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# Representation Results in the Context of Wigner Analysis

ABSTRACT. — In the context of the Wigner analysis introduced by [B-S] we study some representation results for free-Brownian martingales which have analogues in the classical Wiener analysis.

## Risultati di rappresentazione nell'ambito dell'analisi di Wigner

SOMMARIO. — Nell'ambito dell'analisi di Wigner introdotta in [B-S] si studiano alcuni risultati di rappresentazione di martingale del moto browniano libero, analoghi a quelli che valgono nel contesto dell'analisi di Wiener.

### 1. - INTRODUCTION

The analytical theory known as *analysis on Wiener space* is based on the fact that the classical (Boson) Fock space associated with an infinite dimensional Hilbert space can be interpreted as the space of square integrable random variables with respect to the Wiener measure. In a recent paper (see [B-S]) it has been shown that many results from the analysis on Wiener space have analogues when the Boson (i.e. symmetric) Fock space is replaced by the free (i.e. unsymmetrized) Fock space, and the free Brownian motion plays the role of the classical Brownian motion. Stochastic calculus with respect to free noise has been developed in [K-S], [Sp] and [F] and others, inspired by the Hudson and Parthasarathy quantum stochastic calculus [H-P]. In this paper we follow a different approach for a stochastic integration theory with respect to the free Brownian motion proposed in [B-S]. The particular feature of this calculus is that the stochastic integrals are defined with respect to the free Brownian motion, and not

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with respect to creation and annihilation processes. Because the semicircular distribution, or *Wigner distribution*, plays the role of the gaussian distribution in the classical theory, we will refer to this approach as *Wigner analysis*.

The key point for the development of the Wigner analysis is the fact that the free Fock space modelled on $L^2(\mathbb{R}_+)$ can be interpreted as giving chaos decomposition of the $L^2$-space of a free Brownian motion. In this paper we study the chaotic decomposition of the $L^2$-space of a free Brownian motion through free multiple integrals. We define the iterated stochastic integral in the Wigner space and we examine the relationship with the multiple integral. By using the chaotic decomposition it is possible to define a free gradient operator and its adjoint, which plays the role of a free Skorohod integral [B-S]. In this context we prove some representation results: we start from the free Bismut-Clark-Ocone formula (see also [B-S]) and we obtain a second order free representation formula, which we will call free Taylor formula (with a remainder), in analogy with the classical case. This formula suggests a method to obtain a Taylor series expansion; in the context of Wiener analysis this Taylor series expansion allows to identify the symmetric kernels in the chaotic decomposition of a Wiener functional through the mean of iterated Malliavin derivatives of the functional itself [St]. This is due to the fact that in the symmetric Fock space the uniqueness of the decomposition is realized when the deterministic kernels are taken symmetric. We prove that this identification is not true in the free Fock decomposition because the iterated derivatives give symmetric functions which allow to identify only the symmetric part of the integral kernels.

## 2. - Notations and preliminaries

The free Fock space associated with the Hilbert space $L^2(\mathbb{R}_+)$ is denoted by $\mathcal{F}(L^2(\mathbb{R}_+))$ and defined by

$$\mathcal{F}(L^2(\mathbb{R}_+)) = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+)^{\otimes n}.$$

Remark that $\bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+)^{\otimes n} \equiv \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+^n)$. Let

$$l_t = l(I_{[0,t]}), \qquad l_t^* = l^*(I_{[0,t]})$$

be the left annihilation and creation operators defined on the full Fock space as follows: for any $h \in L^2(\mathbb{R}_+)$ and $h_1, \ldots, h_n \in L^2(\mathbb{R}_+)$

$$l(h)\, h_1 \otimes \ldots \otimes h_n = \langle h_1, h \rangle\, h_2 \otimes \ldots \otimes h_n,$$

$$l^*(h)\, h_1 \otimes \ldots \otimes h_n = h \otimes h_1 \otimes \ldots \otimes h_n.$$

For any $h \in L^2(\mathbb{R}_+)$, $l(h)$ and $l(h)^*$ are bounded operators and adjoint of each other on the Fock space. Denote $X_t = l_t + l_t^*$. Let $\mathcal{A}$ be the von Neumann algebra of operators on $\mathcal{F}(L^2(\mathbb{R}_+))$ generated by $\{X_t, t \in \mathbb{R}_+\}$. Then $(X_t)_{t \in \mathbb{R}_+}$ is a free Brownian motion with respect to the filtration $(\mathcal{A}_t)$, where $\mathcal{A}_t$ is the von Neumann algebra of operators generated by $\{X_s : s \leqslant t\}$. This means that $X_t \in \mathcal{A}_t$ for all $t \geqslant 0$ and for all $s$, $t$ with $s \leqslant t$, $X_t - X_s$ is free with $\mathcal{A}_s$ and has semicircular distribution with mean zero and variance $t - s$.

Togheter with $\mathcal{A}$ consider also the opposite algebra $\mathcal{A}^{op}$ (same linear structure and reversed order of multiplication), with the trace $\tau^{op}$ $(\tau = \tau^{op})$. On the spaces $\mathcal{A}$ and $\mathcal{A} \otimes \mathcal{A}$ is given the multiplication on the right and on the left, denote by $\sharp$ the following actions:

$$(a \otimes b) \sharp u = a\, u\, b\,, \qquad (a \otimes b) \sharp (u \otimes v) = au \otimes vb\,.$$

Define the non commutative $L^p$-spaces associated to the free Brownian motion $(X_t)_{t \in \mathbb{R}_+}$: $L^p(\mathcal{A})$, $1 \leqslant p < \infty$, as the completion of $\mathcal{A}$ with respect to the norm

$$\|Y\|_{L^p(\mathcal{A})} = \tau[\,|Y|^p\,]^{1/p}.$$

We introduce now the notion of free stochastic integral as in [B-S] following the classical procedure: first define the integral on piecewise constant processes, then state an isometry property and extend the definition to a class of square integrable processes. The peculiarity of the non commutative case consists in the fact that the integrator does not commute with the process to be integrated. Therefore we have the choice of multiplying the integrand on the left or on the right. This observation leads to consider a more general type of integrand, which we introduce in the following definition.

