$, $\phi_j = 1$ in a neighbourhood of b_j for all j, $1 \leqslant j \leqslant \kappa$. Eventually we assume that, for $1 \leqslant j \leqslant \kappa$, $\psi_j = (\chi_j)_* \phi_j$ only depends on the radial variable r, i.e., $\psi_j = \psi_j(r)$.

Remark 5.7: Before we proceed we give an intrinsic interpretation of (5.4). There is a short split exact sequence

$$(5.15)\qquad 0 \to H_N^2(\omega) \to H_Q^2(\omega) \to \prod_{j=1}^{\kappa} \mathbb{C} \to 0$$

with the surjection assigning to each function $u \in H_Q^2(\omega)$ its sequence $(d_1, \ldots, d_\kappa)$ of singular coefficients. Thereby, d_j is explained as the coefficient appearing in (5.6) in front of S for $v = (\chi_j)_*(\phi_j u)$, $\Gamma = \Gamma_j$.

To see that (5.15) is correctly defined observe that the coefficient d_j is not only independent of the choice of the cut-off function ψ_j (see Remark 5.3 (b)), but also independent of the choice of the diffeomorphism χ_j meeting all of the assumptions above. A splitting of (5.15) is obtained via (5.2) after having fixed the diffeomorphisms χ_j and the cut-off functions ψ_j. More precisely, we may write

$$u = u_0 + \sum_{j=1}^{\kappa} d_j (\chi_j)^* (\psi_j(r)\, r^{\pi/\alpha_j} \cos(\pi\theta/\alpha_j))$$

for $u \in H_Q^2(\omega)$, where $u_0 \in H_N^2(\omega)$, $d_j \in \mathbb{C}$ are uniquely determined. The coefficients d_j can be calculated using the formula

$$(5.16)\qquad d_j = \lim_{r \to 0^+} \beta_j^{-2} (r^{-\pi/\alpha_j}((\chi_j)_*(\phi_j u)(r, \theta) - u(b_j)),\ \cos(\pi\theta/\alpha_j))_{L^2(0,\alpha_j)},$$

where $(\cdot, \cdot)_{L^2(0,\alpha_j)}$ denotes the scalar product in $L^2(0, \alpha_j)$, $u(b_j)$ is the value of u at b_j, and $\beta_j = \left\{ \int_0^{\alpha_j} |\cos(\pi\theta/\alpha_j)|^2 \, d\theta \right\}^{1/2}$. The value $u(b_j) = (\chi_j)_*(\phi_j u)(0)$ is well-defined by Theorem A.13.

Notice that an equivalent norm on $H^2_Q(\mathrm{R}\times\omega)$ is given by

$$\|u\|_{H^2_Q(\mathrm{R}\times\omega)} = \left\{\|\phi_0 u\|^2_{H^2(\mathrm{R}\times\omega)} + \sum_{j=1}^{\kappa}\|(\chi_j)_*(\phi_j u)\|^2_{H^2_Q(\mathrm{R}\times\Gamma_j)}\right\}^{1/2}. \tag{5.17}$$

This follows from the fact that $u\in H^2_Q(\mathrm{R}\times\omega)$ if and only if $\phi_l u\in H^2_Q(\mathrm{R}\times\omega)$ for all l, $0\leq l\leq\kappa$, and obviously $\phi_0 u\in H^2_Q(\mathrm{R}\times\omega)$ if and only if $\phi_0 u\in H^2(\mathrm{R}\times\omega)$, while, for $1\leq j\leq\kappa$, $\phi_j u\in H^2_Q(\mathrm{R}\times\omega)$ if and only if $(\chi_j)_*(\phi_j u)\in H^2_Q(\mathrm{R}\times\Gamma_j)$.

From Lemma 5.5 and (5.17) we conclude that

$$H^2_Q(\mathrm{R}\times\omega) = H^2_N(\mathrm{R}\times\omega)\oplus\left\{\sum_{j=1}^{\kappa}(\chi_j)^*(F^{-1}_{\tau\to t}\{\langle\tau\rangle\psi_j(r(\tau))(r(\tau))^{\pi/\alpha_j}\times\right.$$
$$\left.\times\cos(\pi\theta/\alpha_j)\,\hat{d}_j(\tau)\});\ d_j\in H^2(\mathrm{R}),\ 1\leq j\leq\kappa\right\}. \tag{5.18}$$

Analogously to (5.15) we have the following lemma.

LEMMA 5.8: *For $\omega\subset\mathrm{R}^2$ being a bounded, polyhedral domain as above, there is a short split exact sequence*

$$0\longrightarrow H^2_N(\mathrm{R}\times\omega)\longrightarrow H^2_Q(\mathrm{R}\times\omega)\xrightarrow{(\tau_1,\ldots,\tau_\kappa)}\prod_{j=1}^{\kappa}H^{1-\pi/\alpha_j}(\mathrm{R})\longrightarrow 0, \tag{5.19}$$

where the operators τ_j are given by

$$\tau_j u(t) = \lim_{r\to 0^+}\beta_j^{-2}(r^{-\pi/\alpha_j}((\chi_j)_*(\phi_j u)(t,r,\theta) - u(t,b_j)),\ \cos(\pi\theta/\alpha_j))_{L^2(0,\alpha_j)}. \tag{5.20}$$

Moreover, a splitting of (5.19) is given by the mapping

$$(d_{11},\ldots,d_{\kappa 1})\mapsto\sum_{j=1}^{\kappa}(\chi_j)^*(F^{-1}_{\tau\to t}\{\psi_j(r(\tau))\,\hat{d}_{j1}(\tau)\}\,r^{\pi/\alpha_j}\cos(\pi\theta/\alpha_j)). \tag{5.21}$$

PROOF: According to (5.11) and the short exact sequence (5.15), the functions $d_j\in H^2(\mathrm{R})$ appearing in the representation of $u\in H^2_Q(\mathrm{R}\times\omega)$ as

$$u = u_0 + \sum_{j=1}^{\kappa}(\chi_j)^*(F^{-1}_{\tau\to t}\{\langle\tau\rangle\,\psi_j(r(\tau))(r(\tau))^{\pi/\alpha_j}\cos(\pi\theta/\alpha_j)\,\hat{d}_j(\tau)\}),$$
$$= u_0 + \sum_{j=1}^{\kappa}(\chi_j)^*(F^{-1}_{\tau\to t}\{\psi_j(r(\tau))\,\hat{d}_{j1}(\tau)\}\,r^{\pi/\alpha_j}\cos(\pi\theta/\alpha_j))$$

where $u_0\in H^2_N(\mathrm{R}\times\omega)$, are uniquely determined, independently of the choice of the diffeomorphisms χ_j and the cut-off functions ψ_j. Likewise, the same is then true for the functions $d_{j1} = \langle D\rangle^{1+\pi/\alpha_j}d_j\in H^{1-\pi/\alpha_j}(\mathrm{R})$. Therefore, the surjection in (5.19) is well-defined. Moreover, it becomes clear that (5.19) is exact and a splitting of it is provided by (5.21).

