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Analytic Continuation of the Solutions of Linear Partial Differential Equations (**)(***)

SUMMARY. — We give, for a class of vectorial linear partial differential equations, some theorems of analytic continuation of the solutions, which can be multivalued in open, bounded or unbounded, multiply connected sets of any finite connection order; in a particular case we obtain the analytic continuation to the whole set of analyticity of the coefficients and the known terms. The theorems obtained extend a result by G. Johnsson ([11]) related to the scalar case; the proofs utilize the «globalizing method» due to L. Hörmander and the classical method of the majorant functions.

Prolungamento analitico delle soluzioni di equazioni differenziali lineari a derivate parziali

RIASSUNTO. — Si danno, per una classe di equazioni differenziali a derivate parziali lineari vettoriali, alcuni teoremi di prolungamento analitico delle soluzioni, che possono essere polidrome in campi limitati o illimitati, molteplicemente connessi di ordine di connessione finito, qualsiasi; in un caso particolare si ottiene il prolungamento analitico a tutto il campo di analiticità dei coefficienti e dei termini noti. I teoremi ottenuti estendono un risultato di G. Johnsson ([11]) relativo al caso scalare; le dimostrazioni utilizzano un «metodo globalizzante» dovuto a L. Hörmander e il classico metodo delle funzioni maggioranti.

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1. - Introduction

a) We will study the following vectorial linear equation:

$$\frac{\partial^m z}{\partial y^m} = \sum_{|k|=1}^{m} x^k \lambda_k^{(m)}(x, y) \frac{\partial^m z}{\partial x^k \partial y^{m-|k|}} + \sum_{b=1}^{m} \sum_{|k|=0}^{m-b} \lambda_k^{(m-b)}(x, y) \frac{\partial^{m-b} z}{\partial x^k \partial y^{m-b-|k|}} + f(x, y)$$

with $m \geq 1$ and with the following notations:

$$x = \{x_1, x_2, \dots x_N\} \in \mathbf{C}^N, \quad (x, y) \in \mathbf{C}^N \times \mathbf{C}, \quad k = \{k_1, k_2, \dots k_N\},$$

$$x^k = x_1^{k_1}, x_2^{k_2}, \dots x_N^{k_N}, \quad |k| = k_1 + k_2 + \dots + k_N, \quad \frac{\partial^{|k|} z}{\partial x^k} = \frac{\partial^{|k|} z}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_N^{k_N}},$$

$z = \{z_1, z_2, \dots z_M\} \in \mathbf{C}^M$.

Set, moreover:

$$0 < \alpha, \beta, \gamma, \delta \leq +\infty, \quad R_\alpha = \prod_{i=1,\dots,N} \{x_i : |x_i| < \alpha\}, \quad S_\beta = \{y : |y| < \beta\},$$

$$S_\beta(\bar{y}) = \{y : |y - \bar{y}| < \beta\} \quad (S_\beta = S_\beta(0)), \quad S_{\gamma,\delta}(\bar{y}) = \{y : \gamma < |y - \bar{y}| < \delta\},$$

Ω = an arbitrary set in the y-plane, open and connected of any finite connection order;

$\mathcal{O}(A)$ = the set of the holomorphic functions in an open simply connected set $A \subset R_\alpha \times \Omega$;

$\tilde{\mathcal{O}}(B)$ = the set of the holomorphic (not necessarily univalued) functions in an open multiply connected set $B \subset R_\alpha \times \Omega$;

$d(C, D)$ = distance of the set C, D in the y-plane.

b) The present paper follows to previous studies ([3],...,[11]) on the analytic continuation of the solutions of a given type of partial linear equations in the analytic field.

We shortly recall these previous results leaving out [3], [4] and [5] which furnished only the starting point to these researches.

Paper [6] (which extends the results obtained in [5]) concerns the following linear equation:

(1.1) $\qquad z_y = x^n A(y)\, z_x + B(x, y)\, z + f(x, y) \qquad (z = [z_i],\ n \geq 1)$

assuming the matrix of coefficients $A(y)$ holomorphic in Ω, the matrix of coefficients $B(x, y)$ and the known term $f(x, y)$ holomorphic in $R_\alpha \times \Omega$.

We proved then the following property: *let* $z(x, y)$ *be a branch of any integral which is holomorphic (and necessarily univalued) in the simply connected set* $R_a \times S_\beta^*(\bar{y})$ ($\bar{y} \in \Omega$ is fixed and correspondingly, the univalued branches of the coefficients and the known terms are fixed) *and consider an arbitrary bounded open set* $\Omega' \subset \overline{\Omega}' \subset \Omega$; *there exists then a number* $\alpha(\Omega') \in (0, a]$ *such that the integral branch* $z(x, y)$ *can be analytically continued on the whole of* $R_{\alpha(\Omega')} \times \Omega'$.

Subsequently, in [7], we obtain (again for (1.1)) a further results: *assume the matrix* $A(y)$ *holomorphic in an open set* $\tilde{\Omega} \supset \overline{\Omega}$, $B(x, y)$ *and* $f(x, y)$ *holomorphic in* $R_a \times \Omega$; *we can then associate such a number* $\alpha(\Omega')$ *to every bounded open set* $\Omega' \subsetneq \Omega$ *(instead of* $\overline{\Omega}' \subset \Omega$*) provided that the boundary* $\partial\Omega'$ (with exclusion, at most, of a finite number of isolated points) *is of class* C^1.

A simple example (with a polar singularity on $\partial\Omega$) shows that *this result does not hold, in general, if* $A(y)$ *is holomorphic only in* Ω (cfr. [6] Osservazione 1 pg 30).

It follows that *the singularities of the matrix* $A(y)$ *influence the set of analyticity of the solution more strongly than the singularities of* $B(x, y)$ *and* $f(x, y)$; precisely we cannot approach the singularities of $A(y)$ how much we want, while we can indefinitely approach the singularities of $B(x, y)$ and $f(x, y)$.

The authors of [7] attribute this result to the fact that the characteristics depend only on the leading matrix $A(y)$, but they did not explain the essential role of these characteristics.