Definition 2.1: *A simple biprocess is a piecewise constant map $t \mapsto U_t$ from $\mathbb{R}_+$ into the algebraic tensor product $\mathcal{A} \otimes \mathcal{A}^{op}$, such that $U_t = 0$ for $t$ large. In other words there exists finitely many piecewise constant maps $t \mapsto A_t^j$, $t \mapsto B_t^j$, $j = 1, \ldots, n$ with values in $\mathcal{A}$ such that $A_t^j = B_t^j = 0$ for $t$ large and, for all $t \geqslant 0$*

$$U_t = \sum_{j=1}^{n} A_t^j \otimes B_t^j. \tag{1}$$

The simple biprocesses constitute a vector space which we shall endow with the norms

$$\|U\|_{\mathcal{B}_p} = \left( \int_{\mathbb{R}_+} \|U_s\|^2_{L^p(\tau \otimes \tau^{op})}\, ds \right)^{1/2} \qquad \text{for } 1 \leqslant p \leqslant \infty\,.$$

The completion of the space of simple biprocesses for these norms will be denoted by $\mathcal{B}_p$. In particular $\mathcal{B}_2$ is the Hilbert space associated to the inner product

$$\langle U, V\rangle = \int_{\mathbb{R}_+} \langle U_s, V_s\rangle \, ds\,,$$

where $\langle U_s, V_s\rangle$ is the inner product in $L^2(\mathcal{A}, \tau)\otimes L^2(\mathcal{A}, \tau)$.

Definition 2.2: *A simple biprocess U is adapted if for any $t \geqslant 0$ we have $U_t \in \mathcal{A}_t \otimes \mathcal{A}_t$.*

We remark that if a simple biprocess is adapted, then we can choose a decomposition as in (1) where $A_t^j$ and $B_t^j$ belong to $\mathcal{A}_t$ for any $t \geqslant 0$.

Definition 2.3: *Let U be a simple adapted biprocess with a decomposition as in* (1). *Then the stochastic integral of U is the operator*

$$(2) \qquad \int_{\mathbb{R}_+} U_t \sharp \, dX_t \equiv \sum_{k=0}^{m-1} U_{t_k} \sharp (X_{t_{k+1}} - X_{t_k}) = \sum_{j=1}^{n} \sum_{k=0}^{m-1} A_{t_k}^j (X_{t_{k+1}} - X_{t_k})\, B_{t_k}^j.$$

We will denote by $\mathcal{B}_2^a$ the space of adapted square integrable biprocesses. Then the stochastic integral is extended to the space $\mathcal{B}_2^a$ using the following isometry property for the stochastic integral.

Proposition 2.4: *For all adapted simple biprocesses U and V, we have:*

$$\tau\left[\int_{\mathbb{R}_+} U_t \sharp \, dX_t \left(\int_{\mathbb{R}_+} V_t \sharp \, dX_t\right)^*\right] = \langle U, V\rangle,$$

where $\left(\int_{\mathbb{R}_+} V_t \sharp \, dX_t\right)^* = \int_{\mathbb{R}_+} V_t^* \sharp \, dX_t$.

## 3. - Multiple stochastic integral and chaotic decomposition

We begin this paragraph by recalling the chaotic decomposition in the context of Wiener analysis. We emphasize the role of the structure of the symmetric Fock space, in order to generalize the construction to the free case using the structure of the free Fock space. It results that as for the symmetric Fock space it is a property of the free Fock space the realization of the chaoses by multiple integrals. For more details about the classical case see [N].

Denote by $X$ the Wiener space and let $(x_t)_{t\in\mathbb{R}_+}$ be the Wiener process on R. Denote by $H_n$ the $n$-th Hermite poynomial. It is known that the space $L^2(X)$ can be decomposed into the infinite orthogonal sum of the subspaces $\mathcal{C}_n$, where $\mathcal{C}_n$ are the closed linear subspaces of $L^2(X)$ generated by the random variables $H_n\left(\int_0^\infty h(t)\,dx_t\right)$, where $h \in L^2(\mathbb{R}_+)$ with $L^2$-norm equal to 1. $\mathcal{C}_n$ is called the *chaos* of order $n$. Consider a function $f(t_1, \ldots, t_n)$ which is the indicator function of a rectangle in $\mathbb{R}^n$, e.g. $A = [u_1, v_1] \times \ldots \times [u_n, v_n]$. Suppose that the rectangle $A$ does not intersect any diagonal subspace $\{t_i = t_j, i \neq j\}$. Then the $n$-multiple Itô integral is defined by

$$I^n(f) = (x_{v_1} - x_{u_1}) \ldots (x_{v_n} - x_{u_n}).$$

Clearly this definition extends to simple functions $f(t_1, \ldots, t_n)$ vanishing on the rectangles which intersect the diagonals. Note that $I^n(f) = I^n(\tilde{f})$ where $\tilde{f}$ is the symmetrized function

$$\tilde{f}(t_1, \ldots, t_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma(n)} f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}).$$

Moreover the following property holds:

$$E[I^n(f)^2] = E[I^n(\tilde{f})^2] = n!\|\tilde{f}\|^2_{L^2(\mathbb{R}^n_+)} \leq n!\|f\|^2_{L^2(\mathbb{R}^n_+)}.$$

The space of simple functions as above is dense in $L^2(\mathbb{R}^n_+)$. Therefore by the above inequality the operator $I^n$ can be extended to a linear and continuous operator from $L^2(\mathbb{R}^n_+)$ to $L^2(X)$. Note that the isometry holds when $f$ is a symmetric function vanishing on the rectangles which intersect the diagonals. Denote by $L^2_s(\mathbb{R}^n_+)$ the closed subspace of symmetric functions of $L^2(\mathbb{R}^n_+)$. The map

$$f \mapsto \frac{1}{\sqrt{n!}} I^n(f) = \frac{1}{\sqrt{n!}} \int_{\mathbb{R}^n_+} f(t_1, \ldots, t_n)\, dx_{t_1} \ldots dx_{t_n}$$

defines a surjective isomorphism between $L^2_s(\mathbb{R}^n_+)$ and $\mathcal{C}_n$. Therefore for any square integrable random variable $F$ on the Wiener space the chaotic representation is uniquely determined if the kernel-functions $f$ are taken symmetric.

The following result shows that the multiple integrals can be written in terms of iterated Itô integrals.