Thus it remains to show (5.20). From (5.16), applied to $\Gamma = \Gamma_j$, $v = (\chi_j)_*(\phi_j u)$, and Eq. (5.11), in which $d = d_j$, we conclude that

$$\hat{d}_j(\tau) = \lim_{r \to 0^+} \beta_j^{-2}(r^{-\pi/\alpha_j}(\langle\tau\rangle^{-1}\hat{v}(\tau, r\langle\tau\rangle^{-1}, \theta) - \langle\tau\rangle^{-1}\hat{v}(\tau, 0)), \cos(\pi\theta/\alpha_j))_{L^2(0,\alpha_j)} =$$

$$= \lim_{r \to 0^+} \beta_j^{-2}((r\langle\tau\rangle)^{-\pi/\alpha_j}\langle\tau\rangle^{-1}(\hat{v}(\tau, r, \theta) - \hat{v}(\tau, 0)), \cos(\pi\theta/\alpha_j))_{L^2(0,\alpha_j)},$$

the latter line upon replacing r with $r\langle\tau\rangle$, i.e.,

$$\hat{d}_{j1}(\tau) = \langle\tau\rangle^{1+\pi/\alpha_j}\hat{d}_j(\tau) = \lim_{r \to 0^+} \beta_j^{-2}(r^{-\pi/\alpha_j}(\hat{v}(\tau, r, \theta) - \hat{v}(\tau, 0)), \cos(\pi\theta/\alpha_j))_{L^2(0,\alpha_j)},$$

$$d_{j1}(t) = \lim_{r \to 0^+} \beta_j^{-2}(r^{-\pi/\alpha_j}((\chi_j)_*(\phi_j u)(t, r, \theta) - u(t, b_j)), \cos(\pi\theta/\alpha_j))_{L^2(0,\alpha_j)}.$$

This proves Lemma 5.8 completely. ■

From (5.20) we obtain in particular that the trace operation on an edge is local.

Corollary 5.9: *For* $u \in H^2_Q(\mathrm{R} \times \omega)$, *we have* $\operatorname{supp}(\tau_j u) \subsetneq \operatorname{supp}(u) \cap (\mathrm{R} \times \times \{b_j\})$.

Remark 5.10: (a) To interpret the functions $d_{j1} \in H^{1-\pi/\alpha_j}(\mathrm{R})$, $1 \leq j \leq \kappa$, as coefficients in the asymptotic expansion of $u \in H^2_Q(\mathrm{R} \times \omega)$ close to the edge $\mathrm{R} \times \{b_j\}$, we observe that $F^{-1}_{\tau \to t}\{\psi_j(r\tau)\,\hat{d}_{j1}(\tau)\} = d_{j1}(t)$ when $r = 0$.

(b) It can be shown that

$$\beta_j^{-2}(r^{-\pi/\alpha_j}((\chi_j)_*(\phi_j u)(t, r, \theta) - u(t, b_j)), \cos(\pi\theta/\alpha_j))_{L^2(0,\alpha_j)} \in H^1(\mathrm{R})$$

for $u \in H^2_Q(\mathrm{R} \times \omega)$, and convergence in (5.20) takes place in $H^{1-\pi/\alpha_j}(\mathrm{R})$.

The final goal in this section is to conclude the form of asymptotics when going over from $H^2_Q(\mathrm{R} \times \omega)$ to its quotient space $H^2_Q(\mathrm{R}_+ \times \omega)$. This is achieved by constructing a suitable splitting of (5.19) in terms of a continuous projection Π_2 in $H^2_Q(\mathrm{R} \times \omega)$ by means of a reformulation of the asymptotic information.

Theorem 5.11: *Let* $\omega \subset \mathrm{R}^2$ *be a bounded, polyhedral domain as above. Then there exists a continuous projection* Π_2 *in* $H^2_Q(\mathrm{R} \times \omega)$ *obeying the following properties:*

(a) $\ker \Pi_2 = H^2_N(\mathrm{R} \times \omega)$;

(b) $T_s \Pi_2 = \Pi_2 T_s$ *for all* $s \in \mathrm{R}$;

(c) $\operatorname{supp} u \subsetneq \overline{\mathrm{R}}_-$ implies $\operatorname{supp} \Pi_2 u \subsetneq \overline{\mathrm{R}}_-$;

(d) Π_2 *is* $(H^2_{Q,\delta}(\mathrm{R} \times \omega), H^2_{Q,\delta}(\mathrm{R} \times \omega))$*-continuous;*

(e) Π_2 *is* $(H^2_{Q,\mathrm{loc}}(\mathrm{R} \times \omega), H^2_{Q,\mathrm{loc}}(\mathrm{R} \times \omega))$*-continuous.*

In the proof of Theorem 5.11 we shall make use of the following result.

Lemma 5.12: *Let $\Gamma \subset \mathrm{R}^2$ be an open cone. Further let $\psi \in \mathcal{S}(\mathrm{R})$, $\psi_1 \in \mathcal{S}(\overline{\mathrm{R}}_+)$, $d_1 \in H^{1-\pi/\alpha}(\mathrm{R})$. Then*

$$(5.22) \qquad \psi_1(r)\, F^{-1}_{\tau\to t}((\psi(r\langle\tau\rangle) - \psi(r\tau))\, \hat{d}_1(\tau))\, r^{\pi/\alpha} \cos(\pi\theta/\alpha) \in H^2_N(\mathrm{R} \times \Gamma).$$

Proof: Let $u(t, r) = \psi_1(r)\, F^{-1}_{\tau\to t}((\psi(r\langle\tau\rangle) - \psi(r\tau))\, \hat{d}_1(\tau))\, r^{\pi/\alpha} \cos(\pi\theta/\alpha)$. Then we have

$$(5.23) \qquad \|u\|_{H^2_N(\mathrm{R}\times\Gamma)} =$$

$$= \left\{ \int_{-\infty}^{\infty} \langle\tau\rangle^2 \|\kappa(\tau)^{-1} (\psi_1(r)(\psi(r\langle\tau\rangle) - \psi(r\tau))\, \hat{d}_1(\tau)\, r^{\pi/\alpha} \cos(\pi\theta/\alpha))\|^2_{H^2_N(\Gamma)}\, d\tau \right\}^{1/2} =$$

$$= \left\{ \int_{-\infty}^{\infty} \langle\tau\rangle^2\, |\hat{d}(\tau)|^2 \|\psi_1(r\langle\tau\rangle^{-1})(\psi(r) - \psi(r\tau/\langle\tau\rangle))\, r^{\pi/\alpha} \cos(\pi\theta/\alpha)\|^2_{H^2_N(\Gamma)}\, d\tau \right\}^{1/2} \leq$$

