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Shadow of a Measure

Abstract. — Riesz representation theorem implies conversion from a hyperfinite measure space $(T, 2^T, m)$ to a space $(T, A, \mu)$ with $\sigma$-additive measure $\mu$ in the framework of Nelson’s Internal Set Theory [1-3].

Ombra di una misura

Sunto. — Nel quadro della cosiddetta Internal Set Theory di Nelson, e valendosi del teorema di rappresentazione di Riesz-Markov, si trasforma uno spazio misurato iperfinito $(T, 2^T, m)$ in uno spazio misurato standard $(T, A, \mu)$, con $\mu$ misura $\sigma$-additiva.

An essential notion of Nonstandard Analysis (NSA) is that of the shadow. Let $(X, d)$ be a standard metric space. A point $x \in X$ is said to be nearstandard (write $x \in \mu X$) iff there exists a standard $y \in X$ such that $d(x, y) = 0$ (1). If such $y$ exists, it is unique, is said to be the shadow of $x$ and denoted by $^*x$. The importance of this notion is clear, for instance, from the following example. Let $(x_n)_{n \in \mathbb{N}}$ be a standard sequence in $X$. Then it is convergent iff for any $n = \infty$ (2) $x_n \in \mu X$ and the shadow $^*x_n$ is independent of $n = \infty$. In this case for all $n = \infty$ $^*x_n = \lim_{n \to \infty} x_n$. Note that the map $x \mapsto ^*x$ is noninjective: if $d(x_1, x_2) = 0$, then $x_1 \in \mu X$ implies $x_2 \in \mu X$ and $^*x_1 = ^*x_2$.

A point $x \in X \setminus \mu X$ is said to be remote. The following remoteness theorem is known [4]. Let $(x_n)_{n \in \mathbb{N}}$ be such a sequence in $X$ that $\forall p, q \in \mathbb{N} p \neq q \Rightarrow d(x_p, x_q) \gg 0$ (3). Then $x_n$ is remote for some $n = \infty$. For instance, let $X = H$ be a

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(1) Read “is infinitesimal” for “$= 0$”.

(2) Read “is infinite” for “$= \infty$”.

(3) Read “is not infinitesimal” for “$\gg 0$”.
standard Hilbert space and \((e_n)_{n=1}^\infty\) a standard orthonormal basis of \(H\). Then for each \(n = \infty\) \(e_n\) is remote in spite of \(\|e_n\| = 1\). (Note that each finite real number is nearstandard.)

It is worthwhile to extend the notion of nearstandardness as follows. Let \(X\) be a standard normed space and \(l \in X^*\) (adjoint space).

1. **Proposition:** Suppose \(\|l\| \ll \infty\). Then \(l\) is weakly nearstandard, i.e. there exists \(k \in \mathcal{U}(X^*)\), for which

\[
\forall x \in X \quad l(x) = k(x).
\]

Such \(k\) is unique.

**Proof:** Let \(l_0 \in \mathcal{U}(C)^{\mathcal{W}(X)}\) be the map defined by \(\forall x \in X \quad l_0(x) = \mathfrak{g}([l(x)])\). Since \(\|l\| \ll \infty\), we have \(\|l(x)\| \ll \|l\| - \|x\| \ll \infty\) for standard \(x\), therefore \(\mathfrak{g}([l(x)])\) is defined (as shadow of a finite complex number). Obviously, for standard \(\alpha_1, \alpha_2 \in C, x_1, x_2 \in X\) we have

\[
l_0(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 l_0(x_1) + \alpha_2 l_0(x_2), \quad |l_0(x_1)| \leq \|\alpha_1\| \cdot \|x_1\|.
\]

Recall that each map \(f_0 \in \mathcal{U}(F)^{\mathcal{W}(E)}\), where \(E, F\) are arbitrary standard sets, has a unique extension \(f \in \mathcal{U}(F^E)\), which is said to be the **standard extension** of \(f_0\) (1.3)). Define \(k\) as the standard extension of \(l_0\). Transfer principle (7) and (2) imply that \(k\) is a linear continuous functional on \(X\), that is \(k \in X^*\). For standard \(x\) we have \(k(x) = l_0(x) = l(x)\), i.e. (1) holds. Uniqueness of \(k\) is evident: a standard function which equals zero at standard points is identically zero.

2. **Remark:** The functional \(k\) described above is said to be the shadow of \(l\) (in the weak sense) and will be denoted by \(\mathfrak{l}\). Emphasize that if \(l\) is nearstandard in the strong sense, i.e. \(\|l - k\| \approx 0\) for some \(k \in \mathcal{U}(X^*)\), then it is weakly nearstandard and the last \(k\) is the same as \(k\) in (1), \(k = \mathfrak{l}\).