Theorem 3.1: *Given $f$ in $L^2_s(\mathbb{R}^n_+)$ and given $0 < t_1 < t_2 < \ldots < t_n$ consider the sequence of functions defined on the Wiener space by the following recursion formula:*

$$f_1(x, t_2, \ldots, t_n) = \int_0^{t_2} f(t_1, t_2, \ldots, t_n)\, dx_{t_1},$$

$$f_2(x, t_3, \ldots, t_n) = \int_0^{t_3} f_1(x, t_2, \ldots, t_n)\, dx_{t_2},$$

$$f_n(x) = \int_0^{+\infty} f_{n-1}(x, t_n)\, dx_{t_n}.$$

*Then all the stochastic integrals are well defined and*

(3) $$\frac{1}{n!}\int_{\mathbb{R}^n_+} f(t_1, \ldots, t_n)\, dx_{t_1} \ldots dx_{t_n} = f_n.$$

This result leads to an equivalent formula as for the Lebesgue integral of symmetric function $f$

$$\frac{1}{n!}\int_{\mathbb{R}^n_+} f dt^{\otimes n} = \int_{\Delta_n} f dt^{\otimes n};$$

where $\Delta_n$ is the cone $\{(t_1, \ldots, t_n) \in \mathbb{R}^n_+ : t_1 < t_2 < \ldots < t_n\}$.

We describe now the chaotic decomposition theory for the Wigner space following the same procedure as for the Wiener space and the symmetric Fock space (see [B-S]): for $f \in L^2(\mathbb{R}^n_+)$, we will define the multiple stochastic integral

(4) $$\int f(t_1, \ldots, t_n)\, dX_{t_1} \ldots dX_{t_n};$$

this will give an explicit description of the isometry between $\mathcal{F}(L^2(\mathbb{R}_+))$ and $L^2(\mathfrak{A})$, which is referred as *chaotic decomposition.*

First it is defined the multiple integral for indicator functions of rectangles $A = [u_1, v_1] \times \ldots \times [u_n, v_n]$ which does not intersect the diagonals. Then the multiple integral for $f = I_A$ is defined as

(5) $$I^n(f) = (X_{v_1} - X_{u_1}) \ldots (X_{v_n} - X_{u_n})$$

and it is extended by linearity to simple functions vanishing on the rectangles which intersect the diagonals. Note that the increments on the right hand side of (5) do not commute. Then $I^n(f) \neq I^n(\tilde{f})$ where $\tilde{f}$ is the symmetrized function.

It is easily seen that if $f, g$ are simple functions in $\mathbb{R}^n_+$, then

$$\langle I^n(f), I^n(g)\rangle_{L^2(\mathfrak{A})} = \langle f, g\rangle_{L^2(\mathbb{R}^n_+)}.$$

Since the simple functions as above are dense in $L^2(\mathbb{R}^n_+)$, we can extend the definition of $I^n(f)$ to any $f \in L^2(\mathbb{R}^n_+)$.

Let $f = \bigoplus_{n=0}^{\infty} f^{(n)} \in \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^n_+)$. Then

(6) $$\langle I^n(f^{(n)}), I^m(g^{(m)})\rangle_{L^2(\mathfrak{A})} = 0 \quad \text{if } n \neq m.$$

This implies

$$\left\| \sum_{n=0}^{\infty} I^n(f^{(n)}) \right\|_{L^2(\mathcal{Cl})}^2 = \sum_{n=0}^{\infty} \| f^{(n)} \|_{L^2(\mathbb{R}_+^n)}^2 = \| f \|_{\mathcal{F}(L^2(\mathbb{R}_+))}^2 .$$

Therefore with any $f = \bigoplus_{n=0}^{\infty} f^{(n)} \in \mathcal{F}(L^2(\mathbb{R}_+))$ it is associated the series of multiple integrals

$$I(f) = \sum_{n=0}^{\infty} I^n(f^{(n)}) = \sum_{n=0}^{\infty} \int f^{(n)}(t_1, \ldots, t_n)\, dX_{t_1} \ldots dX_{t_n}$$

and the map

$$I \colon \mathcal{F}(L^2(\mathbb{R}_+)) \to L^2(\mathcal{Cl})$$

which associates $f$ with $I(f)$ is an isomorphism. This isomorphism yields the chaotic decomposition of the space $L^2(\mathcal{Cl})$: any element in $L^2(\mathcal{Cl})$ can be represented in a unique way as a series of multiple integrals

$$I(f) = \sum_{n=0}^{\infty} \int f^{(n)}(t_1, \ldots, t_n)\, dX_{t_1} \ldots dX_{t_n}, \qquad f = \bigoplus_n f^{(n)} \in \mathcal{F}(L^2(\mathbb{R}_+)) .$$

Remark 3.2: It is not possible to state the analogous of (3) in the context of the Wigner space. Consider a function $f = I_A$ where $A$ is a rectangle in $\mathbb{R}_+^n$ which does not intersect the diagonals, for instance $A = [u_1, v_1] \times \ldots \times [u_n, v_n]$ with $u_1 < v_1 < \ldots < u_n < v_n$. We can compute the iterated integral of $f$ in the following way

$$\int_0^{\infty} \cdots \int_0^{t_2} I_{[u_1, v_1]}(t_1) \ldots I_{[u_n, v_n]}(t_n) \sharp\, dX_{t_1} \sharp \ldots \sharp\, dX_{t_n} .$$

Since $L^2(\mathbb{R}_+^n) \cong L^2(\mathbb{R}_+)^{\otimes n}$, we can consider the indicator function of $A$ as the biprocess $I_{[u_1, v_1]}(t_1) \otimes [I_{[u_2, v_2]}(t_2) \ldots I_{[u_n, v_n]}(t_n)]$. Then the free stochastic integral is well defined and we have:

$$\int_0^{t_2} I_{[u_1, v_1]}(t_1) \otimes [I_{[u_2, v_2]}(t_2) \otimes \ldots \otimes I_{[u_n, v_n]}(t_n)] \sharp\, dX_{t_1} =$$
$$= (X_{v_1} - X_{u_1})\, I_{[u_2, v_2]}(t_2) \otimes \ldots \otimes I_{[u_n, v_n]}(t_n) .$$

Iterating the procedure we obtain

$$\int_0^{\infty} \cdots \int_0^{t_2} I_{[u_1, v_1]}(t_1) \ldots I_{[u_n, v_n]}(t_n) \sharp\, dX_{t_1} \sharp \ldots \sharp\, dX_{t_n} = (X_{v_1} - X_{u_1}) \ldots (X_{v_n} - X_{u_n}) .$$