Successively, in [8] and [9] *we generalized* the first type of results ($\overline{\Omega}' \subset \Omega$) to the more general equation:

$$(1.2) \quad z_{yy} + \sum_{i=1}^{N} x_i \lambda_i(x, y)\, z_{x_i y} + \mu(x, y)\, z_y + \nu(x, y)\, z + f(x, y) =$$

$$= \sum_{ij=1}^{N} x_i x_j a_{ij}(x, y)\, z_{x_i x_j} + \sum_{i=1}^{N} x_i b_i(x, y)\, z_{x_i} \quad (z = [z_\ell],\ \ell = 1, \dots M)$$

assuming the matrix of coefficients $\lambda_i(x, y)$, $\mu(x, y)$, $\nu(x, y)$, $a_{ij}(x, y)$, $b_i(x, y)$ $(i, j = 1, 2, \dots N)$ *and the known term* $f(x, y)$ *holomorphic in* $R_a \times \Omega$.

A subsequent improvement has been obtained by G. Johnsson in [11] (with reference to [6], [7], [8]) for a scalar equation ($M = 1$) of m-order of analogous type.

The author gave in fact a new sufficient condition in order *to continue the solutions as near as we want to the singular points of the leading coefficients: he assume precisely, for the equation* (1.2), *that the coefficients and the known term are holomorphic in* $R_a \times \Omega$ *and moreover the leading coefficients* $\lambda_i(x, y)$ *and* $a_{ij}(x, y)$ *are bounded there.*

It is then possible to continue analytically every branch of a solution from the simply-connected set $R_a \times S_\beta^*(\bar{y})$ *to the whole of* $R_{\alpha(\Omega')} \times \Omega'$, $\forall \Omega' \subsetneq \Omega$ *(instead of* $\overline{\Omega}' \subset \Omega$*) provided that the boundary* $\partial\Omega'$ *satisfies some hypotheses of regularity* (cfr. [11]

Th. 4.4, previously printed in Pre-prints of the Dep. of Math. of the Royal Institute of tecnology - Stockholm 1989).

Observe that, by the example given in [6] *the hypothesis of boundedness cannot be completely eliminated.*

In [10] *we extended,* at first, *these results to the following vectorial equation* ($z = [z_r]$, $m \geqslant 1$)

$$(1.3)\quad \frac{\partial^m z}{\partial y^m} = \sum_{|k|=1}^{m} x^k \lambda_k^{(m)}(x, y) \frac{\partial^m z}{\partial x^k \partial y^{m-|k|}} + \\ + \sum_{|k|=0}^{m-1} x^k \lambda_k^{(m-1)}(x, y) \frac{\partial^{m-1} z}{\partial x^k \partial y^{m-1-|k|}} + \ldots + \sum_{|k|=0}^{1} x^k \lambda_k^{(1)}(x, y) \frac{\partial z}{\partial x^k \partial y^{1-|k|}} + \\ + \lambda_0^{(0)}(x, y) z + f(x, y).$$

Subsequently we examined for (1.3) the case where *the coefficients and the known term are holomorphic in* $R_\alpha \times \Omega$ *with* Ω *open, multiply connected of any finite connection order, which contains the exterior part of a circle* $\{y: 0 < \beta < |y - y^*| < + \infty\}$.

The leading coefficients, moreover, are supposed not only bounded in $R_\alpha \times \Omega$, *but also infinitesimal when* $|y| \to +\infty$ *of a sufficiently high order. There exists, then,* $\forall \Omega' \subsetneq \Omega$, *open, bounded or unbounded set, a number* $\alpha(\Omega') \in (0, \alpha]$ *such that* $z(x, y) \in \tilde{\mathcal{O}}\,(R_{\alpha(\Omega')} \times \Omega')$.

c) *In the present paper we extend,* at first, *the statements of* [10] *to the equation:*

$$(1.4)\quad \frac{\partial^m z}{\partial y^m} = \sum_{|k|=1}^{m} x^k \lambda_k^{(m)}(x, y) \frac{\partial^m z}{\partial x^k \partial y^{m-|k|}} + \\ + \sum_{|k|=0}^{m-1} \lambda_k^{(m-1)}(x, y) \frac{\partial^{m-1} z}{\partial x^k \partial y^{m-1-|k|}} + \ldots + \sum_{|k|=0}^{1} \lambda_k^{(1)}(x, y) \frac{\partial z}{\partial x^k \partial y^{1-|k|}} + \\ + \lambda_0^{(0)}(x, y) z + f(x, y) \quad (z = [z_r],\ m \geqslant 1).$$

Equation (1.4) *is different from* (1.3) *for the coefficients of the derivatives of order* $< m$, *which are completely arbitrary: the factor* x^k *for the coefficients of the derivatives of order* $|k|$ $(1 \leqslant |k| \leqslant m)$ *with respect to* x, *appear only for the leading coefficients.*

The new proof is more simple than the previous; we utilize the «globalizing method» (due to Hormander and previously applied by Johnsson in [11]) and the classical method of the majorant functions in different ways.

Subsequently *we consider the equation*

$$\frac{\partial^m z}{\partial y^m} = \sum_{|k|=1}^{m} x^k \lambda_k^{(m)}(y) \frac{\partial^m z}{\partial x^k \partial y^{m-|k|}} + \tag{1.5}$$

$$+ \sum_{|k|=0}^{m-1} \lambda_k^{(m-1)}(x, y) \frac{\partial^{m-1} z}{\partial x^k \partial y^{m-1-|k|}} + \ldots + \sum_{|k|=0}^{1} \lambda_k^{(1)}(x, y) \frac{\partial z}{\partial x^k \partial y^{1-|k|}} +$$

$$+ \lambda_0^{(0)}(x, y) z + f(x, y) \quad (z = [z_l] \; l = 1, \ldots M, \; m \geq 1)$$

((1.5) is different from (1.4) only for the leading coefficients) *with coefficients and known term holomorphic in $R_\infty \times \Omega$ and we prove that every branch of a solution holomorphic in $R_\infty \times S_{\bar\beta}(\bar y)$ is analytically prolongeable on the whole of $R_\infty \times \Omega$* (as in the case of the linear ordinary equations).

Observe that we previously obtained this last result in the case $N = m = 1$ in [5], [6], [7].

We enunciate, in conclusion, the following theorems.