$$\leq C \left\{ \int_{-\infty}^{\infty} \langle\tau\rangle^2\, |\hat{d}(\tau)|^2\, d\tau \right\}^{1/2},$$

where $d = \langle D\rangle^{-1-\pi/\alpha} d_1 \in H^2(\mathrm{R})$. Thereby,

$$\|\psi_1(r\langle\tau\rangle^{-1})(\psi(r) - \psi(r\tau/\langle\tau\rangle))\, r^{\pi/\alpha} \cos(\pi\theta/\alpha)\|_{H^2_N(\Gamma)} \leq C$$

for a certain constant $C > 0$ independent of τ is seen from the fact that $\psi_2(r) \mapsto \psi_2(r)\, r^{\pi/\alpha} \cos(\pi\theta/\alpha)$ constitutes a bounded map from $\{\psi_2 \in \mathcal{S}(\overline{\mathrm{R}}_+);\ \psi_2(0) = 0\}$ into $H^2_N(\Gamma)$, while $\{\psi_1(r\langle\tau\rangle^{-1})(\psi(r) - \psi(r\tau/\langle\tau\rangle));\ \tau \in \mathrm{R}\}$ for $\psi \in \mathcal{S}(\mathrm{R})$, $\psi_1 \in \mathcal{S}(\overline{\mathrm{R}}_+)$ is bounded in $\{\psi_2 \in \mathcal{S}(\overline{\mathrm{R}}_+);\ \psi_2(0) = 0\}$. Hence the right-hand side in (5.23) is finite proving that $u \in H^2_N(\mathrm{R} \times \Gamma)$. ■

Proof of Theorem 5.11: Lemma 5.12 allows to replace $F^{-1}_{\tau\to t}\{\langle\tau\rangle \psi_j(r\langle\tau\rangle)(r\langle\tau\rangle)^{\pi/\alpha_j} \cos(\pi\theta/\alpha_j)\, \hat{d}_j(\tau)\}$ in (5.18) by $\psi_{j1}(r)\, F^{-1}_{\tau\to t}\{\psi_j(r\tau)\, \hat{d}_{j1}(\tau)\}\, r^{\pi/\alpha} \cos(\pi\theta/\alpha_j)$ i.e., we have

$$H^2_Q(\mathrm{R} \times \omega) = H^2_N(\mathrm{R} \times \omega) \oplus$$

$$\oplus \left\{ \sum_{j=1}^{\kappa} (\chi_j)^* (\psi_{j1}(r)\, F^{-1}_{\tau\to t}\{\psi_j(r\tau)\, \hat{d}_{j1}(\tau)\}\, r^{\pi/\alpha_j} \cos(\pi\theta/\alpha_j));\ d_{j1} \in H^{1-\pi/\alpha_j}(\mathrm{R}) \right\},$$

where, for each j, $1 \leq j \leq \kappa$, $\psi_j \in \mathcal{S}(\mathrm{R})$, $\psi_{j1} \in C_0^\infty(\overline{\mathrm{R}}_+)$, $\psi_j(0) = \psi_{j1}(0) = 1$, and ψ_{j1} is supported in V_j when considered as a function on Γ_j. If especially the ψ_j are chosen in a way such that $\operatorname{supp} F^{-1}\psi_j \subseteq \overline{\mathrm{R}}_-$ holds for all j, then

$$(5.24) \qquad \Pi_2 u = \sum_{j=1}^{\kappa} (\chi_j)^* (\psi_{j1}(r)\, F^{-1}_{\tau\to t}\{\psi_j(r\tau)(\tau_j u)^\frown(\tau)\}\, r^{\pi/\alpha_j} \cos(\pi\theta/\alpha_j))$$

for $u \in H_Q^2(\mathbb{R} \times \omega)$ is a projection in $H_Q^2(\mathbb{R} \times \omega)$ meeting all the requirements (a)-(e). That Π_2 is a projection follows from the fact that $\tau_j \Pi_2 u = \tau_j u$ holds for $u \in H_Q^2(\mathbb{R} \times \omega)$, (a), (c) are immediate, (b) is the locality of the trace operator τ_j (see Corollary 5.9) and the translation invariance of the pseudodifferential operator $d_1 \mapsto F^{-1}_{\tau \to t}(\psi_j(r\tau)\, \hat{d}_1(\tau))$, where $r > 0$ is regarded as a parameter, and (d), (e) come from the observation that $\psi_{j1}(r)\, F^{-1}_{\tau \to t}\{\psi_j(r\tau)(\tau_j u)^{\wedge}(\tau)\}\, r^{\pi/\alpha_j} \cos(\pi\theta/\alpha_j)$ belongs to $H^2_{Q,b}(\mathbb{R} \times \Gamma_j)$ and $H^2_{Q,\mathrm{loc}}(\mathbb{R} \times \Gamma_j)$, respectively, for u belonging to $H^2_{Q,b}(\mathbb{R} \times \Gamma_j)$ and $H^2_{Q,\mathrm{loc}}(\mathbb{R} \times \Gamma_j)$, as an easy calculation reveals. ■

The following consequences of Theorem 5.11 supply the projection Π_2^+ in $H^2_{Q,b}(\mathbb{R}_+ \times \omega)$ onto its closed subspace comprising the asymptotic information as well as the short exact sequences used in § 6.

THEOREM 5.13: *Let $\omega \subset \mathbb{R}^2$ be a bounded, polyhedral domain as above. Then there exists a continuous projection Π_2^+ in $H^2_{Q,b}(\mathbb{R}_+ \times \omega)$ obeying the following properties:*

(a) $\ker \Pi_2^+ = H^2_{N,b}(\mathbb{R}_+ \times \omega)$;

(b) $T_s \Pi_2^+ = \Pi_2^+ T_s$ *for all* $s \geq 0$.

Moreover, Π_2^+ *is* $(H^2_{Q,\mathrm{loc}}(\overline{\mathbb{R}}_+ \times \omega), H^2_{Q,\mathrm{loc}}(\overline{\mathbb{R}}_+ \times \omega))$*-continuous.*

PROOF: The theorem follows from Theorem 5.11 (a)-(e) by continuous extension of the projection Π_2 to $H^2_{Q,b}(\mathbb{R} \times \omega)$ and its subsequent factorization to $H^2_{Q,b}(\mathbb{R}_+ \times \omega)$. ■

Notice that a projection Π_2^+ satisfying the requirements of Theorem 5.13 is

$$\Pi_2^+ u = \sum_{j=1}^{\kappa} (\chi_j)^* \left(\psi_{j1}(r)\, F^{-1}_{\tau \to t}\{\psi_j(\tau r)((\tau_j^+ u)_{\mathrm{ext}})^{\wedge}(\tau)\}\, r^{\pi/\alpha_j} \cos(\pi\theta/\alpha_j)\right), \tag{5.25}$$

$u \in H^2_{Q,b}(\mathbb{R}_+ \times \omega)$, where ψ, ψ_{j1} are as in (5.24). Here $(\tau_j^+ u)_{\mathrm{ext}}$ means an arbitrary extension of $\tau_j^+ u \in H_b^{1-\pi/\alpha_j}(\mathbb{R}_+)$ to a function in $H_b^{1-\pi/\alpha_j}(\mathbb{R})$.