3. **Remark:** Each \(x \in X\) is in a natural way some element of \(X^{**}\), with the same norm \(\|x\|\). Therefore, if \(\|x\| \ll \infty\), then \(x\) has the shadow \(x^*\) which belongs to \(X^{**}\). If \(X\) is reflexive, then \(x^*\) can be regarded as some standard element \(\mathfrak{x}\) of \(X\), which is uniquely defined by \(\forall y \in X^* \quad (x,y) = \mathfrak{x}\). For instance, if \(X = H\) is a standard Hilbert space, \(x \in H\), and \(\|x\| \ll \infty\), then the shadow \(x^*\) of \(x\) is uniquely defined by conditions: \(\mathfrak{x} \in H\) and \(\forall y \in H \quad (x,y) = (\mathfrak{x},y)\).

Now consider an interesting special case. In what follows \((T, d)\) denotes a stan-

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(\*4) Read \(\mathfrak{g}\) is finite or \(\mathfrak{g}\) noninfinite for \(\ll \infty\).

(\*5) Read \(\mathfrak{g}\) is a standard member of the set \(\mathcal{E}\) for \(\ll \mathcal{E}\).

(\*6) \(F^E\) denotes the set of functions defined on the set \(E\) with values in the set \(F\).

(\*7) Transfer principle is: \(\mathcal{U}(x) \approx \mathcal{U}(y)\), where property \(p(x)\) can be expressed in language of usual mathematics. Linearity and boundness are such properties.
standard compact metric space, $A$ the algebra of borelian subsets $\mathcal{B} \subset T$. By definition a charge on $T$ is a $\sigma$-additive function defined on $A$ with values in $\mathbb{C}$. Each charge $c$ generates a linear functional $l = l_c$ on the space $C(T)$ of continuous functions $x \in C^T$ (with norm $\|x\|_\infty := \max_{t \in T} |x(t)|$) by $l(x) := \int_T x \, dc$. Note that $\|l\| = \text{var}_T c (\ast)$.

4. Theorem: If $\text{var}_T c \ll \infty$, then $c$ is weakly nearstandard, i.e. there exists a unique standard charge $\nu^* c$ such that

$$\forall \xi \in C(T) \int_T \xi \, d(\nu^* c) = \ast \int_T \xi \, dc.$$  

Proof: Define $\forall \xi \in C(T) \, l(\xi) := \int_T \xi \, dc$. Then $l \in (C(T))^\ast$, and $\|l\| = \text{var}_T c \ll \infty$.

By proposition 1 $l$ has the shadow $\hat{l} \in (C(T))^\ast$. By Riesz-Markov representation theorem $\hat{l}$ is of the form $\hat{l}(\xi) = \int_T \xi \, d(\nu^* c)$ for some $\sigma$-additive charge $\nu^* c$ on $A$. Transfer principle (in the form $\exists x \, p(x) \Rightarrow \exists^* x \, p(x)$) and uniqueness of $\nu^* c$ imply that $\nu^* c$ is standard.

5. Remark: Obviously, theorem 4 can be generalized to the case of locally compact $(T, d)$. Theorem 4 prompts a method to transform a hyperfinite measure space to a standard measure space. The reader is invited to compare this construction with use of the Loeb measure [5-8]. Let us explain that a hyperfinite measure space is a triple $(T, 2^T, m)$ where $T$ is a set such that $\text{card} \, T \in \mathbb{N} \setminus ^\ast \mathbb{N}$ and $m$ is an additive function with positive values on the algebra $2^T$ of all (internal) subsets $E \subset T$. If $m_i$ denotes the value $m\{i\}$ of $m$ at one-point set $\{i\}$, then $\forall E \in 2^T \, mE := \sum_{i \in E} m_i$.

6. Theorem: Let a hyperfinite set $T$ be a subset of $T$ where $(T, d)$ is a standard compact metric space. Any additive measure $m$ defined on $2^T$ such that $mT \ll \infty$ generates on the algebra $A$ of borelian subsets of $T$ a standard $\sigma$-additive measure $\mu$ which is uniquely determined by

$$\forall \xi \in C(T) \int_T \xi(t) \, d\mu(t) = \ast \sum_{i \in I} \xi(t_i) \, m_i,$$

Proof: The measure $m$ on $2^T$ induces the measure $\hat{m}$ on $A$ by the formula: $\forall \delta \in A \, \hat{m}(\delta) = m(\delta \cap T)$. This $\hat{m}$ is trivially $\sigma$-additive. Indeed, let $\delta \in A$ be a disjunctive union $\delta = \bigcup_{n \in \mathbb{N}} \delta_n$, $\delta_n \in A$. Only finite quantity of $\delta_n \cap T$ is not empty. Therefore for some $k \in \mathbb{N} \forall n > k \, \hat{m}(\delta_n) = 0$. Since $\hat{m}T = mT \ll \infty$, by theorem 4 there exists a unique stan-

$\ast$ denotes the variation of $c$ on the set $E$. 