THEOREM I: *Assume that, in equations* (1.4) *the coefficients and the known term satisfy the following conditions:*

$$\lambda_k^{(h)}(x, y) \quad (h = 1, 2, \ldots m), \; f(x, y) \in \tilde{\mathcal{O}}(R_\alpha \times \Omega) \tag{1.6}$$

$$\lambda_k^{(m)}(x, y) \textit{ are bounded in } R_\alpha \times \Omega\,. \tag{1.7}$$

Then, if $z(x, y) \in \mathcal{O}(R_\alpha \times S_{\bar\beta}(\bar y))$ ($\bar y \in \Omega$ is fixed and, corrispondingly, the univalued branches of the coefficients and the known terms are fixed, $\bar\beta \leq d(\bar y, \partial\Omega)$) *is an univalued branch of a solution of* (1.4), *taken an arbitrary bounded open set $\Omega' \subsetneq \Omega$ with $\partial\Omega'$ of class C^1* (with the exclusion, at most, of a finite number of isolated points) *there exists a number $\alpha(\Omega') \in (0, \alpha]$ such that $z(x, y)$ can be analytically continued on the whole of $R_{\alpha(\Omega')} \times \Omega'$.*

THEOREM II: *Assume that, in equation* (1.4), (1.6) *and* (1.7) *hold.*
Moreover let $y^* \notin \Omega$ be such that $\Omega \supset S_{\beta, +\infty}(y^*)$ and

$$\lambda_k^{(m)}(x, y) = \frac{1}{(y - y^*)^2} \sum_{h=0}^{+\infty} \lambda_{kh}^{(m)}(x)(y - y^*)^{-h} \quad \textit{for } |k| = 1$$

$$\lambda_k^{(m)}(x, y) = \frac{1}{(y - y^*)^4} \sum_{h=0}^{+\infty} \lambda_{kh}^{(m)}(x)(y - y^*)^{-h} \quad \textit{for } |k| = 2$$

$$\vdots$$

$$\lambda_k^{(m)}(x, y) = \frac{1}{(y - y^*)^{2m}} \sum_{h=0}^{+\infty} \lambda_{kh}^{(m)}(x)(y - y^*)^{-h} \quad \textit{for } |k| = m\,.$$

Then, if $z(x, y) \in \mathcal{O}(R_\alpha \times S_{\bar\beta}(\bar y))$ ($\bar y \in \Omega$ is fixed and corrispondingly, the univalued branches of the coefficients and the known terms are fixed, $\bar\beta \leq d(\bar y, \partial\Omega)$) *is an univalued branch of a solution of* (1,4), *taken an arbitrary open, bounded or unbounded set* $\Omega' \subseteq \Omega$ *with* $\partial\Omega'$ *of class* C^1 (with the exclusion at most, of a finite number of isolated points) *there exists a suitable number* $\alpha(\Omega') \in (0, \alpha]$ *such that* $z(x, y)$ *can be analytically continued on the whole of* $R_{\alpha(\Omega')} \times \Omega'$.

THEOREM III: *Assume that, in the equation* (1.5) *the coefficients and the known terms satisfy the following condition:*

(1.8) $\lambda_k^{(m)}(x, y) = \lambda_k^{(m)}(y)$ *are independent of* x *and* $\lambda_k^{(m)}(y) \in \tilde{\mathcal{O}}(\Omega)$

(1.9) $\lambda_k^{(h)}(x, y)$ $(h = 1, \dots m-1)$, $f(x, y) \in \tilde{\mathcal{O}}(R_{+\infty} \times \Omega)$.

Then if $z(x, y) \in \mathcal{O}(R_{+\infty} \times S_{\bar\beta}(\bar y))$ ($\bar y \in \Omega$ is fixed and, corrispondingly, the univalued branches of the coefficients and of the known terms are fixed, $\bar\beta \leq d(\bar y, \partial\Omega)$) *is an univalued branch of a solution of* (1,5), $z(x, y)$ *can be analytically continued on the whole of* $R_{+\infty} \times \Omega$.

The following example shows that some hypotheses of the enunciated theorems cannot be completely eliminated.

The Cauchy problem $\begin{cases} z_y = (x^2/y)\, z_x \\ z(x, 1) = x \end{cases}$ has the solution $z(x, y) = x/(1 - x \log y)$ singular in the points $(1/\log y, y)$ $\forall y \in S_{0,\infty}(0)$; the considered equation is of type (1.4) with $m = 1$ and $\lambda_1^{(1)}(x, y) = x/y$.

We have that $\underset{y \in S_{0,\delta}(0)}{\mathrm{Inf}} |1/\log y| = 0$ $(\delta > 0)$; then it follows that the hypothesis of the Theorem I of boundness of $\lambda_1^{(1)}(x, y)$ cannot be eliminated.

We have that $\underset{y \in S_{\gamma,\infty}(0)}{\mathrm{Inf}} |1/\log y| = 0$ $(\gamma > 0)$; that if follows that the hypothesis of the Theorem I of boundess of Ω' and, moreover the hypothesis of the Theorem II that $\lambda_1^{(1)}(x, y)$ must be infinitesimal of second order for $y \to \infty$ cannot be eliminated.

Finally $z(x, y) \notin \tilde{\mathcal{O}}(R_\infty \times S_{0,\infty}(0))$; then it follows that the hypothesis of the Theorem III that $\lambda_1^{(1)}(x, y)$ must be independent of x cannot be eliminated.

We can apply to the equation considered only the Theorem I with $\Omega = S_{\gamma,\infty}(0)$ $(\gamma > 0)$ and $\Omega' \subseteq S_{\gamma,\delta}(0)$ $(\gamma, \delta > 0)$; if $\alpha(\Omega') = \underset{y \in S_{\gamma,\delta}(0)}{\mathrm{Inf}} |1/\log y|$ we have that $z(x, y) \in \tilde{\mathcal{O}}(R_{\alpha(\Omega')} \times \Omega')$.

2. - PROOFS OF THEOREMS I, II, III

a) Let us recall, for the reader's convenience, the following definitions and statements which refer to the equation (1.4) for $N = M = 1$.