COROLLARY 5.14: *The short exact sequence (5.19) extends by continuity and factors subsequently to a short split exact sequence*

$$0 \longrightarrow H^2_{N,b}(\mathbb{R}_+ \times \omega) \longrightarrow H^2_{Q,b}(\mathbb{R}_+ \times \omega) \xrightarrow{(\tau_1^+, \ldots, \tau_\kappa^+)} \prod_{j=1}^{\kappa} H_b^{1-\pi/\alpha_j}(\mathbb{R}_+) \longrightarrow 0,$$

where $(\tau_1^+, \ldots, \tau_\kappa^+)$ is the vector of trace operators. A splitting is obtained from (5.25) by replacing $\tau_j^+ u$ by $d_{1j} \in H_b^{1-\pi/\alpha_j}(\mathbb{R}_+)$.

6. - Regular and singular part of the trajectory attractor

In this final section we show that the trajectory attractor A of the problem (0.1) decomposes into a regular part A_{reg} and a singular part A_{sing}. For brevity we suppose that the right-hand side g of the problem (0.1) is strongly translation-compact in Ξ^+. The case of weak translation-compactness is treated analogously.

Let $K^+ = K_\Sigma^+$ be the union of all solutions to the family (3.4) (see Definition 3.7). Let Π_2^+ be the same as in Theorem 5.13. The regular and the singular part of the union K^+ are introduced by

$$K_{reg}^+ = \Pi_1^+ K^+, \; K_{sing}^+ = \Pi_2^+ K^+, \quad \text{where } \Pi_1^+ = \mathrm{Id} - \Pi_2^+. \tag{6.1}$$

Notice that

$$K_{reg}^+ \subset [H^2_{N,\,loc}(\Omega_+)]^k, \tag{6.2}$$

and the topology on K_{reg}^+ induced by the embedding $K_{reg}^+ \subset \Theta_0^+$ coincides with the topology induced by the embedding (6.2).

From Theorem 5.13 it follows that the semigroup $\{T_s, s \geq 0\}$ of positive shifts acts both in K_{reg}^+ and in K_{sing}^+, i.e.,

$$T_s K_{reg}^+ \subseteq K_{reg}^+ \quad \text{and} \quad T_s K_{sing}^+ \subseteq K_{sing}^+ \quad \text{for all } s \geq 0. \tag{6.3}$$

Definition 6.1: The attractor A_{reg} of the semigroup $\{T_s, s \geq 0\}$ acting in the topological space K_{reg}^+ is called the regular trajectory attractor for the problem (0.1) (see Definition 3.7).

Analogously, the attractor A_{sing} of the semigroup $\{T_s, s \geq 0\}$ acting in the topological space K_{sing}^+ is called the singular trajectory attractor for the problem (0.1).

Theorem 6.2: *Under the above assumptions, the problem* (0.1) *possesses a regular trajectory attractor* A_{reg} *as well as a singular trajectory attractor* A_{sing}. *Moreover,*

$$A_{reg} = \Pi_1^+ A, \quad A_{sing} = \Pi_2^+ A, \tag{6.4}$$

where A *is the trajectory attractor for the problem* (0.1).

Proof: We only verify that $A_{sing} = \Pi_2^+ A$. $A_{reg} = \Pi_1^+ A$ is completely analogous.

First we are concerned with the attracting property. Let $\mathcal{O} = \mathcal{O}(\Pi_2^+ A)$ be a neighbourhood of $\Pi_2^+ A$ in K_{sing}^+. By Theorem 5.13, $(\Pi_2^+)^{-1}\mathcal{O}$ is a neighbourhood of A in K^+. Hence from the attracting property for A we obtain that there exists a $s_0 > 0$ such that

$$T_s K^+ \subset \Pi_2^{-1} \mathcal{O} \quad \text{for } s \geq s_0. \tag{6.5}$$

Applying Π_2^+ to both sides of (6.5) and using assertion (b) of Theorem 5.13, we find

$$T_s K_{\text{sing}}^+ \subset \Pi_2^+ (\Pi_2^+)^{-1} \mathcal{O} \subsetneq \mathcal{O} \quad \text{for } s \geq s_{\mathcal{O}}.$$

This is the attracting property for Π_2^+ A. Secondly, by definition, T_s A = A for all $s \geq 0$. Applying Π_2^+ to both sides and using (b) of Theorem 5.13 again we get

$$T_s \Pi_2^+ \mathrm{A} = \Pi_2^+ \mathrm{A} \quad \text{for } s \geq 0.$$

This is the strict invariance of Π_2^+ A under the action of $\{T_s, s \geq 0\}$. Finally, the compactness of Π_2^+ A in K_{sing}^+ is immediate from the compactness of the attractor A and the continuity of Π_2^+.

Thus Π_2^+ A is the singular trajectory attractor for the problem (0.1). Theorem 6.2 is proved. ■

Corollary 6.3: *Let τ_j^+, $1 \leq j \leq \kappa$, be the trace operators supplied by* Corollary 5.14. *Then the semigroup $\{T_s, s \geq 0\}$ of positive shifts acts in the spaces $\tau_j^+ K^+ \subset [H_{\text{loc}}^{1-\pi/\alpha_j}(\overline{\mathbb{R}}_+)]^k$ and possesses the attractors $\mathrm{A}_j = \tau_j^+ \mathrm{A}$ there.*

This is immediate from the topological isomorphism

$$(\tau_1^+, \ldots, \tau_\kappa^+): \Pi_2^+ H_{Q,\text{loc}}^2(\overline{\mathbb{R}}_+ \times \omega) \to \bigoplus_{j=1}^{\kappa} H_{\text{loc}}^{1-\pi/\alpha_j}(\overline{\mathbb{R}}_+)$$

derived in § 5. Note that τ_j^+ A has an invariant meaning, while Π_2^+ A depends on the choice of the projection Π_2^+.

Finally we are concerned with the question of stabilization of asymptotics in the case when stabilization of solutions takes place (see § 4). For that we impose all assumptions of § 4, in particular, that $f(u) = -\nabla F(u)$ is gradient-like (see (4.5)) and the limit equation

$$a\Delta v_+ - f(v_+) = g_+, \qquad \partial_n v_+ |_{\partial\omega} = 0 \tag{6.6}$$

has only a finite number of solutions $v_+ = v_+^N$ in $[H_Q^2(\omega)]^k$, $N = 1, \ldots, l$.