$(\ast)$ var$^*_T c$ denotes the variation of $c$ on the set $E$. 

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standard charge $\mu$ defined on $A$ such that (4) holds. Since for $\xi \geq 0$ the right side in (4) is $\geq 0$, this charge is a measure.  

7. **Example:** Let $T := \{-1, +1\} \subset \mathbb{R}$ and $T$ be some hyperfinite subset of $T$. For each $t \in T$ denote by $t + dt$ the member of $T$ which follows immediately after $t$ in the sense of $<$. Define $\forall E \in 2^T m_E := \sum_{t \in E} dt$ where $dt := (t + dt) - t$. Then $(T, 2^T, m)$ is a hyperfinite measure space. Suppose that the first $t$ equals $-1$, the last $t + dt$ equals $+1$, and $\forall t \in T \ dt = 0$. Then the standard measure space $(T, A, \mu)$ generated by $(T, 2^T, m)$ according to theorem 6 is $([-1, +1], A, \mu)$ where $\mu$ is the standard Lebesgue measure. This is clear from (4).

8. **Example:** Let $T = [-1, +1]$ and $\varepsilon$ be some positive infinitesimal number. For any borelian $\delta \subset T$ put $m \delta := (2\varepsilon)^{-1} \mu(\delta \cap [-\varepsilon, +\varepsilon])$ where $\mu$ is the standard Lebesgue measure on $T$. The shadow of $\mu$ (in the weak sense; see theorem 4) is the Dirac measure concentrated at 0. Indeed, $\forall \xi \in C[-1, +1]$ 

$$\int_T \xi(t) d(\mu) = \left( \frac{1}{2\varepsilon} \int_{-\varepsilon}^{+\varepsilon} \xi(t) \ dt \right) = \xi(0); \quad \text{see (3).}$$

9. **Example:** Let $H$ be a standard (separable or not) Hilbert space and $A \in \mathcal{B}(H)$ a standard selfadjoint operator. Denote by $H_0$ a subspace of $H$ such that $^a H \subset H_0$ and $\dim H_0 \in \mathbb{N}$. (The existence of such $H_0$ follows from the idealization principle of IST [1, 4, 9].) Let $P$ be the orthoprojector $H \rightarrow H_0$. Denote by $A'$ the restriction of $PAP$ to $H_0$. Since $A'$ is an operator in a finite-dimensional space, its spectrum $\sigma(A')$ consists of eigenvalues only. For each eigenvalue $\lambda \in \sigma(A')$ denote by $P_{\lambda}$ the corresponding eigenprojector of $A'$. Since $A'$ is selfadjoint, $P_{\lambda}$ is an orthoprojector in $H_0$. Now for any $x \in H$ there arises a hyperfinite measure space $(T, 2^T, m)$, where $T := \sigma(A')$ and

$$\forall E \in 2^T m_E = m_E = \sum_{\lambda \in \sigma(A')} (P_{\lambda} x \mid x).$$

Suppose that $\|x\| < \infty$. Then $\|m\| = m T = \|x\|^2 < \infty$. According to theorem 6 $(T, 2^T, m)$ generates a standard measure space $(T, A, \mu)$ with a standard $\sigma$-additive measure $\mu(= \mu_x)$ such that $T = [-\|A\|, +\|A\|]$ and $A$ is the family of borelian sets $\delta \subset T$. Just the standard $\mu$ is uniquely determined by

$$\forall \xi \in C(T) \int_T \xi(t) d\mu = \sum_{\lambda \in \sigma(A')} \xi(\lambda)(P_{\lambda} x \mid x).$$
We claim that
\[(6) \quad \forall \lambda \in H \quad \forall \delta \in A \quad \mu \delta = (P(\delta) x | x),\]
where \(\{P(\delta)\}_{\delta \in A}\) is the spectral family of \(A\) (compare with [7]).

Proof: It is easy to see that for any standard polynomial \(p(\lambda)\) and any \(x \in H\) we have \(p(A^*) x = p(A) x\); note that \(\forall x \in H \quad P x = x\). Thus for \(x \in H\)
\[(p(A) x | x) = \sum_{\lambda \in T} p(\lambda) (P_\lambda x | x) = \int \mu d\mu.\]
Since the first and the last members of this chain are standard, they are equal. By transfer principle, the equality \(p(A) x | x) = \int \mu d\mu\) holds for all (optionally standard) \(p\) and \(x\). This proves our assertion. \(\blacksquare\)

Now we want to give some complementary information about the measure \(\mu\) defined by (4). The notation below are as in theorem 6.