DEFINITION 1: *A vector* $N(\xi, \tau) \in \mathbb{C}^2$, $N \neq (0, 0)$ *is said to be characteristic with* *respect to* (1.4) *at the point* $(x_0, y_0) \in \mathbb{C}^2$ *if* $N(\xi, \tau)$ *solves the characteristic equation*

$$\tau^m + \sum_{k=1}^{m} x_0^k \lambda_k^{(m)}(x_0, y_0)\, \xi^k \tau^{m-k} = 0\,. \tag{2.1}$$

DEFINITION 2: *Let* S *be a surface in* $\mathbb{C}^2$ *defined by the equation* $\varphi(x, y) = 0$, *with* φ *analytic.*

S is said to be characteristic with respect to (1.4) *at a point* (x_0, y_0) *if* $N(\varphi_x(x_0, y_0), \varphi_y(x_0, y_0)) \neq (0, 0)$ *is a vector characteristic with respect to* (1.4) *at the point* (x_0, y_0); *S is said to be non-characteristic with respect to* (1.4) *at the point* (x_0, y_0) *if* $N(\varphi_x(x_0, y_0), \varphi_y(x_0, y_0))$ *does not solve* (2.1); *that implies* $N \neq (0, 0)$.

This last definition can be extended, formally in the same way, to the case of a surface S defined by an equation $\psi(x, y) = 0$, with real ψ, of class C^1.

DEFINITION 3 (¹): *The surface* $S \subset \mathbb{C}^2$ *defined by an equation* $\psi(x, y) = 0$ *with real* ψ, *of class* C^1, *is said to be characteristic (or Zerner characteristic) at a point* (x_0, y_0), *if the vector* $N(\psi_x(x_0, y_0), \psi_y(x_0, y_0)) \neq (0, 0)$ *solves* (2.1).

DEFINITION 4: *Let* $V \subset \mathbb{C}^2$. *Let* H *be a closed half-space in* $\mathbb{C}^2$ *and* h *the corresponding real hyperplane which is boundary of* H.

The complex normal cone of V *at* $(x_0, y_0) \in \partial V$, $N_v(x_0, y_0)$, *is defined as the closure of the set* $\{N_H: N_H$ *is the complex normal of* $h = \partial H$ *such that* $(x_0, y_0) \in h$ *and for a suitable open neighbourhood* Ω_H *of* (x_0, y_0) $V \cap \Omega_H \subset H \cap \Omega_H\}$.

Theorem of Zerner (²) *Assume that:*

– *the coefficients and the known term of* (1.4) *are bolomorphic in the open set* $R_a \times \Omega$;

– $z(x, y)$ *is a solution holomorphic in an open set* $V \subset R_a \times \Omega$;

– $(x_0, y_0) \in \partial V \cap \{R_a \times \Omega\}$;

– ∂V *is of class* C^1 *and non-characteristic at* (x_0, y_0).

Then $z(x, y)$ *can be analytically continued to a suitable neighborhood of* (x_0, y_0).

When the boundary is non-smooth we can utilize the following theorem of local-continuation.

Theorem of Bony-Shapira (³) *Assume that:*

(¹) Cfr. [12] pp. 349-350.
(²) Cfr. [12] Th. 9.4.7, p. 350.
(³) Cfr. [13] Th. 4.2.

– *the coefficients and the known term of* (1.4) *are holomorphic in a neighbourhood of an open convex cone* $\Gamma \subset \mathbb{C}^2$ *with vertex* (x_0, y_0);

– $z(x, y)$ *is a solution of* (1.4), *holomorphic in* Γ;

– *the complex normal cone of* Γ *at* (x_0, y_0) *does not include characteristic directions.*

Then $z(x, y)$ *can be analytically continued to a suitable neighbourhood of* (x_0, y_0).

Before proving the theorems given in § 1 we can observe that the solutions of (1.4) and (1.5) with $m = 1$ satisfy necessarily equations of the same type but of higher order; it is therefore sufficient to prove all the theorems for $m \geq 2$.

b) Proof of Theorem I: b_1) We assume firstly $m \geq 2$, $N = M = 1$ and we refer, for the symbols, to (1.4); in this case we can easily prove the following lemma which is fundamental for both the Theorems I and II.

Lemma: *Assume that:*

– $z(x, y)$ *be a branch of any integral which is holomorphic (and necessarily univalued) in the simply connected set* $R_\alpha \times S_{\bar\beta}(\bar y) \subset R_\alpha \times \Omega$,

– $y \in S_{\bar\beta}(\bar y)$,

– $0 < \beta \leq d(y, \partial\Omega)$.

Then $z(x, y)$ *is analytically continuable to* $R_{\alpha(\beta)} \times \{S_{\bar\beta}(\bar y) \cup S_\beta(y)\}$ *with suitable* $\alpha(\beta) \in (0, \alpha]$ *independent of* y *and dependent only of* β.

To prove the Lemma we can obviously suppose that $y = 0$ (the translation which move $(0, y)$ into $(0, 0)$ does not change the type of equation); consequently we will prove (by the following propositions 1) 2) 3) 4) 5)) that $z(x, y) \in \mathcal{O}(R_\alpha \times S_{\beta'})$ (with arbitrary β' such that $S_{\beta'} \subset S_{\bar\beta}(\bar y)$) is analitically continuable to $R_{\alpha(\beta)} \times S_\beta$ with arbitrary β such that $S_\beta \subset \Omega$ and suitable $\alpha(\beta) \in (0, \alpha]$ dependent only of β.

1) *Let* $K > \max\{\sqrt[k]{|\lambda_k^m(x, y)|}\, k = 1, \dots m\}$ $((x, y) \in R_\alpha \times \Omega)$ *and let* $N(\xi, \tau) \neq (0, 0)$ *be a characteristic vector with respect to* (1.4) *at the point* $(x_0, y_0) \in R_\alpha \times \Omega$; *we have then:* $|\tau/\xi| < mK|x_0|$. (4).

[*Indeed if* $(\xi, \tau) \neq (0, 0)$ *is characteristic, we have* $\xi \neq 0$ ($\xi = 0$ *implies, by* (2.1), *that* $\tau = 0$); *dividing* (2.1) *by* $|\xi|^m$ *and setting* $\eta = \tau/|\xi|$, *we have*

$$\eta^m - \sum_{k=1}^{m} x_0^k \lambda_k^{(m)}(x_0, y_0) \left(\frac{\xi}{|\xi|}\right)^k \eta^{m-k} = 0 .$$

(4) Cfr. [11] Lemma 4.1.