Because the elementary functions vanishing on the rectangles which intersect the diagonals are dense in $L^2(\mathbb{R}_+^n)$, the definition can be extended to any function in $L^2(\mathbb{R}_+^n)$. Differently from the classical case we cannot write the analogous of (3) because we cannot restrict to symmetric functions. The relation between the iterated integral and the multiple integral is the following:

$$(7) \quad \int_{\mathbb{R}_+^n} f(t_1, \ldots, t_n)\, dX_{t_1} \ldots dX_{t_n} = \sum_{t_1, \ldots, t_n \in \mathfrak{S}(n)} \int_{0 \leq t_{i_1} < \cdots < t_{i_n}} f(t_1, \ldots, t_n) \updownarrow dX_{t_{i_1}} \ldots \updownarrow dX_{t_{i_n}}.$$

The notion of multiple integrals can be extended to «bi-multiple» integrals (essentially this means that we work with $I \otimes I$ instead of just $I$): let $f \in L^2(\mathbb{R}_+; L^2(\mathbb{R}_+^n) \otimes L^2(\mathbb{R}_+^m))$ and consider

$$t \to f_t \in L^2(\mathbb{R}_+^n) \otimes L^2(\mathbb{R}_+^m),$$

then we denote

$$(I^n \otimes I^m)(f_t) = \int f_t(t_1, \ldots, t_n, s_1, \ldots, s_m)\, dX_{t_1} \ldots dX_{t_n} \otimes dX_{s_1} \ldots dX_{s_m},$$

that is $(I^n \otimes I^m)(f_t) \in L^2(\mathbb{R}_+; L^2(\mathcal{A}) \otimes L^2(\mathcal{A})) = \mathcal{B}_2$. As above it is easy to see that the map $I^n \otimes I^m$ gives an isometry between $L^2(\mathbb{R}_+; \mathcal{F}(L^2(\mathbb{R}_+)) \otimes \mathcal{F}(L^2(\mathbb{R}_+)))$ and $\mathcal{B}_2$, therefore any biprocess $U_t$ can be written as

$$(8) \quad U_t = \sum_{n,m} (I^n \otimes I^m)(f_t).$$

Definition 3.3: *Let* $f \in L^2(\mathbb{R}_+; \mathcal{F}(L^2(\mathbb{R}_+)) \otimes \mathcal{F}(L^2(\mathbb{R}_+)))$, *the process*

$$f = \bigoplus_{n,m \geq 0} f^{(n,m)} \in L^2(\mathbb{R}_+; \mathcal{F}(L^2(\mathbb{R}_+)) \otimes \mathcal{F}(L^2(\mathbb{R}_+)))$$

*is adapted if the following holds:*

$$f_t^{(n,m)}(t_1, \ldots, t_n, s_1, \ldots, s_m) = 0 \quad a.s. \quad \text{if } \max(t_1, \ldots, t_n, s_1, \ldots, s_m) > t.$$

The following result connects the notion of multiple integral in the context of the free Fock space and that of stochastic integral on the Wigner space.

Proposition 3.4: *For a process* $f = \bigoplus_{n,m \geq 0} f^{(n,m)} \in L^2(\mathbb{R}_+; \mathcal{F}(L^2(\mathbb{R}_+)) \otimes \mathcal{F}(L^2(\mathbb{R}_+)))$ *the following statements are equivalent:*

(i) *$f$ is adapted*

(ii) $(I \otimes I)(f) \in \mathcal{B}_2^a$.

*Moreover, if one of the above conditions is satisfied, then*

$$\int (I^n \otimes I^m)(f_t) \updownarrow dX_t = \tag{9}$$

$$= \sum_{n,m=0}^{\infty} \int f_t^{(n,m)}(t_1, \dots, t_n, s_1, \dots, s_m)\, dX_{t_1} \dots dX_{t_n} dX_t dX_{s_1} \dots dX_{s_m}.$$

In analogous way we extend the notion of bi-multiple integral to $n$-multiple integral, for $n \geq 2$, that is we consider $f \in L^2(\mathrm{R}_+^{n-1}; L^2(\mathrm{R}_+^{m_1}) \otimes \dots \otimes L^2(\mathrm{R}_+^{m_n}))$ and we set

$$(I^{m_1} \otimes \dots \otimes I^{m_n})(f_{t_1, \dots, t_{n-1}}) =$$

$$= \int f_{t_1, \dots, t_{n-1}}(r_1, \dots, r_{m_1}, \dots, s_1, \dots, s_{m_n})\, dX_{r_1} \dots dX_{r_{m_1}} \otimes \dots \otimes dX_{s_1} \dots dX_{s_{m_n}}.$$

## 4. - Gradient operator

In the Wiener analysis the procedure to define the Malliavin derivative of a Wiener functional is the following (see [M] for details). First define the derivative for a class $\mathcal{S}$ of smooth random variables, then observe that the derivative operator $D$ is closable on $\mathcal{S}$ and therefore define the domain of $D$ in $L^2(P)$ as the closure of the class $\mathcal{S}$ with respect to an appropriate norm. It is possible to give a characterization of the domain of $D$ in terms of Wiener chaos decomposition. The action of the derivative operator on the $n$-th chaos in the classical Wiener analysis is the following:

Proposition 4.1: *Let $f$ be a function in $L_s^2(\mathrm{R}_+^n)$. Then we have:*

$$D_t\left(\int f(t_1, \dots, t_n)\, dx_{t_1} \dots dx_{t_n}\right) = n \int f(t_1, \dots, t_{n-1}, t)\, dx_{t_1} \dots dx_{t_{n-1}}.$$

We introduce now the notion of gradient operator as in [B-S] and we generalize it to higher order derivatives. It is possible to follow the same procedure as in the classical case. Moreover, in virtue of the isomorphism between the free Fock space and the space of square integrable functionals on the Wigner space, we can specify the action of the gradient operator and its domain in terms of the chaotic decomposition. Let $Y$ be a function in $L^2(\mathcal{A})$ and suppose that it has a representation of the form (4), then we let

$$D_t\left(\int f(t_1, \dots, t_n)\, dX_{t_1} \dots dX_{t_n}\right) = \tag{10}$$

$$= \sum_{k=1}^{n} \int f(t_1, \dots, t_{k-1}, t, t_{k+1}, \dots, t_n)\, dX_{t_1} \dots dX_{t_{k-1}} \otimes dX_{t_{k+1}} \dots dX_{t_n}.$$