Let $\{d_j^N\}_{j=1}^{\kappa}$ be the sequence of singular coefficients to v_+^N, i.e.,

$$v_+^N = v_0^N + \sum_{j=1}^{\kappa} d_j^N (\chi_j)^* (\psi_j(r) r^{\pi/\alpha_j} \cos(\pi\theta/\alpha_j)) \tag{6.7}$$

where $v_0^N = \Pi_1^+ v_+^N \in [H_N^2(\omega)]^k$ and $d_j^N \in \mathbb{C}^k$ (see Remark 5.7).

Theorem 6.4: *Let the assumptions of* Theorem 4.4 *be fulfilled. Then, for each solution u to the problem* (0.1), *there exists an equilibrium v_+^N such that*

$$T_s u \to v_+^N \quad \text{in } \Theta_0^+ \quad \text{as } s \to \infty. \tag{6.8}$$

Moreover,

$$T_s \Pi_1^+ u \to \Pi_1^+ v_+^N = v_0^N, \quad T_s \Pi_2^+ u \to \Pi_2^+ v_+^N$$

and

(6.9)
$$T_s \tau_j^+ u \to d_j^N \quad \text{in } H_{loc}^{1-\alpha/\alpha_j}(\bar{\mathbb{R}}_+) \text{ as } s \to \infty .$$

PROOF: (6.8) is a consequence of Theorem 4.4. The other assertions are immediate from the continuity of the operators Π_1^+, Π_2^+, and τ_j^+ in the appropriate spaces. ∎

APPENDIX A. – ELLIPTIC REGULARITY

We formulate auxiliary results concerning the regularity of solutions to linear elliptic equations of the form (0.1).

Let $\omega \subset \mathbb{R}^n$ be a bounded polyhedral domain. Then $H_Q^2(\omega)$ denotes the space of all variational solutions $v \in H^1(\omega)$ to the problem

(A.1)
$$(1 - \Delta)\, v = g \,, \qquad \partial_\nu v|_{\partial\omega} = 0 \,,$$

where the right-hand side g belongs to $L^2(\omega)$. $H_Q^2(\omega)$ is a Hilbert space in a natural manner.

For the moment, let $\mathcal{G}_0$ be the space of all variational solutions to the problem (A.1), with ω replaced by $\Omega_{T_1 - 1, T_2 + 1} = (T_1 - 1, T_2 + 1) \times \omega$. When speaking on Ω_{T_1, T_2}, we shall not consider Ω_{T_1, T_2} as a bounded polyhedral domain itself, but as piece of the cylinder $\mathbb{R} \times \omega$ over the bounded polyhedral domain ω. In slight abuse of notation we define:

DEFINITION A.1: $H_Q^2(\Omega_{T_1, T_2})$ is the space of all restrictions of functions from $\mathcal{G}_0$ to Ω_{T_1, T_2} with the norm

(A.2)
$$\|v, \Omega_{T_1, T_2}\|_{2, Q} \equiv \inf \{ \|u\|_{\mathcal{G}_0} : \, u \in \mathcal{G}_0 , \, u|_{\Omega_{T_1, T_2}} = v \} .$$

Let $H_Q^{3/2}(\omega)$ denote the space of traces on the set $\{t = 0\}$ of functions belonging to $H_Q^2(\Omega_0)$ with the norm

$$\|u_0, \omega\|_{3/2, Q} = \inf \{ \|u, \Omega_0\|_{2, Q} : \, u|_{t=0} = u_0 \} .$$

DEFINITION A.2: $H_{Q, \text{loc}}^2(\Omega_+)$ denotes the Fréchet space of all distributions u on Ω_+ such that $u|_{\Omega_T}$ belongs to $H_Q^2(\Omega_T)$ for every $T \geq 0$.

$H_{Q, b}^2(\Omega_+)$ denotes the Banach space of all functions from $H_{Q, \text{loc}}^2(\Omega_+)$ for which the following norm is finite:

(A.3) $$\|u\|_b = \sup_{T \geq 0} \|u, \Omega_T\|_{2,Q}.$$

Set $V_0 = [H_Q^{3/2}(\omega)]^k$, $\Theta_0^+ = [H^2_{Q,\mathrm{loc}}(\Omega_+)]^k$, $F_0^+ = [H^2_{Q,b}(\Omega_+)]^k$.

Lemma A.3: *Let u be a variational solution to the problem*

(A.4) $$\begin{cases} \partial_t^2 u + \Delta u = g, & \partial_n u|_{\partial\omega} = 0, \\ u|_{t=T_1} = u_1, & u|_{t=T_2} = u_2, \end{cases}$$

where $u_1, u_2 \in V_0$ and $g \in L^2(\Omega_{T_1,T_2})$. Then $u \in H^2_Q(\Omega_{T_1,T_2})$. Moreover, the following estimate holds:

$$\|u, \Omega_{T_1,T_2}\|_{2,Q} \leq C(\|u_1, \omega\|_{3/2,Q} + \|u_2, \omega\|_{3/2,Q} + \|g, \Omega_{T_1,T_2}\|_{0,2}).$$

Proof: By definition, there is a function $v \in H^2_Q(\Omega_{T_1,T_2})$ with $v|_{t=T_1} = u_1$, $v|_{t=T_2} = u_2$ such that

$$\|v, \Omega_{T_1,T_2}\|_{2,Q} \leq C(\|u_1, \omega\|_{3/2,Q} + \|u_2, \omega\|_{3/2,Q}).$$

We show that $w = u - v \in H^2_Q(\Omega_{T_1,T_2})$. The function w satisfies the equation

$$\begin{cases} \partial_t^2 w + \Delta w = g_1 \equiv g - (\partial_t^2 v + \Delta v) \in L^2(\Omega_{T_1,T_2}), \\ w|_{t=T_1} = w|_{t=T_2} = 0, \quad \partial_n w|_{\partial\omega} = 0. \end{cases}$$

Take a cut-off function $\phi \in C_0^\infty(\mathbb{R})$ with $\phi(t) = 1$ for $t \in (T_1, T_2)$ and $\phi(t) = 0$ for $t \notin (T_1 - \varepsilon, T_2 + \varepsilon)$, where $0 < \varepsilon < T_2 - T_1$. It is readily seen that the function

$$W(t) = \phi(t)\,\tilde{w}(t) \equiv \phi(t) \begin{cases} -w(2T_1 - t) & \text{for } t \in (T_1 - 1, T_1), \\ w(t) & \text{for } t \in (T_1, T_2), \\ -w(2T_2 - t) & \text{for } t \in (T_2, T_2 + 1) \end{cases}$$

belongs to $\mathfrak{G}_0$. Indeed,

$$\partial_t^2 W + \Delta W = \phi(t)\,\tilde{g}_1(t) + 2\phi'(t)\,\partial_t \tilde{w}(t) + \phi''(t)\,\tilde{w}(t) \in L^2(\Omega_{T_1-1,T_2+1}),$$

where $\tilde{g}_1$ is defined similarly to $\tilde{w}$. In addition, W satisfies the appropriate boundary conditions. Hence, according to Definition A.1, $w \in H^2_Q(\Omega_{T_1,T_2})$. ■