For $|\eta| = mK|x_0|$ *we have*

$$\left| \sum_{k=1}^{m} x_0^k \lambda_k^m(x_0, y_0) \left(\frac{\xi}{|\xi|} \right)^k \eta^{m-k} \right| \leq (mK|x_0|)^m = |\eta|^m$$

then (*for the Theorem of Rouché*) *if* η_j $(j = 1, \ldots m)$ *are solutions* $|\eta_j| < mK|x_0|$ $(j = 1, \ldots m)$].

2) *The set*

$$V_b = \{(x, y) \in \mathbf{C}^2 \colon |x| < \alpha e^{-mK|y|},\ 0 \leq |y| \leq b\} \quad b \in (0, \beta)$$

has boundary ∂V_b *non-characteristic with respect to* (2.1) *at any point* (x_0, y_0) *with* $y_0 \neq 0$.

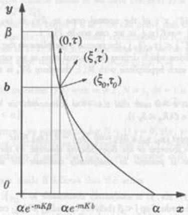

Fig. 1.

[The Fig. 1, which contains the intersection of V_b with $\{(x, y) \in \mathbf{R}^2_+\}$ shows that ∂V_b consists, for $|y| \neq 0$, of two parts; the first one has equation

$$\psi(x, y) = \alpha e^{-mK|y|} - |x| = 0 \quad (0 < |y| \leq b).$$

If $(x_0, y_0) \neq (x_0, 0)$ belongs to this part the normal $N(\xi, \tau)$ satisfies the condition:

$$\left| \frac{\tau}{\xi} \right| = \left| \frac{\psi_y(x_0, y_0)}{\psi_x(x_0, y_0)} \right| = \left| \frac{\alpha e^{-mK|y_0|}(-mK(y_0/|y_0|))}{-x_0/|x_0|} \right| = mK\alpha e^{-mK|y_0|} = mK|x_0|$$

and therefore ∂V_b is non-characteristic at (x_0, y_0).

The second part of ∂V_b consists of the points (x_0, y_0) with $|y_0| = b$ and $|x_0| \leq \alpha e^{-mKb}$.

If (x_0, y_0) belongs to this part and $|x_0| < \alpha e^{-mKb}$ the normal vector (ξ, τ) will be of the type $(0, 1)$ and therefore does not satisfy (2.1); if, on the contrary $|x_0| = \alpha e^{-mKb}$ and (ξ_0, τ_0) is the vector limit of (ξ, τ) normal to the surface $|x| = \alpha e^{-mK|y|}$ at the points (x, y) when $(x, y) \to (x_0, y_0)$ then

$$\left| \frac{\tau_0}{\xi_0} \right| = mK\alpha e^{-mK|y_0|} = mK|x_0| ;$$

the other vectors (ξ', τ') of the normal cone to ∂V_b in (x_0, y_0) are such that $|\tau'/\xi'| > |\tau_0/\xi_0| = mK|x_0|$ as we can see in Fig. 1.

(In fact, if $|\tau'/\xi'| < |\tau_0/\xi_0|$, the normal hyperplane to the vector (ξ', τ') cannot satisfy the condition which defines the normal cone as we can see for the straight line intersection of such hyperplane with $\mathbb{R}^2_+$); therefore ∂V_b is non-characteristic at (x_0, y_0)].

3) *There exists $\bar{b} > 0$ such that $z(x, y)$ is holomorphic in $V_{\bar{b}}$.*
[Indeed $z(x, y) \in \mathcal{O}(R_\alpha \times S_{\beta'})$].

4) *To every point $(x_0, y_0) \in \partial V_{\bar{b}}$ with $|y_0| = \bar{b}$, $|x_0| \leq \alpha e^{-mK|y_0|} = \alpha e^{-mK\bar{b}}$, we can apply the Zerner theorem or the Bony-Shapira theorem; by the compactness of this part of boundary it is possible to analytically continue the solution to V_b with $b > \bar{b}$.*

5) *Let $J = \{|y| : z(x, y)$ is analytically continuable to $V_{|y|}\}$; it results* $\operatorname{Sup} J = \beta$.
[Indeed if, ab absurdo, $\operatorname{Sup} J < \beta$ then the boundary $\partial V_{\operatorname{Sup} J}$ contains, at most, one point y_0 with $|y_0| = \operatorname{Sup} J$ to which it is not possible to apply any of the two theorems of local continuation recalled; which is not true].

We conclude therefore that $z(x, y) \in \mathcal{O}(R_{\alpha(\beta)} \times S_\beta)$ where $\alpha(\beta) = \alpha e^{-mK\beta}$ depends only on β (and obviously on α).

b_2) By the connection of Ω we obtain that $\forall S_\beta(y) \in \Omega$ there exists a suitable $\alpha(\beta) \in (0, \alpha]$ such that $z(x, y)$ is analytically continuable to $R_{\alpha(\beta)} \times S_\beta(y)$ and univalued in an open simply connected set of the type

$$R_{\alpha(\beta)} \times \{S_{\beta_1=\bar{\beta}}(y_1 = \bar{y}) \cup S\,\beta_2(y_2) \cup \ldots \cup S_{\beta_n=\beta}(y_n = y)\}\,.$$

Let now $\Omega' \subset \overline{\Omega}' \subset \Omega$ open, bounded and simply connected; by the compactness of $\overline{\Omega}'$ we can cover $\overline{\Omega}'$ with a finite number of sets $S_\beta(y)$ and than we obtain a suitable $\alpha(\overline{\Omega}') \in (0, \alpha]$ such that $z(x, y) \in \mathcal{O}(R_{\alpha(\overline{\Omega}')} \times \overline{\Omega}')$ and necessarily univalued.

If $\Omega' \subset \overline{\Omega}' \subset \Omega$ is connected of a finite connection order we can cover Ω' with a finite number of simply connected sets $\Omega'_k (k = 1, \ldots n)$ such that $\Omega'_k \subset \overline{\Omega}'_k \subset \Omega$; than we have a suitable $\alpha(\Omega') \in (0, \alpha]$ such that $z(x, y) \in \tilde{\mathcal{O}}(R_{\alpha(\Omega')} \times \Omega')$ but non necessarily univalued.