The domain of $D$ can be characterized in terms of the chaotic decomposition. We denote by $\mathbf{D}_1^2$ the domain of $D$. The following holds (see [B-S]):

Proposition 4.2: *Let $Y \in L^2(\mathcal{A})$ have chaos decomposition $Y = \sum_{n=0}^{\infty} I^n(f^{(n)})$ with $f^{(n)} \in L^2(\mathrm{R}_+^n)$. Then*

$$\int_{\mathrm{R}_+} \|D_t Y\|_{\mathscr{B}_2}^2 \, dt = \sum_{n=0}^{\infty} n \|f^{(n)}\|_{L^2(\mathrm{R}_+^n)}^2$$

*and $Y$ belongs to $\mathbf{D}_1^2$ if and only if the sum converges to a finite value.*

Due to the non commutativity of the setting it turns out that the derivative of an element of $L^2(\mathcal{A})$ is a biprocess

$$D \colon L^2(\mathcal{A}) \to \mathscr{B}_2 = L^2(\mathrm{R}_+ ; L^2(\mathcal{A}) \otimes L^2(\mathcal{A})).$$

The derivative operator has been defined in [B-S] for any element in $L^2(\mathcal{A})$. We need to extend this definition. For any element $A \otimes B$ belonging to $L^2(\mathcal{A}) \otimes L^2(\mathcal{A})$ we extend it in a canonical way:

$$D_t(A \otimes B) = D_t A \otimes B + A \otimes D_t B\,. \tag{11}$$

Therefore by iteration we can define the second derivative:

$$D^2 \colon L^2(\mathcal{A}) \to L^2(\mathrm{R}_+^2 ; L^2(\mathcal{A}) \otimes L^2(\mathcal{A}) \otimes L^2(\mathcal{A}))$$

$$D_{st}^2(\cdot) = D_s(D_t \cdot)\,.$$

For istance, let $Y$ have chaotic decomposition of the form (4). Then we write the second order derivative by applying $D_s$ to (10) with $t$ fixed:

$$D_s\left(\sum_{k=1}^{n} \int f(t_1, \ldots, t_{k-1}, t, t_{k+1}, \ldots, t_n)\, dX_{t_1} \ldots dX_{t_{k-1}} \otimes dX_{t_{k+1}} \ldots dX_{t_n}\right) =$$

$$= \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k-1} \int f(t_1, \cdot, t_{j-1}, s, t_{j+1}, \cdot, t_{k-1}, t, t_{k+1}, \cdot, t_n)\, dX_{t_1} \ldots dX_{t_{j-1}} \otimes dX_{t_{j+1}} \cdots dX_{t_{k-1}} \otimes dX_{t_{k+1}} \cdots dX_{t_n} + \right.$$

$$\left. + \sum_{j=k+1}^{n} \int f(t_1, \cdot, t_{k-1}, t, t_{k+1}, \cdot, t_{j-1}, s, t_{j+1}, \cdot, t_n)\, dX_{t_1} \cdots dX_{t_{k-1}} \otimes dX_{t_{k+1}} \cdots dX_{t_{j-1}} \otimes dX_{t_{j+1}} \cdots dX_{t_n} \right\}.$$

Then for any $k > 1$ we define the operator

$$D^k \colon L^2(\mathcal{A}) \to L^2(\mathrm{R}_+^k ; L^2(\mathcal{A})^{\otimes k+1})$$

as $D^k = D(D^{k-1})$. Moreover, we can characterize the domain of $D^k$, which we will denote by $\mathbf{D}_k^2$ by using the chaotic decomposition.

PROPOSITION 4.3: *Let* $Y \in L^2(\mathfrak{A})$ *have chaos decomposition* $Y = \sum_{n=0}^{\infty} I^n(f^{(n)})$ *with* $f^{(n)} \in L^2(\mathbb{R}^n_+)$. *Then*

$$\int_{\mathbb{R}^k_+} \| D^k_{t_1 \dots t_k} Y \|^2_{L^2(\mathbb{R}^k_+ ; L^2(\mathfrak{A})^{\otimes (k+1)})} dt_1 \dots dt_k = \sum_{n=k}^{\infty} n(n-1) \dots (n-k+1) \| f^{(n)} \|^2_{L^2(\mathbb{R}^n_+)}$$

*and* $Y$ *belongs to* $\mathbf{D}^2_k$ *if and only if the sum converges to a finite value.*

The following result holds.

PROPOSITION 4.4: *Let* $Y \in L^2(\mathfrak{A})$ *then* $D_{s_1} D_{s_2} Y = D_{s_2} D_{s_1} Y$. *In other words the function from* $\mathbb{R}^2_+$ *to* $L^2(\mathfrak{A}) \otimes L^2(\mathfrak{A})$ *defined by* $(s_1, s_2) \mapsto D_{s_1} D_{s_2} Y$ *is symmetric.*

PROOF: Because $Y$ belongs to $L^2(\mathfrak{A})$, it has the representation $Y = \sum_n I^n(f^{(n)})$. It is enough to consider $Y = I^n(f^{(n)})$. Then we have

$D_{s_1} D_{s_2} Y =$

$$= \sum_{k=1}^{n} \left( \sum_{h=1}^{k-1} \int f(t_1, \cdot, t_{h-1}, s_1, t_{h+1}, \cdot, t_{k-1}, s_2, t_{k+1}, \cdot, t_n)\, dX_{t_1} \cdot dX_{t_{h-1}} \otimes dX_{t_{h+1}} \cdot dX_{t_{k-1}} \otimes dX_{t_{k+1}} \cdot dX_{t_n} + \right.$$

$$\left. + \sum_{h=k+1}^{n} \int f(t_1, \cdot, t_{k-1}, s_2, t_{k+1}, \cdot, t_{h-1}, s_1, t_{h+1}, \cdot, t_n)\, dX_{t_1} \cdot dX_{t_{k-1}} \otimes dX_{t_{k+1}} \cdot dX_{t_{h-1}} \otimes dX_{t_{h+1}} \cdot dX_{t_n} \right).$$

