Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
117° (1999), Vol. XXIII, fasc. 1, pagg. 57-82

MARCO BIROLI (*) - NICOLETTA ANNA TCHOU (**)

Nonlinear Subelliptic Problems with Measure Data (***)

SUMMARY. — We consider nonlinear subelliptic problems with measure data and we prove a generalization of an estimate given by Malý in the elliptic case. As consequence we give a proof of the necessity of Wiener criterion for the regularity of boundary points and a generalization to our subelliptic framework on a result given by Ziemer concerning the continuity of local solution of a nonlinear elliptic problem with measure data.

Problemi subellittici nonlineari con dato misura (***)

SUNTO. — Si considerano problemi sottoellittici non lineari con dato misura e si prova una generalizzazione di una stima data da Malý nel caso ellittico. Da tale risultato si deduce una prova della parte necessaria del criterio di Wiener per punti del bordo e una generalizzazione al caso subellittico di un precedente risultato di Ziemer concernente la continuità di soluzioni locali di un problema ellittico non lineare con dato misura.

1. - INTRODUCTION

The necessity part of the Wiener criterion for the regularity of boundary points in the case of nonlinear elliptic problems has been proved in a recent paper by Malý, [17][18], using an estimate on positive subsolutions of the problem. The goal of the present paper is to prove a generalization of the previous estimate to the subelliptic case. In the subelliptic case the sufficient part of the Wiener criterion for the regularity of boundary points has been proved in [9] for operators that are homogeneous of or-

(*) Indirizzo dell'autore: Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci, 32 - 20133 Milano, Italia. The A. has been supported by the MURST research project 9801262841.
(**) Indirizzo dell'autore: IRMAR, Université de Rennes 1, Beaulieu - 35042 Rennes Cedex, France.
(***) Memoria presentata il 2 marzo 1999 da Marco Biroli, socio dell'Accademia.

der $(p-1)$, in the general case we can also obtain a proof using the same methods considered in [6] in the case of relaxed Dirichlet problem for the subelliptic p-Laplacian (then we omit this part in the present paper); in the following we give a proof of the necessity part by the methods of Malý, [17], [18]. Moreover we deduce from our estimate also a criterion for continuity or Hölder continuity of a local solution to a nonlinear subelliptic problem with measure data; this last result generalizes a previous one in [20] concerning the Hölder continuity of the local solution to a nonlinear elliptic problem with measure data (see also [2] for the related notion of Kato measure in the nonlinear case).

We state now the result which is the main goal of the paper.

Let Ω be a bouned open set of R^N and X_i, $i = 1, \ldots, m$, be vector fields defined in R^N with C^∞ coefficients satisfying an Hörmander condition (i.e. the vector fields X_i and their commutators up to a fixed order span at every point R^N); we denote by X the vector $(X_1, \ldots, X_m)$.

We recall that we have a distance associated to the vector fields that may be defined as

$$d(x, y) = \sup\{\phi(x) - \phi(y);\ \phi \in C_0^\infty(R^N),\ |X\phi| \leq 1 \text{ a.e.}\}.$$

The distance d defines a topology equivalent to the euclidean one. We denote by $B(x, r)$ the set $\{y;\ d(x, y) < r\}$, [9], [19], [21].

For $x \in \Omega$ there exists R_0 such that for $s < r < R_0$ we have

$$|B(x, s)| \geq c_0 \left(\frac{s}{r}\right)^\nu |B(x, r)| \tag{1.1}$$

where c_0 is a constant depending on Ω, but independent of $s < r < R_0$, [9], [10], [19], [21] (here and in the following we denote, for a Lebesgue measurable set E, by $|E|$ the Lebesgue measure of E); we say that $\nu > 0$ is the (or an estimate of) intrinsic dimension of our problem.

We denote by $H^{1,p}(\Omega, X)$, $p \in (1, +\infty)$ the closure of $C^\infty(\Omega)$ for the norm

$$\|u\|_{1,p}^p = \int_\Omega |Xu|^p\, dx + \int_\Omega |u|^p\, dx$$

where, here and in the following, dx denotes the Lebesgue measure on R^N. By $H^{1,p}_{loc}(\Omega, X)$ we denote the space of functions in $H^{1,p}(O, X)$ for every relatively compact open set O with closure in Ω. The space $H_0^{1,p}(\Omega, X)$, $p \in (1, +\infty)$, is the closure of $C_0^\infty(\Omega)$ in $H^{1,p}(\Omega, X)$ and $H^{-1,p'}(\Omega, X)$, $1/p + 1/p' = 1$, denotes the dual space of $H_0^{1,p}(\Omega, X)$. Finally $H^{-1,p'}_{loc}(\Omega, X)$ denotes the space of all distributions on Ω, that are in $H^{-1,p'}(O, X)$ for every relatively compact open set O with closure in Ω.

We denote by $\mathcal{M}^q(R^N)$ the space of the function $f \in L^1(R^N)$ such that

$$\|f\|_{\mathcal{M}^q} = \sup_{B(x,r)} \frac{r^q}{|B(x,r)|} \int_{B(x,r)} |f| \, dx < +\infty$$

and by $\mathcal{M}^q(\Omega)$ the space of the functions whose extension by 0 to R^N is in $\mathcal{M}^q(R^N)$, $\|f\|_{\mathcal{M}^q(\Omega)}$ will be the norm in $\mathcal{M}^q(R^N)$ of the extension of f by 0 to R^N.