Let \(Q\) be an embedding \(2^T \to A\) such that \(\forall t, s \in T \quad t \neq s \Rightarrow Q t \cap Q s = \emptyset\) where \(Q t := Q t\), \(\forall E \in 2^T\), \(Q E := \bigcup_{t \in E} Q t\), and \(Q T = T\) (or more generally \(T\) coincides with the standardization \(\hat{Q}(Q T)\) of the set \(Q T\); see [10] where \((T, A, Q)\) is named the standard filling of the (hyper)finite set \(T\)). Then each \(\sigma\)-additive measure \(\nu\) defined on \(A\) induces some additive measure \(n\) on \(2^T\) which is given by
\[(7) \quad \forall E \in 2^T \quad n E = \nu Q E.\]

10. Theorem: Suppose that the measure \(\nu\) is standard and
\[1) \quad \forall t \in T \quad \text{diam} \quad Q t = 0, \quad 2) \quad \max_{t \in T} \frac{m_t}{\nu} \ll \infty,\]
where \(m_t := m\{t\}, \quad n_t := \nu Q t\). Then the measure \(\mu\) generated by \(m\) (see (4)) is absolutely continuous relative to the measure \(\nu\).

For proof we need the following

11. Lemma: Put for \(\xi \in C(T)\), \(t \in T\),
\[(8) \quad N \xi(t) := \frac{1}{\nu Q t} \int_{Q t} \xi d\nu.\]
Then \(N\) is continuous as a map \(L_2(T, \nu) \to L_2(T, n)\).

Proof: By Bunyakowski-Cauchy inequality \(|N \xi(t)|^2 \leq (1/n_t) \int_{Q t} |\xi|^2 d\nu\). Therefore \(\|N \xi\|_{L_2(T, n)}^2 \leq \sum_{t \in T} |N \xi(t)|^2 n_t \leq \sum_{t \in T} \int_{Q t} |\xi|^2 d\nu \leq \|\xi\|_{L_2(T, \nu)}^2.\) \(\blacksquare\)
Proof of Theorem 10: For $\xi \in {}^uC(T)$ the condition $\text{diam}Qt = 0$ implies $\mathcal{N}\xi(t) = \xi(t)$. Since $mT \ll \infty$, we have $\sum_{t \in T} \xi(t) m_t = \sum_{t \in T} \mathcal{N}\xi(t) m_t$. Once more by Bunyakowski-Cauchy inequality and by lemma 11,

$$\left| \sum_{t \in T} \xi(t) m_t \right|^2 = \left| \sum_{t \in T} \mathcal{N}\xi(t) m_t \right|^2 \leq mT \sum_{t \in T} m_t \left| \mathcal{N}\xi(t) \right|^2 n_t \leq$$

$$\leq \gamma^2 \left\| \mathcal{N}\xi \right\|_{L^2(T, \nu)} \leq \gamma'^2 \left\| \xi \right\|_{L^2(T, \nu)},$$

where $\gamma^2 := mT \cdot \max(m_t/n_t) \ll \infty$. Since $\forall \xi \in {}^uC(T) \sum_{t \in T} \xi(t) m_t = \int_T \xi \, d\mu$, we see that

$$\left| \int_T \xi \, d\mu \right| \leq \gamma_1 \left\| \xi \right\|_{L^2(T, \nu)}$$

for some standard $\gamma_1 > 0$. By transfer this inequality holds for all (optionally standard) $\xi \in C(T)$. This means that the functional $\xi \mapsto \int_T \xi \, d\mu$ belongs to $L_2(T, \nu)^*$. By Riesz representation theorem (for Hilbert spaces)

$$\int_T \xi \, d\mu = \int_T \xi \, \eta \, dv,$$

for some $\eta \in L_2(T, \nu)$. Taking into account that $C(T)$ is dense in $L(T, \mu)$ and $L_2(T, \nu)$ we replace here $\xi$ by characteristic function $\chi_\delta$ of an arbitrary $\delta \in A$. We get

$$\forall \delta \in A \quad \mu \delta = \int_\delta \mu \, dv,$$

where $\mu_\delta := \eta$ is proved to be the Radon-Nikodym derivative of $\mu$ with respect to $\nu$.  

REFERENCES


The Trajectory Attractor for a Nonlinear Elliptic System in a Cylindrical Domain with Piecewise Smooth Boundary

Consider a nonlinear elliptic system of the form

\[ \begin{cases} \Delta u - \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial y_i} = 0, \\ \frac{\partial u}{\partial y_i} + \sum_{i=1}^{n} c_i u = 0, \\ u = 0 \quad \text{on } \partial \Omega, \end{cases} \]

where \( \Delta u \) denotes the Laplacian of \( u \), \( a_i, b_i, c_i \) are coefficients, and \( \partial \Omega \) is the boundary of the domain \( \Omega \). The system describes the evolution of the unknown function \( u \) over time in a cylindrical domain with piecewise smooth boundary.

The behavior of the solutions to such problems is crucial in understanding the long-term dynamics of the system.