Finally let $\Omega' \subsetneq \Omega$ open bounded and connected with $d(\partial\Omega', \partial\Omega) = 0$: we can prove the thesis of Theorem 1 utilizing the hypotheses on $\partial\Omega'$.

By these hypotheses we can construct a bounded open connected set Ω'' with $\Omega'' \subset \overline{\Omega}'' \subset \Omega' \subsetneq \Omega$ such that $\Omega' - \Omega''$ can be covered by discs contained in Ω' with center $\tilde{y} \in \Omega''$, of equal suitable radius β; we have that $z(x, y)$ is analytically continuable to $R_{\alpha(\overline{\Omega}'')} \times \overline{\Omega}''$ $(\overline{\Omega}'' \subset \Omega)$.

Consider now a disc with center $\tilde{y} \in \Omega''$; we can prove that there exists a suitable $\alpha(\beta) \in (0, \alpha]$ such that $z(x, y)$ is analytically continuable to $R_{\alpha(\beta)} \times (\Omega'' \cup S_\beta(\tilde{y}))$ (by proposition analogous to 1) 2) 3) 4) 5)); at the same time we can prove that $\alpha(\beta)$ does not depend on $\tilde{y}$ and then that $z(x, y) \in \tilde{\mathcal{O}}\,(R_{\alpha(\Omega') = \alpha(\beta)} \times \Omega')$.

b_5) The proof for equation (1.4) with $m \geq 2, N \geq 1, M = 1$ is identical to the case $N = 1$ after the proof of Lemma (it is sufficient to set $R_\alpha = \prod_{i=1,\ldots N} \{x_i\colon |x_i| < \alpha\}$).

To prove the Lemma we suppose (as for $N = 1$) $y = 0$, $z(x, y) \in \mathcal{O}(R_\alpha \times S_{\beta'})$ (with arbitrary β' such that $S_{\beta'} \subset S_{\bar{\beta}}(\bar{y})$); we will obtain that $z(x, y)$ is analytically continuable to $R_{\alpha(\beta)} \times S_\beta$ with arbitary β such that $S_\beta \subset \Omega$ and suitable $\alpha(\beta) \in (0, \alpha)$ dependent only of β.

By the hypotheses made if follows that the series

$$\lambda_k^{(m)}(x, y) = \sum_{b,n=0}^{+\infty} \lambda_{k,b,n}^{(m)} x^b y^n$$

$$\left(k = \{k_1, k_2, \ldots k_N\},\quad b = \{b_1, b_2, \ldots b_N\},\quad x^b = x_1^{b_1} x_2^{b_2} \ldots x_N^{b_N},\quad \sum_{p=h}^{k} = \sum_{p_1=h_1}^{k_1} \sum_{p_2=h_2}^{k_2} \ldots \sum_{p_N=h_N}^{k_N}\right)$$

converges in $R_\alpha \times S_\beta$ and analogously for the other coefficients and known term; moreover $z(x, y) = \sum_{h,n=0}^{+\infty} \zeta_{hn} x^h y^n$ converges in $R_\alpha \times S_{\beta'}$ and satisfies the following

Cauchy problem

$$
(2.2)\qquad \begin{cases} (1,4) \\ \dfrac{\partial^b z}{\partial y^b}(x,0) = \displaystyle\sum_{k=0}^{+\infty} \zeta_{kb} x^k \quad b = 0, 1 \dots m-1 . \end{cases}
$$

It is sufficient to construct a majorant problem with solution convergent in $R_{\alpha(\beta)} \times S_\beta$ to prove the thesis.

If we set

$$
\Lambda^m(x,y) = \sum_{h,n=0}^{+\infty} \left\{ \sum_{i=1}^{m} \sum_{|k|=i} |\lambda_{k,h,n}^{(m)}| \right\} x^h y^n
$$

analogously $\Lambda^{m-1}(x,y), \dots \Lambda^0(x,y), F(x,y) \in \mathcal{O}(R_\alpha \times S_\beta)$ we can consider the following Cauchy problem, which is majorant of (2.2)

$$
(2.3)\qquad \begin{cases} \dfrac{\partial^m Z}{\partial y^m} = \Lambda^{(m)}(x,y) \displaystyle\sum_{|k|=1}^{m} x^k \dfrac{\partial^m z}{\partial x^k \partial y^{m-|k|}} + \\ \qquad + \Lambda^{m-1}(x,y) \displaystyle\sum_{|k|=0}^{m-1} \dfrac{\partial^{m-1} z}{\partial x^k \partial y^{m-1-|k|}} + \dots + \Lambda^{(0)}(x,y) z + F(x,y) \\ \dfrac{\partial^b Z}{\partial y^b}(x,0) = \displaystyle\sum_{k=0}^{+\infty} |\zeta_{kb}| x^k \quad (b = 0, 1, \dots m-1). \end{cases}
$$

Observe now that setting $x_1 = x_2 = \dots = x_N = t$ in (2.3) the function $Z(t, t \dots t, y) = \tilde{Z}(t,y)$ satisfies the following Cauchy problem (for an equation of type (1.4) for $N = M = 1$):

$$
(2.4)\qquad \begin{cases} \dfrac{\partial^m \tilde{Z}}{\partial y^m} = \tilde{\Lambda}^{(m)}(t,y) \displaystyle\sum_{|k|=1}^{m} \binom{N+|k|-1}{|k|} t^{|k|} \dfrac{\partial^m \tilde{Z}}{\partial t^{|k|} \partial y^{m-|k|}} + \\ + \tilde{\Lambda}^{m-1}(t,y) \displaystyle\sum_{|k|=0}^{m-1} \binom{N+|k|-1}{|k|} \dfrac{\partial^{m-1} \tilde{Z}}{\partial t^{|k|} \partial y^{m-1-|k|}} + \dots + \tilde{\Lambda}^{(0)}(t,y) \tilde{Z} + \tilde{F}(t,y) \\ \dfrac{\partial^b \tilde{Z}}{\partial y^b}(t,0) = \displaystyle\sum_{i=0}^{+\infty} \left\{ \sum_{|k|=i} |\zeta_{kb}| \right\} t^i \quad (b = 0, 1, \dots m-1) \end{cases}
$$

where $\widetilde{A}^{(m)}, \widetilde{A}^{(m-1)}, \dots \widetilde{A}^{(0)}, \widetilde{F}$ are defined similarly to $\widetilde{Z}$ and $\binom{N+|k|-1}{|k|}$ is the number of N-ples $\{k_1, k_2, \dots k_N\}$ such that $k_1 + k_2 + \dots + k_N = |k|$.