Theorem A.4: *The space $H^2_Q(\Omega_{T_1,T_2}) \cap L^\infty(\Omega_{T_1,T_2})$ is dense in $H^2_Q(\Omega_{T_1,T_2})$.*

Proof: It suffices to prove that $\mathfrak{G}_0 \cap L^\infty(\Omega_{T_1-1,T_2+1})$ is dense in $\mathfrak{G}_0$. Take a function $u \in \mathfrak{G}_0$ and a function $g \in L^2(\Omega_{T_1-1,T_2+1})$ which satisfy (A.1). Let $\{g_m\} \subset$

$\subset L^\infty(\Omega_{T_1-1,T_2+1})$ be a sequence such that

$$g_m \to g \quad \text{in } L^2(\Omega_{T_1-1,T_2+1}) \text{ as } m \to \infty . \tag{A.5}$$

Let $u_m \in \mathcal{G}_0$ be the variational solution to (A.1) with right-hand side g_m. It follows from (A.5) and the definition of the space $\mathcal{G}_0$ that

$$u_m \to u \quad \text{in } \mathcal{G}_0 \text{ as } m \to \infty .$$

Hence Theorem A.4 will be proved when we will have been shown that the functions $u_m \in L^\infty(\Omega_{T_1-1,T_2+1})$.

We use the following maximum principle.

LEMMA A.5: *Let $u_i \in H^1(\Omega_{T_1-1,T_2+1})$ for $i = 1, 2$ be variational solutions to* (A.1) *with right-hand sides $g_i \in H^1(\Omega_{T_1-1,T_2+1})^*$. Assume, in addition, that*

$$\langle g_1, \Phi\rangle \geq \langle g_2, \Phi\rangle \quad \text{for all } \Phi \in H^1(\Omega_{T_1-1,T_2+1}),\ \Phi \geq 0 . \tag{A.6}$$

Then

$$u_1(t,x) \geq u_2(t,x) \quad \text{for } (t,x) \in \Omega_{T_1-1,T_2+1} \text{ a.e.} \tag{A.7}$$

PROOF: Consider the function $u = u_1 - u_2$. Then

$$\langle \partial_t u, \partial_t \Phi\rangle + \langle \nabla u, \nabla \Phi\rangle + \langle u, \Phi\rangle \geq 0 \quad \text{for all } \Phi \in H^1(\Omega_{T_1-1,T_2+1}),\ \Phi \geq 0 . \tag{A.8}$$

Now introduce $u_+(t,x) = \max\{u, 0\}$, $u_-(t,x) = \max\{-u, 0\}$. Then $u = u_+ - u_-$. It is known that $u_+, u_- \in H^1(\Omega_{T_1-1,T_2+1})$ and

$$\langle u_+, u_-\rangle = 0 , \quad \langle \partial_t u_+, \partial_t u_-\rangle + \langle \nabla u_+, \nabla u_-\rangle = 0$$

(see [21]). Upon replacing Φ in (A.8) with u_-, we obtain

$$-\langle \partial_t u_-, \partial_t u_-\rangle - \langle \nabla u_-, \nabla u_-\rangle - \langle u_-, u_-\rangle \geq 0 .$$

This implies that $\langle u_-, u_-\rangle = 0$ or $u_-(t,x) = 0$ for $(t,x) \in \Omega_{T_1-1,T_2+1}$ a.e. The lemma is proved. ∎

COROLLARY A.6: *Let $u \in H^1(\Omega_{T_1-1,T_2+1})$ be a variational solution to* (A.1) *with right-hand side $g \in L^\infty(\Omega_{T_1-1,T_2+1})$. Then $u \in L^\infty(\Omega_{T_1-1,T_2+1})$.*

This completes the proof of Theorem A.4.

THEOREM A.7: *The embedding*

$$H_0^2(\Omega_{T_1,T_2}) \subset L^\infty(\Omega_{T_1,T_2}) \tag{A.9}$$

holds for

$$q \leqslant q_0 = 2\,\frac{n+1}{n-3}. \tag{A.10}$$

If $q < q_0$*, then this embedding is compact. Furthermore, if* $u \in H^2_Q(\Omega_{T_1, T_2})$*, then* $u|u|^{(q_0-2)/2} \in H^1(\Omega_{T_1, T_2})$ *and*

$$\|u|u|^{(q_0-2)/2}, \Omega_{T_1, T_2}\|_{1,2} \leqslant C\|u, \Omega_{T_1, T_2}\|^{q_0/2}_{2, Q}. \tag{A.11}$$

Proof. Let $u \in H^2_Q(\Omega_{T_1, T_2})$. By definition, there is a function $\tilde{u} \in H^1(\Omega_{T_1-1, T_2+1})$, $\tilde{u}|_{\Omega_{T_1, T_2}} = u$, such that

$$\langle \partial_t \tilde{u}, \partial_t \Phi\rangle + \langle \nabla \tilde{u}, \nabla \Phi\rangle + \langle \tilde{u}, \Phi\rangle = \langle \tilde{g}, \Phi\rangle \quad \text{for all } \Phi \in H^1(\Omega_{T_1-1, T_2+1}) \tag{A.12}$$

for a certain right-hand side $\tilde{g} \in L^2(\Omega_{T_1-1, T_2+1})$ and

$$\|\tilde{g}, \Omega_{T_1-1, T_2+1}\|_{0,2} \leqslant C\|u, \Omega_{T_1, T_2}\|_{2, Q}.$$

Approximate $\tilde{g} \in L_2(\Omega_{T_1-1, T_2+1})$ by a sequence $\{\tilde{g}_m\} \subset L^\infty(\Omega_{T_1-1, T_2+1})$, $\tilde{g}_m \to \tilde{g}$ in $L^2(\Omega_{T_1-1, T_2+1})$ as $m \to \infty$. Let $\tilde{u}_m$ be the solution to the variational problem (A.12) with the right-hand side $\tilde{g}$ replaced with $\tilde{g}_m$. Then $\tilde{u}_m \in L^\infty(\Omega_{T_1-1, T_2+1})$ by Corollary A.6. Hence the function $\tilde{u}_m|\tilde{u}_m|^{l-2}$ belongs to $H^1(\Omega_{T_1-1, T_2+1})$, where $l_0 - 2 = 4/(n-3) = (q_0-2)/2$. Replacing $\tilde{u}$ by $\tilde{u}_m$ and Φ by $\tilde{u}_m|\tilde{u}_m|^{l_0-2}$ in (A.12) and arguing as in the proof of estimate (1.19) we obtain the inequality

$$\|\tilde{u}_m|\tilde{u}_m|^{(l_0-2)/2}, \Omega_{T_1-1, T_2+1}\|^2_{1,2} \leqslant C(1 + |\langle \tilde{g}_m, \tilde{u}_m|\tilde{u}_m|^{l_0-2}\rangle|). \tag{A.13}$$