Analogously

$D_{s_2} D_{s_1} Y =$

$$= \sum_{k=1}^{n} \left( \sum_{h=1}^{k-1} \int f(t_1, \cdot, t_{h-1}, s_2, t_{h+1}, \cdot, t_{k-1}, s_1, t_{k+1}, \cdot, t_n)\, dX_{t_1} \cdot dX_{t_{h-1}} \otimes dX_{t_{h+1}} \cdot dX_{t_{k-1}} \otimes dX_{t_{k+1}} \cdot dX_{t_n} + \right.$$

$$\left. + \sum_{h=k+1}^{n} \int f(t_1, \cdot, t_{k-1}, s_1, t_{k+1}, \cdot, t_{h-1}, s_2, t_{h+1}, \cdot, t_n)\, dX_{t_1} \cdot dX_{t_{k-1}} \otimes dX_{t_{k+1}} \cdot dX_{t_{h-1}} \otimes dX_{t_{h+1}} \cdot dX_{t_n} \right).$$

Then interchange $h$ in $k$ and obtain the previous sum. ■

### 5. - REPRESENTATION FORMULAS

Classical results in the Wiener analysis concern the representation of the Brownian martingales as stochastic integrals. If some regularity assumptions are imposed, the integrand process is well identified as the Malliavin derivative of the Brownian martingale. This result is the Bismut-Clark-Ocone formula. Moreover this representation procedure can be iterated leading to a stochastic Taylor formula (Stroock formula [8]) which is connected with the multiple Itô integrals decomposition.

In the previous paragraph we have introduced the operator $D$ defined on the space $L^2(\mathfrak{A})$ with values in $\mathcal{B}_2$. As in the Wiener analysis, we can consider the adjoint of the free gradient operator, i.e. the *divergence* operator $\delta$ which acts on the biprocesses

$$\delta : \mathcal{B}_2 \to L^2(\mathcal{A}).$$

Thanks to the isometry (8) the action of $\delta$ can be specified on the bi-multiple integrals in the following way:

$\delta : \mathcal{B}_2 \to L^2(\mathcal{A})$

$$\delta\left(\int f_t(t_1, \ldots, t_n, s_1, \ldots, s_m)\, dX_{t_1} \ldots dX_{t_n} \otimes dX_{s_1} \ldots dX_{s_m}\right) =$$

$$= \int f_t(t_1, \ldots, t_n, s_1, \ldots, s_m)\, dX_{t_1} \ldots dX_{t_n} dX_t dX_{s_1} \ldots dX_{s_m}.$$

It can be shown (see [B-S]) that the divergence operator can be seen as in the classical case as a (Skorohod) stochastic integral. The following result holds [B-S].

PROPOSITION 5.1: *Let $U$ be an adapted biprocess $\in \mathcal{B}_2^a$. Then $U$ belongs to the domain of $\delta$ and*

$$\int_{\mathbb{R}_+} U_t \sharp\, dX_t = \delta(U).$$

PROOF: From the isomorphism there exists an adapted process $f_t = \bigoplus_{n,m} f_t^{(n,m)}$ such that $U_t = (I \otimes I)\, f_t$. Then we have

$$\int U_t \sharp\, dX_t = \int (I \otimes I)\, f_t \sharp\, dX_t =$$

$$= \sum_{n,m} \int f_t^{(n,m)}(t_1, \ldots, t_n, s_1, \ldots, s_m)\, dX_{t_1} \ldots dX_{t_n} dX_t dX_{s_1} \ldots dX_{s_m} =$$

$$= \sum_{n,m} \delta(I^n \otimes I^m)(f_t^{(n,m)}) = \delta((I \otimes I)(f_t)) = \delta(U).$$

Since $\int U_t \sharp\, dX_t \in L^2(\mathcal{A})$ the above equality also shows that $U$ belongs to the domain of $\delta$. ■

We introduce the notion of martingale in this context. Following the classical notation we denote by $\tau[\cdot\,|\mathcal{B}]$ the conditional expectation with respect to the closed $^*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$. Since it extends to a contraction on all the $L^p$ spaces, then a map $t \mapsto N_t$ from $[0, +\infty[$ to $L^p(\mathcal{A})$ will be called an $L^p$-martingale with respect to the filtration $(\mathcal{A}_t)_{t \geq 0}$ if for every $s \leq t$ one has $\tau[N_t\,|\mathcal{A}_s] = N_s$.

We start with the representation formula for the martingales of the free Brownian motion.

Theorem 5.2: *Let* $N = (N_t)_{t \geq 0}$ *be a bounded martingale in* $L^2(\mathcal{A})$ *relative to the filtration* $(\mathcal{A}_t)_{t \geq 0}$ *and such that* $N_0 = 0$. *Then there exists* $Y \in \mathcal{B}_2^a$ *such that for any* $t \geq 0$

$$N_t = \int_0^t Y_s \sharp \, dX_s.$$

Proof: For any $T \geq 0$ and $t \leq T$, by the martingale property we have

$$N_t = \tau[N_T \,|\, \mathcal{A}_t].$$

Since $N_T \in L^2(\mathcal{A})$ and $\tau[N_T] = N_0 = 0$, $N_T$ can be represented by the multiple integral decomposition

$$N_T = I(f) = \sum_{n \geq 1} I^n(f^{(n)}),$$

where $f = \bigoplus_n f^{(n)}$. Fix $n$ and define

$$f_t^{(k,\, n-k-1)} = f(t_1, \ldots, t_k, t, s_1, \ldots, s_{n-k-1}) \quad \text{if } \max\{t_1, \ldots, t_k, s_1, \ldots, s_{n-k-1}\} \leq t$$

and equal to 0 otherwise. Clearly they are adapted processes. Then

$$I^n(f^{(n)}) = \int_0^T \left( \sum_{k=0}^{n-1} (I^k \otimes I^{n-k-1})\, f_t^{(k,\, n-k-1)} \right) \sharp \, dX_t.$$

Let $A_n(t) = \sum_{k=0}^{n-1} (I^k \otimes I^{n-k-1})\, f_t^{(k,\, n-k-1)}$ and $B_m(t) = \sum_{n=0}^{m} A_n(t)$.