Setting now:

$$Z(x, y) = \sum_{k, n=0}^{+\infty} \xi_{kn} x^k y^n, \quad \text{solution of (2.3)}, \tag{2.5}$$

$$\widetilde{Z}(t, y) = \sum_{t, n=0}^{+\infty} \left\{ \sum_{|k|=t} \xi_{kn} \right\} t^t y^n, \quad \text{solution of (2.4)}, \tag{2.6}$$

there exists (by the previous proof for $N=1$) $\alpha(\beta) \in (0, \alpha]$ such that (2.6) and consequently (2.5) converges in $R_{\alpha(\beta)} \times S_\beta$ (infact $\forall \overline{\alpha} < \alpha(\beta)$ and $\overline{\beta} < \beta$ (2.6) converges absolutely and uniformly for $|t| \leq \overline{\alpha}$, $|y| \leq \overline{\beta}$; hence the series (2.5) converges absolutely and uniformly in $\overline{R_{\overline{\alpha}}} \times \overline{S_{\overline{\beta}}}$ with $R_{\overline{\alpha}} = \prod_{i=1,\dots N} \{x_i\colon |x_i| < \overline{\alpha}\}$) then the series (2.5) converges in $R_{\alpha(\beta)} \times S_\beta$ where $\alpha(\beta)$ does not depend on y.

b_4) If $m \geq 2$, $N \geq 1$, $M > 1$ the proof is formally the same after the proof of the Lemma.

To prove the Lemma we suppose (as for $M=1$) $y=0$, the vector $z(x, y) \in \mathcal{O}(R_\alpha \times S_{\beta'})$ (with arbitrary β' such that $S_{\beta'} \subset S_{\overline{\beta}}(\overline{y})$) and we will prove that $z(x, y)$ is analytically continuable to $R_{\alpha(\beta)} \times S_\beta$ with arbitrary β such that $S_\beta \subset \Omega$ and suitable $\alpha(\beta) \in (0, \alpha]$ depending only of β.

Consider now the matrices $\lambda_k^{(m)}(x, y) = [\lambda_k^{(m)(l_1, l_2)}(x, y)]$ $(l_1, l_2 = 1, \dots M)$ with $\lambda_k^{(m)(l_1, l_2)}(x, y) = \sum_{hn=0}^{+\infty} \lambda_{khn}^{(m)(l_1, l_2)} x^h y^n$ and set

$$\Lambda^{(m)}(x, y) = \sum_{hn=0}^{+\infty} \left\{ \sum_{l_1, l_2=1}^{M} \sum_{t=1}^{M} \sum_{|k|=t} |\lambda_{khn}^{(m)(l_1 l_2)}| \right\} x^h y^n \in \mathcal{O}(R_\alpha \times S_\beta);$$

analogously we will define $\Lambda^{(m-1)}, \dots \Lambda^0(x, y), F(x, y)$.

We can then associate to the Cauchy problem:

$$\begin{cases} (1.4) \text{ with } z = [z_l] \\ \dfrac{\partial^h z}{\partial y^h}(x, 0) = \left[\sum_{k=0}^{+\infty} \zeta_{k, h, l} x^k \right] \quad (l = 1, \dots M,\ h = 0, \dots m-1), \end{cases}$$

the following majorant Cauchy problem:

$$\begin{cases} \dfrac{\partial^m Z}{\partial y^m} = A^{(m)}(x,y) \displaystyle\sum_{|k|=1}^{m} x^k \begin{bmatrix} 1, & 1, & \dots & 1 \\ 1, & 1, & \dots & 1 \\ \cdot & \cdot & \dots & \cdot \\ 1, & 1, & \dots & 1 \end{bmatrix} \dfrac{\partial^m Z}{\partial x^k \partial y^{m-|k|}} + \\ \qquad + A^{m-1}(x,y) \displaystyle\sum_{|k|=0}^{m-1} \begin{bmatrix} 1, & 1, & \dots & 1 \\ 1, & 1, & \dots & 1 \\ \cdot & \cdot & \dots & \cdot \\ 1, & 1, & \dots & 1 \end{bmatrix} \dfrac{\partial^{m-1} Z}{\partial x^k \partial y^{m-1-|k|}} + \dots \\ \qquad \dots + A^{(0)}(x,y) \begin{bmatrix} 1, & 1, & \dots & 1 \\ 1, & 1, & \dots & 1 \\ \cdot & \cdot & \dots & \cdot \\ 1, & 1, & \dots & 1 \end{bmatrix} Z + F(x,y) \begin{bmatrix} 1 \\ 1 \\ \cdot \\ 1 \end{bmatrix}, \\ \dfrac{\partial^b Z}{\partial y^b}(x,0) = \phi^b(x) \begin{bmatrix} 1 \\ 1 \\ \cdot \\ 1 \end{bmatrix} \quad (b = 0, 1, \dots m-1) \end{cases}$$

where $\phi^b(x) = \sum_{k=0}^{+\infty} \left\{ \sum_{l=1}^{M} |\zeta_{k,b,l}| \right\} x^k \ (b = 0, \dots m-1)$.

It is easy to verify that the solution of this majorant Cauchy problem is a vector having all the rows equal to the function $\tilde{Z}(x, y)$ which is solution of the following scalar Cauchy problem for an equation of type (1.4)

$$(2.7) \quad \begin{cases} \dfrac{\partial^m \tilde{Z}}{\partial y^m} = M \Bigg\{ A^m(x,y) \displaystyle\sum_{|k|=1}^{m} x^k \dfrac{\partial^m \tilde{Z}}{\partial x^k \partial y^{m-|k|}} + \\ \qquad + A^{m-1}(x,y) \displaystyle\sum_{|k|=0}^{m-1} \dfrac{\partial^{m-1} \tilde{Z}}{\partial x^k \partial y^{m-1-|k|}} + \dots + A^{(0)}(x,y)\, \tilde{Z} \Bigg\} + F(x,y) \\ \dfrac{\partial^b \tilde{Z}}{\partial y^b}(x,0) = \phi^b(x) \quad (b = 0, 1, \dots m-1). \end{cases}$$

The hypotheses stated for the scalar equation (1.4) hold for (2.7); then the thesis precedently proved, holds too.