By Sobolev's embedding theorem, $H^1(\Omega_{T_1-1, T_2+1}) \subset L^{2n/(n-2)}(\Omega_{T_1-1, T_2+1})$. It follows that

$$\|\tilde{u}_m, \Omega_{T_1-1, T_2+1}\|^{l_0}_{0, q_0} = \|\tilde{u}_m|\tilde{u}_m|^{(l_0-2)/2}, \Omega_{T_1-1, T_2+1}\|^2_{0, 2n/(n-2)} \leqslant$$
$$\leqslant C\|\tilde{u}_m|\tilde{u}_m|^{(l_0-2)/2}, \Omega_{T_1-1, T_2+1}\|^2_{1,2}.$$

By Hölder's inequality, we further obtain

$$|\langle \tilde{g}_m, \tilde{u}_m|\tilde{u}_m|^{l_0-2}\rangle| \leqslant \|g, \Omega_{T_1-1, T_2+1}\|_{0,2}\|\tilde{u}_m, \Omega_{T_1-1, T_2+1}\|^{l_0-1}_{0, q_0} \leqslant$$
$$\leqslant \mu\|\tilde{u}_m, \Omega_{T_1-1, T_2+1}\|^{l_0}_{0, q_0} + C_\mu\|g_m, \Omega_{T_1-1, T_2+1}\|^{l_0}_{0,2},$$

where $\mu > 0$ is arbitrary. Applying these estimates to (A.13) for a sufficiently small $\mu > 0$, we get

$$\|\tilde{u}_m, \Omega_{T_1-1, T_2+1}\|^{l_0}_{0, q_0} + \|\tilde{u}_m|\tilde{u}_m|^{(l_0-2)/2}, \Omega_{T_1-1, T_2+1}\|^2_{1,2} \leqslant \tag{A.14}$$
$$\leqslant C\|g_m, \Omega_{T_1-1, T_2+1}\|^{l_0}_{0,2}.$$

We know that $\tilde{g}_m \to \tilde{g}$ in $L^2(\Omega_{T_1-1,T_2+1})$, hence the sequence $\{\tilde{u}_m\}$ is bounded in the space $L^{q_0}(\Omega_{T_1-1,T_2+1})$. Without loss of generality, we may assume that $\tilde{u}_m \to \tilde{u}$ weakly in $L^{q_0}(\Omega_{T_1-1,T_2+1})$. Thus $\tilde{u} \in L^{q_0}(\Omega_{T_1-1,T_2+1})$ and

$$\|u, \Omega_{T_1,T_2}\|_{0,q_0} \leq \|\tilde{u}, \Omega_{T_1-1,T_2+1}\|_{0,q_0} \leq C\|\tilde{g}, \Omega_{T_1-1,T_2+1}\|_{0,2} \leq C_1\|u, \Omega_{T_1,T_2}\|_{2,Q}.$$

The statement $u|u|^{l_0-2} \in H^1(\Omega_{T_1,T_2})$ is shown analogously.

We finally prove the compactness of embedding (A.9) for $q < q_0$. By the interpolation inequality between H^1 and L^{q_0},

$$H_Q^2(\Omega_{T_1,T_2}) \subset H^{\varepsilon,q}(\Omega_{T_1,T_2})$$

for some $\varepsilon > 0$. The assertion follows, since the embedding $H^{\varepsilon,q} \subset L^q$ is compact.

The proof of Theorem A.7 is complete. ■

COROLLARY A.8: *The embedding*

$$H_Q^2(\Omega_{T_1,T_2}) \subset C([T_1, T_2], L^{p_0}(\omega)) \tag{A.15}$$

holds. Here $p_0 = 2l_0 = 2 + 4/(n-3)$ *is the maximum value of* p *in* (0.2).

In fact, the estimate (A.11) together with Sobolev's embedding theorem imply that $u|u|^{(l_0-2)/2} \in C([T_1, T_2], L^2(\omega))$ if $u \in H_Q^2(\Omega_{T_1,T_2})$. Moreover, we infer from the embedding $H_Q^2 \subset H^1$ that $u \in C([T_1, T_2], L^2(\omega))$. Now, employing standard arguments, we finally obtain that $u \in C([T_1, T_2], L^{p_0}(\omega))$.

THEOREM A.9: *Let* $u \in H_Q^2(\Omega_{T_1,T_2})$. *Then* $\partial_t^2 u \in L^2(\Omega_{T_1,T_2})$, $\partial_t \nabla u \in L^2(\Omega_{T_1,T_2})$. *Moreover, the following estimate holds:*

$$\|\partial_t^2 u, \Omega_{T_1,T_2}\|_{0,2} + \|\partial_t \nabla u, \Omega_{T_1,T_2}\|_{0,2} \leq C\|u, \Omega_{T_1,T_2}\|_{2,Q}. \tag{A.16}$$

PROOF: By definition, there exists a function $\tilde{u} \in \mathcal{G}_0$ that satisfies $\tilde{u}|_{\Omega_{T_1,T_2}} = u$ and

$$\begin{cases} \partial_t^2 \tilde{u} + \Delta \tilde{u} - \tilde{u} = g(x), & \partial_n \tilde{u}|_{\partial\omega} = 0, \\ \tilde{u}|_{t=T_1-1} = 0, & \tilde{u}|_{t=T_2+1} = 0 \end{cases} \tag{A.17}$$

for some function $g \in L^2(\Omega_{T_1-1,T_2+1})$, $\|g, \Omega_{T_1-1,T_2+1}\|_{0,2} \leq C\|u, \Omega_{T_1,T_2}\|_{2,Q}$. Below we only give the formal arguments for deriving estimate (A.16). A rigorous proof can be supplied by exploiting the Galerkin approximation method.

Multiplying Eq. (A.17) by $\partial_t^2 \tilde{u}$ and integrating over Ω_{T_1-1,T_2+1}, we obtain after an integration by parts

$$\langle |\partial_t^2 \tilde{u}|^2, 1\rangle + \langle |\partial_t \nabla \tilde{u}|^2, 1\rangle + \langle |\partial_t \tilde{u}|^2, 1\rangle = \langle g, \partial_t^2 \tilde{u}\rangle. \tag{A.18}$$

Applying Hölder's inequality

$$(g, \partial_t^2 \bar{u}) \leq \frac{1}{2}(|g|^2, 1) + \frac{1}{2}(|\partial_t^2 \bar{u}|^2, 1)$$

to the right-hand side in (A.18), we then find (A.16).

The proof is finished. ■

From Theorem A.9 we conclude:

COROLLARY A.10: *We have*

(A.19) $$H_Q^2(\Omega_{T_1, T_2}) \subset H^1((T_1, T_2), H^1(\omega)) \cap H^2((T_1, T_2), L^2(\omega)).$$

In particular, the functions $t \mapsto \|u(t)\|_{1,2}$ *and* $t \mapsto \|\partial_t u(t)\|_{0,2}$ *are defined and continuous for every* $u \in H_Q^2(\Omega_{T_1, T_2})$.