Using the free isometry property we have for any $m, m'$

$$\int_0^T \|B_m(t) - B_{m'}(t)\|^2_{L^2(\mathcal{A}) \otimes L^2(\mathcal{A})}\, dt =$$

$$= \tau\left[ \left( \int_0^T \sum_{m < n \leq m'} A_n(t) \sharp \, dX_t \right)^2 \right] = \tau\left[ \left( \sum_{m < n \leq m'} I^n(f^{(n)}) \right)^2 \right].$$

Since the series $\sum_n I^n(f^{(n)})$ converges to $N_T$ in $L^2(\mathcal{A})$, we have obtained that $(B_m)_m$ is a Cauchy sequence in $\mathcal{B}_2$. Let $Y^{(T)}$ be the limit. Then $N_T = \int_0^T Y_t^{(T)} \sharp \, dX_t$.

Finally let $Y = \lim_{T \to \infty} Y^{(T)}$. Since $N$ is a bounded martingale, we have

$$\sup_{T > 0} \|N_T\|^2_{L^2(\mathcal{A})} < \infty,$$

therefore for any $T$

$$\int_0^T \|Y_s\|^2_{L^2(\mathfrak{A})}\, ds = \|N_T\|^2_{L^2(\mathfrak{A})} < \infty .$$

This implies that

$$\|Y\|^2_{\mathbb{B}_2} = \int_0^\infty \|Y_s\|^2_{L^2(\mathfrak{A})}\, ds < \infty . \quad \blacksquare$$

As in the classical case, under additional smoothness hypothesis on the function we can compute more explicitly the integrand in the representation formula.

We begin with an extension of the *free Bismut-Clark-Ocone formula* stated in [B-S] to differentiable biprocesses. Denote by $L^2_1$ the class of biprocesses $U$ such that for any $t$ fixed $U_t \in D^2_1$ and $DU$ is a two parameters process belonging to $L^2(\mathbb{R}^2_+ ; L^2(\mathfrak{A}) \otimes L^2(\mathfrak{A}) \otimes L^2(\mathfrak{A}))$. Remark that $L^2_1 \subset \mathrm{dom}\,(\delta)$.

Proposition 5.3: *For any biprocess $Z$ such that $Z \in L^2_1$, we have that for any $t$*

$$Z_t = \tau[Z_t] + \delta_s(\tau[D_s Z_t \mid \mathfrak{A}_s]) \tag{12}$$

*(where $\delta_s$ is used to indicate that the integration acts with respect to the $s$-variable).*

Proof: It is enough to prove the formula in the case of a biprocess $Z$ having the form

$$Z_t = \int f_t(t_1, \ldots, t_n, s_1, \ldots, s_m)\, dX_{t_1} \ldots dX_{t_n} \otimes dX_{s_1} \ldots dX_{s_m}.$$

We have:

$$D_s Z_t = \sum_{k=1}^{n} \int f_t(t_1, \ldots, t_{k-1}, s, t_{k+1}, \ldots, t_n, s_1, \ldots, s_m)\, dX_{t_1} \cdots dX_{t_{k-1}} \otimes dX_{t_{k+1}} \cdots dX_{t_n} \otimes dX_{s_1} \cdots dX_{s_m} +$$

$$+ \sum_{j=1}^{m} \int f_t(t_1, \ldots, t_n, s_1, \ldots, s_{j-1}, s, s_{j+1}, \ldots, s_m)\, dX_{t_1} \cdots dX_{t_n} \otimes dX_{s_1} \cdots dX_{s_{j-1}} \otimes dX_{s_{j+1}} \cdots dX_{s_m}.$$

Then the projection of the above multiple integrals on the $\mathfrak{A}_s$-measurable processes gives that the integrals are computed respectively over the sets

$S_1 = \{s \geqslant \max(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n, s_1, \ldots, s_m)\}$ and

$S_2 = \{s \geqslant \max(t_1, \ldots, t_n, s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_m)\}$.

Finally compute $\delta_t(\tau[D_t Z_t \mid \mathcal{C}_t])$. From Proposition 3.4 we have

$$\delta\Bigg(\sum_{k=1}^{n}\int_{S_1} f_t(t_1,\cdots,t_{k-1},s,t_{k+1},\cdots,t_n,s_1,\cdots,s_m)\,dX_{t_1}\ldots dX_{t_{k-1}}\otimes dX_{t_{k+1}}\cdots dX_{t_n}\otimes dX_{s_1}\ldots dX_{s_m}+$$

$$+\sum_{j=1}^{m}\int_{S_2} f_t(t_1,\cdots,t_n,s_1,\cdots,s_{j-1},s,s_{j+1},\cdots,s_m)\,dX_{t_1}\cdots dX_{t_n}\otimes dX_{s_1}\cdots dX_{s_{j-1}}\otimes dX_{s_{j+1}}\cdots dX_{s_m}\Bigg)=$$

$$=\sum_{k=1}^{n}\int_{\tilde S_1} f_t(t_1,\cdot,t_{k-1},s,t_{k+1},\cdot,t_n,s_1,\cdot,s_m)\,dX_{t_1}\cdot dX_{t_{k-1}}dX_s dX_{t_{k+1}}\cdot dX_{t_n}\otimes dX_{s_1}\cdot dX_{s_m}+$$

$$+\sum_{j=1}^{m}\int_{\tilde S_2} f_t(t_1,\cdot,t_n,s_1,\cdot,s_{j-1},s,s_{j+1},\cdot,s_m)\,dX_{t_1}\cdot dX_{t_n}\otimes dX_{s_1}\cdot dX_{s_{j-1}}dX_s dX_{s_{j+1}}\cdot dX_{s_m}$$

where the sets $\tilde S_1$ and $\tilde S_2$ are given respectively by

$$\{s=\max(t_1,\ldots,t_{k-1},t_{k+1},\ldots,t_n,s_1,\ldots,s_m)\}$$

$$\{s=\max(t_1,\ldots,t_n,s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_m)\}.$$

Finally the above sum is equal to

$$\int f_t(t_1,\ldots,t_n,s_1,\ldots,s_m)\,dX_{t_1}\ldots dX_{t_n}\otimes dX_{s_1}\ldots dX_{s_m},$$

that is $Z_t$. In order to conclude the proof, observe that for fixed $t$, $\tau[Z_t]$ is zero, because of (6). Therefore Bismut-Clark-Ocone formula for differentiable biprocesses is proved. ■

The above result suggests a method to obtain a Taylor formula. If we assume that the random variable $Y$ admits second order derivative in the sense introduced before, the above procedure can be iterated.