Therefore $\tilde{Z}(x, y)$ and the vectors $Z(x, y)$ and $z(x, y)$ are holomorphic in $R_{\alpha(\beta)} \times S_\beta$ with $\alpha(\beta) \in (0, \alpha]$ independent of y.

c) Proof of Theorem II: We suppose $m \geq 2$, $N \geq 1$, $M \geq 1$ and we refer to equation (1.4). We can obviously set $y^* = 0$.

Observe now that the change of variables

$$\begin{cases} X_i = x_i \\ Y = y^{-1} \end{cases} \quad (i = 1, 2, \dots N)$$

assuming $Z(X, Y) = z(X, Y^{-1})$ (and analogously for the coefficients and known term) transform (1.4) in the vector equation:

$$(2.8) \quad (-1)^m Y^{2m} \frac{\partial^m Z}{\partial y^m} = \sum_{|k|=1}^{m} (-1)^{m-|k|} Y^{2(m-|k|)} X^k A_k^{(m)}(X, Y) \frac{\partial^m Z}{\partial X^k \partial Y^{m-|k|}} +$$

$$+ \sum_{|k|=0}^{m-1} (-1)^{m-1-|k|} Y^{2(m-1-|k|)} A_k^{(m-1)}(X, Y) \frac{\partial^{m-1} Z}{\partial X^k \partial Y^{m-1-|k|}} + \dots$$

$$\dots + \sum_{|k|=0}^{1} (-1)^{1-|k|} Y^{2(1-|k|)} A_k^{(1)}(X, Y) \frac{\partial Z}{\partial X^k \partial Y^{1-|k|}} +$$

$$+ A^{(0)}(X, Y) Z(X, Y) + F(X, Y).$$

If we set (2.8) in normal form, the coefficients, sign excluded, are equal to

$$X^k A_k^{(m)}(x, y) Y^{-2|k|} \ (|k| = 1, \dots m), \quad A_k^{(m-1)}(X, Y) Y^{-2-2|k|} \ (|k| = 0, \dots m-1),$$

$$\dots A^{(0)}(X, Y) Y^{-2m};$$

by the hypotheses the leading coefficients are holomorphic, and then bounded, in $R_\alpha \times S_{1/\beta}$ while the other coefficients and the known term are, generally, holomorphic in $R_\alpha \times [S_{1/\beta} - \{0\}]$.

Let now $\Omega' \subseteq \Omega$ be the open set considered; if Ω' is bounded the thesis follows by the Th. I applied to (1.4).

If Ω' is unbounded we set $\Omega' = \Omega'_1 \cup \Omega'_2$ where $\Omega'_1 \subset \{\Omega \cap S_{\beta' > \beta}\}$ has boundary of class C^1, with exclusion, at most, of a finite number of isolated points.

The Th. I, applied to (1.4), gives the existence of a number $\alpha(\Omega'_1) \in (0, \alpha]$ such that $z(x, y) \in \tilde{C}(R_{\alpha(\Omega'_1)} \times \Omega'_1)$.

The change of variables above considered transforms $S_{\beta, +\infty}(0)$ to $S_{1/\beta}$; by Th. I applied to (2.8) $Z(X, Y)$ is analytically continuable to $R_{\alpha(1/\beta)} \times [S_{1/\beta} - \{0\}]$ and then, assuming $\alpha(\Omega') = \min(\alpha(\Omega'_1), \alpha(1/\beta))$ $z(x, y)$ will be analytically continuable to $R_{\alpha(\Omega')} \times \Omega'$.

b) Proof of Theorem III: We previously proved the thesis for $m = N = 1$, $M \geq 1$ (Cfr. [5] pg. 137, Teorema 1, [6] pg. 30, Teorema 3, [7] pg. 11, Teorema 1).

For the general case it is sufficient to prove the thesis for $m \geq 2$, $N = M = 1$, because we can reconduct all the other cases to this one by the method of the majorant functions as we did for Theorem I. We refer to (1.4) and to an arbitrary bounded open set $\Omega' \subset \overline{\Omega}' \subset \Omega$; we can prove the following lemma.

LEMMA: *Assume that*

– *$z(x, y)$ is a branch of any integral which is holomorphic (and necessarily univalued) in the simply-connected set* $R_\infty \times S_{\bar{\beta}(\bar{y})} \subset R_\infty \times \Omega'$;

– $y \in S_{\bar{\beta}}(\bar{y})$;

– $0 < \beta < d(y, \partial\Omega')$;

then $z(x, y)$ is analytically continuable to $R_\infty \times S_\beta(y)$.

Indeed the propositions 1) 2) 3) 4) 5) of b_1) hold $\forall \alpha \in R_+$ and $\alpha(\beta) = \alpha e^{-mK\beta}$ with $K > \max_{\Omega'}\{\sqrt[k]{|\lambda_k^m(y)|}\ (k = 1, \dots m)\}$ is independent of α; then $\operatorname{Sup} \alpha(\beta) = +\infty$ $(0 < \alpha < +\infty)$.

With a finite number of continuations, we obtain that $z(x, y)$ is holomorphic (not necessarily univalued) on the whole of $R_\infty \times \Omega'$ and finally, for the arbitrariness of Ω' on the whole of $R_\infty \times \Omega$.

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$$z_{yy} + \sum_{i=1}^{N} x_i \lambda_i(x,y)\, z_{x_i y} + \mu(x,y)\, z_y + \nu(x,y)\, z + f(x,y) = \sum_{ij=1}^{N} x_i x_j a_{ij}(x,y)\, z_{x_i x_j} + \sum_{i=1}^{N} x_i b_i(x,y)\, z_{x_i}$$

$(z = \{z_1, \dots z_M\} \in \mathbb{C}^M)$, Rend. Accad. Naz. Sci. XL Mem. Mat. 107°, vol XIII (1989).

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$$+ \sum_{|k|=0}^{1} x^k \lambda_k^{(1)}(x,y) \frac{\partial z}{\partial x^k \partial y^{1-|k|}} + \lambda_0^{(0)}(x,y)\, z + f(x,y) \quad (z = [z_i],\ m \geq 1),$$

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