COROLLARY A.11: *Furthermore,*

(A.20) $$H_Q^2(\Omega_{T_1, T_2}) = H^2((T_1, T_2), L^2(\omega)) \cap L^2((T_1, T_2), H_Q^2(\omega)).$$

REMARK A.12: For smooth domains $\omega \subset \mathbb{R}^n$, all previous results of this appendix are consequences of L^2-elliptic regularity for the Laplace operator (see, e.g., [19]) especially of the fact

(A.21) $$H_Q^2(\Omega_{T_1, T_2}) = \{u \in H^2(\Omega_{T_1, T_2}) : \partial_\nu u|_{\partial\omega} = 0\}$$

and Sobolev's embedding theorem. For polyhedral domains ω, however, (A.21) is in general not fulfilled (see § 5).

The following deep result may be found in [9].

THEOREM A.13: *Let* $\omega \subset \mathbb{R}^n$ *be a bounded polyhedral domain. Then there exists an* ε *satisfying* $0 < \varepsilon \leq 1/2$ *such that*

(A.22) $$H_Q^2(\omega) \subsetneq H^{3/2+\varepsilon}(\omega).$$

COROLLARY A.14: *Let* $\omega \subset \mathbb{R}^n$ *be a bounded polyhedral domain. Then*

(A.23) $$H_Q^2(\Omega_{T_1}, T_2) \subsetneq H^{3/2+\varepsilon}(\Omega_{T_1, T_2}),$$

where ε *is the same as in* Theorem A.13.

(A.23) is actually a consequence of (A.20), (A.22).

COROLLARY A.15: *Let* $u \in H^2_Q(\Omega_{T_1, T_2})$. *Then*

$$\partial_n u|_{\partial\omega} \in H^s((T_1, T_2) \times \partial\omega).$$

This follows from (A.23) and Sobolev's embedding theorem. In particular, using Green's formula (see [12]), we obtain that $\partial_n u|_{\partial\omega} = 0$ for $u \in H^2_Q(\Omega_{T_1, T_2})$. Thus solutions u to the problem (0.1) which belong to Θ_0^+ satisfy the homogeneous Neumann boundary condition in the proper sense.

Acknowledgement. The second and the fourth author were partially supported by the Russian Fond of Fundamental Research under grand 96-01-00354 and US CRDF RM-1-186.

REFERENCES

[1] A. V. BABIN, *Attractor of the generalized semigroup generated by an elliptic equation in a cylindrical domain*, Russian Acad. Sci. Izv. Math., 44 (1995), 207-223.

[2] A. V. BABIN - M. I. VISHIK, *Attractors of Evolution Equations*, Stud. Math. Appl., vol. 25, North Holland, Amsterdam, 1992.

[3] P. BRUNOVSKY - X. MORA - P. POLACIK - J. SOLÀ-MORALES, *Asymptotic behavior of solutions of semilinear elliptic equations on an unbounded strip*, Acta Math. Univ. Comenian., LX (1995), 163-183.

[4] Á. CALSINA - X. MORA - J. SOLÀ-MORALES, *The dynamical approach to elliptic problems in cylindrical domains, and a study of their parabolic singular limit*, J. Differential Equations, 102 (1993), 244-304.

[5] V. V. CHEPYZHOV - M. I. VISHIK, *Attractors of non-autonomous dynamical systems and their dimension*, J. Math. Pures Appl., 73 (1994), 279-333.

[6] V. V. CHEPYZHOV - M. I. VISHIK, *Non-autonomous evolutionary equations with translation-compact symbols and their attractors*, C. R. Acad. Sci. Paris Sér. I Math., 321 (1995), 153-158.

[7] V. V. CHEPYZHOV - M. I. VISHIK, *Trajectory attractors for evolution equations*, C. R. Acad. Sci. Paris Sér. I Math., 321 (1995), 1309-1314.

[8] V. V. CHEPYZHOV - M. I. VISHIK, *Evolution equations and their trajectory attractors*, J. Math. Pures Appl., 76 (1997), 913-964.

[9] M. DAUGE, *Elliptic Boundary Value Problems on Corner Domains*, Lecture Notes in Math., vol. 1341, Springer, Berlin, 1988.

[10] W. HUTSON - J. S. PYM, *Applications of Functional Analysis and Operator Theory*, Math. Sci. Engrg., vol. 146, Academic Press, London, 1980.

[11] M. A. KRASNOSELSKI, *Topological Methods in the Theory of Nonlinear Integral Equations*, Gostechizdat, Moscow, 1956 (Russian).

[12] J.-L. LIONS - E. MAGENES, *Problèmes aux limites non homogènes et applications*, Vol. I Dunod, Paris, 1968.

[13] S. A. NAZAROV - B. A. PLAMENEVSKY, *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, de Gruyter Exp. Math., vol. 13, de Gruyter, Berlin, 1994.

[14] E. Schrohe - B.-W. Schulze, *A symbol algebra for pseudodifferential boundary value problems on manifolds with edges*, Differential Equations, Asymptotic Analysis, and Mathematical Physics (M. Demuth and B.-W. Schulze, eds.), Math. Res., vol. 100, Akademie Verlag, Berlin, pp. 292-324.

[15] B.-W. Schulze, *Pseudo-differential operators on manifolds with edges*, Symposium «Partial Differential Equations» Holzhau 1988 (B.-W. Schulze and H. Triebel, eds.), Teubner-Texte Math., vol. 112, Teubner, Leipzig, 1989, pp. 259-288.

[16] B.-W. Schulze, *Boundary value problems and edge pseudo-differential operators*, Microlocal Analysis and Spectral Theory (L. Rodino, ed.), NATO Adv. Sci. Inst. Ser. C. Math. Phys. Sci., vol. 490, Kluwer Acad. Publ., Dordrecht, 1997, pp. 165-226.

[17] B.-W. Schulze, *Boundary Value Problems and Singular Pseudo-differential Operators*, Wiley Ser. Pure Appl. Math., J. Wiley, Chichester, 1998.

[18] S. L. Sobolev, *Some Applications of Functional Analysis to Mathematical Physics*, Nauka, Moscow, 1988.

[19] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[20] M. I. Vishik - S. V. Zelik, *The trajectory attractor for a nonlinear elliptic system in a cylindrical domain*, Russian Acad. Sci. Sb. Math., 187 (1996), 1755-1789.

[21] A. I. Volpert - S. I. Khudyaev, *Analysis in Classes of Discontinuous Functions and Equations of Mathematical Physics*, Nijhoff, Dordrecht, 1985.

[22] S. V. Zelik, *The trajectory attractor for a nonlinear elliptic system in an unbounded domain*, Math. Notes, 63 (1998), 135-137.

Direttore responsabile: Prof. A. Ballio - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5/L - Bologna