Theorem 5.4: *Let $Y$ be in $\mathbf{D}_2^2$. Then*

$$Y=\tau[Y]+\int \tau\otimes\tau[D_t Y]\,\sharp\, dX_t+\int\int_{s<t}(\tau[D_{ts}^2 Y\mid \mathcal{C}_s])\,\sharp\, dX_s\,\sharp\, dX_t.$$

Proof: It is enough to prove the result in the case where

$$Y=I^n(f^{(n)})=\int f(t_1,\ldots,t_n)\,dX_{t_1}\ldots dX_{t_n}.$$

Let $n>1$. Then

$$D_t Y=\sum_{k=1}^{n}\int f_t(t_1,\ldots,t_{k-1},t_{k+1},\ldots,t_n)\,dX_{t_1}\ldots dX_{t_{k-1}}\otimes dX_{t_{k+1}}\ldots dX_{t_n}.$$

Remark that $D_t Y$ is a biprocess. Fix $t$ and apply Bismut-Clark-Ocone formula to $D_t Y$:

$$D_t Y - \tau \otimes \tau[D_t Y] = \delta_s(\tau[D_{st}^2 Y | \mathfrak{a}_s]), \tag{13}$$

where $\delta_s$ is used to indicate that the integration acts with respect to the $s$-variable. Now apply Bismut-Clark-Ocone formula to $Y$

$$Y - \tau[Y] = \delta_t(\tau[D_t Y | \mathfrak{a}_t]) \tag{14}$$

and substitute (13) in (14)

$$Y - \tau[Y] = \delta_t(\tau[\tau \otimes \tau[D_t Y] | \mathfrak{a}_t]) + \delta_t(\tau[\delta_s(\tau[D_{st}^2 Y | \mathfrak{a}_s]) | \mathfrak{a}_t]).$$

Observe that

$$\tau[\tau \otimes \tau[D_t Y] | \mathfrak{a}_t] = \tau \otimes \tau[D_t Y].$$

applying (3.4) to the adapted process $\tau[D_{st}^2 Y | \mathfrak{a}_s]$, we have

$$\delta_s(\tau[D_{st}^2 Y | \mathfrak{a}_s]) = \sum_{k=1}^{n} \int f_t(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)\, dX_{t_1} \ldots dX_{t_{k-1}} \otimes dX_{t_{k+1}} \ldots dX_{t_n}.$$

Then

$$\tau[\delta_s(\tau[D_{st}^2 Y | \mathfrak{a}_s]) | \mathfrak{a}_t] =$$

$$= \sum_{k=1}^{n} \int_{S_t} f_t(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)\, dX_{t_1} \ldots dX_{t_{k-1}} \otimes dX_{t_{k+1}} \ldots dX_{t_n},$$

where $S_t = \{t \geq \max(t_1, \ldots, t_{k-1}, s, t_{k+1}, \ldots, t_n)\}$. Finally

$$\delta_t(\tau[\delta_s(\tau[D_{st}^2 Y | \mathfrak{a}_s]) | \mathfrak{a}_t]) = \sum_{k=1}^{n} \int_{\{t_k = \max(t_1, \ldots, t_n)\}} f(t_1, \ldots, t_n)\, dX_{t_1} \ldots dX_{t_n} =$$

$$= \int f(t_1, \ldots, t_n)\, dX_{t_1} \ldots dX_{t_n} = Y.$$

In order to conclude it is enough to observe that if $n \geq 2$, then $\tau[Y] = 0$ and $\tau \otimes \tau[D_t Y] = 0$ because of the orthogonality property (6). ∎

Remark 5.5: Under further smoothness hypothesis the procedure can be iterated. In the commutative case it is possibile to express the coefficients of the multiple integral decomposition of the random variable by its iterated derivatives. This leads to the following result due to Stroock [8] which is essentially based on the Taylor formula and the following relations:

$$D_{t_1 \ldots t_n}^n \int f^{(n)}(s_1, \ldots, s_n)\, dx_{s_1} \ldots dx_{s_n} = n!\, f^{(n)}(t_1, \ldots, t_n) \tag{15}$$

and

$$\int_{\mathbb{R}_+^n} f^{(n)}(t_1, \ldots, t_n)\, dx_{t_1} \ldots dx_{t_n} = n! \int_{\Delta_n} f^{(n)}(t_1, \ldots, t_n)\, dx_{t_1} \ldots dx_{t_n}.$$

The result of Stroock is the following: let $F$ be a random variable with chaotic decomposition

$$F = \sum_{n=0}^{\infty} \int f^{(n)}(t_1, \ldots, t_n)\, dx_{t_1} \ldots dx_{t_n}. \tag{16}$$

Suppose moreover that $F$ has Malliavin derivatives of all orders belonging to $L^2$. Then

$$n!\, f^{(n)}(t_1, \ldots, t_n) = E[D^n_{t_1 \ldots t_n} F].$$

In the case of Wigner space such identification is not possible. This is clear from the relation (7) which we have found between the iterated and the multiple integral and the fact that the iterated derivatives identify the symmetrized kernel functions.

## REFERENCES


[B-S] P. Biane - R. Speicher, *Stochastic Calculus with respect to Free Brownian Motion and Analysis on Wigner space*, Prob. Th. Rel. Fields, 112 (1998), 373-409.

[F] F. Fagnola, *On quantum stochastic integration with respect to «free» noises*, Quantum prob. rel. topics, vol. VI (1991), 285-304.

[H-P] R. L. Hudson - K. R. Parthasarathy, *Quantum Itô's formula and stochastic evolution*, Comm. Math. Phys., 93 (1994), 301-323.

[K-S] B. Kümmerer - R. Speicher, *Stochastic integration on the Cuntz algebra*, J. Funct. An., 103 (1992), 372-408.

[M] P. Malliavin, *Stochastic Analysis*, Springer, A Series of Comprehensive Studies in Mathematics, Volume 313.

[N] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer Series in Applied Mathematics, Springer Berlin, Heidelberg, New York (1995).

[Sp] R. Speicher, *Stochastic Integration on the full Fock space with respect to a kernel calculus*, Pubbl. R.I.M.S., 27 (1991), 149-184.

[St] D. W. Stroock, *Homogeneous chaos revisited*, Sem. Prob. XXI Springer, Lecture Notes in Math. 1247 (1987).