For $x \in \Omega$ there exists $\overline{R}_0$ such that for $r < \overline{R}_0$ the following scaled Poincaré inequality holds, [15] [16],

$$\int_{B(x,r)} |u - u_r|^p \, dx \leq c_1 r^p \int_{B(x,r)} |Xu|^p \, dx \tag{1.2}$$

where c_1 is a constant depending on Ω, but independent of $r < \overline{R}_0$. As consequence we have the Sobolev inequalities for the intrinsic dimension ν, [5], [6], [12].

We recall that a variational *p-capacity* of a set E relative to the open set Ω, $E \subset\subset \Omega$ may be defined (generalizing the usual Newtonian definition) as

$$p - \operatorname{cap}(E, \Omega) = \inf_{\{v \in C_0^\infty(\Omega),\ v = 1 \text{ in a neighbourhood of } E\}} \left\{ \int_\Omega |Xv|^p \right\}$$

and we refer for the properties of such a type of capacity to [8]. We denote $p - \operatorname{cap}(E) = p - \operatorname{cap}(E, R^N)$ and we recall that, if $\overline{E} \subset \Omega$, $p - \operatorname{cap}(E, \Omega) = 0$ iff $p - \operatorname{cap}(E) = 0$.

Using the same methods as in [8] Theorem 4.1 and the estimates on the cut-off functions between balls, see [1], we obtain easily

$$p - \operatorname{cap}(B(x_0, r), B(x_0, 2r)) \simeq \frac{r^p}{|B(x_0, r)|}$$

for $\overline{B(x_0, 4r)} \subseteq \Omega$, where Ω is a bounded open set and the constants may depend on Ω.

A function u defined in an open set O with $\overline{O} \subseteq \Omega$ is *quasi-continuous* if for every $\varepsilon > 0$ there exists an open set V with $p - \operatorname{cap}(V, \Omega) \leq \varepsilon$ such that u is continuous on $E \setminus V$; a function u in $H^{1,p}(O, X)$ has a quasi-continuous q.e. representative (i.e. there exists a quasi-continuous function $\tilde{u}$ such that $\tilde{u} = u$ up to sets of zero p-capacity); the proof follows the same methods as in [13] for the case $p = 2$. In the following we identify u with its quasi-continuous representative.

Let now $A: \Omega \times R \times R^m \to R^m$ and $B: \Omega \times R \times R^m \to R$ be (Lebesgue) measurable functions such that the following structure conditions hold:

$$\text{(1.3)}\qquad \begin{cases} |A(x,\zeta,\xi)| \leqslant a_1|\xi|^{p-1} + a_2|\zeta|^{p-1} + a_3 \\ |B(x,\zeta,\xi)| \leqslant b_0|\xi|^p + b_1|\xi|^{p-1} + b_2|\zeta|^{p-1} + b_3 \\ A(x,\zeta,\xi).\ \xi \geqslant c_1|\xi|^p - c_2|\zeta|^p - c_3, \end{cases}$$

a.e. in Ω for every $\zeta \in R$, $\xi \in R^m$ and $p \in (1, \nu]$.

Here c_1 is a positive constant, a_1 and b_0 are nonnegative constants and

$$\text{(1.4)}\qquad a_2, a_3 \in L^{\nu/(p-1)}(R^N)$$

if $p \in (1, \nu)$ or

$$\text{(1.4')}\qquad a_2, a_3 \in L^{(\nu+\sigma)/(\nu-1)}(R^N)$$

where $\sigma > 0$, if $p = \nu$,

$$b_1^p, b_2, b_3, c_2, c_3 \in \mathcal{M}^{p-\varepsilon}(R^N)$$

$$b_1^p, b_2, c_2 \in L^{\nu q/(\nu\varepsilon+p)\varepsilon}(R^N)$$

where $q > p$ is a fixed exponent such that $q < \nu p/(\nu - p)$ and ε is defined by the relation

$$\varepsilon = (\nu - p)\left(\frac{q}{p} - 1\right).$$

We also define τ and γ by

$$\frac{1}{\tau} = \frac{p-1}{q} + \frac{1}{p}, \qquad \gamma = (p-1)\,\tau.$$

We assume

$$\sup_{x\in\overline{\Omega}}[r^{\gamma}\|b_1^p + b_2 + c_2\|_{\mathcal{M}^{p-\varepsilon}(B(x,r))}] \leqslant C$$

Let $a(r)$, $b(r)$, $\kappa(r)$ be defined as

$$a(r) = \sup_{x\in\overline{\Omega}}\|a_2\|_{L^{\nu/(p-1)}(B(x,r))}$$

if $p \in (1, \nu)$ or

$$a(r) = \sup_{x\in\overline{\Omega}}\|a_2\|_{L^{(\nu+\sigma)/(\nu-1)}(B(x,r))}$$

if $p = \nu$,

$$b(r) = \sup_{x \in \overline{\Omega}} [(r^{\varepsilon} \|b_1^p + b_2\|_{\mathcal{M}^{p-\varepsilon}(B(x,r))})^{1/(p-1)} + (r^{\varepsilon} \|c_2\|_{\mathcal{M}^{p-\varepsilon}(B(x,r))})^{1/p}]$$

$$\kappa(r) = \sup_{x \in \overline{\Omega}} [(\|a_3\|^{1/(p-1)}_{L^{\frac{p}{p-1}}(B(x,r))} + \|b_3 + b_0 c_3\|_{\mathcal{M}^{p-\varepsilon}(B(x,r))})^{1/(p-1)} + (r^{\varepsilon} \|c_3\|_{\mathcal{M}^{p-\varepsilon}(B(x,r))})^{1/p}]$$

if $p \in (1, \nu)$ or

$$\kappa(r) = \sup_{x \in \overline{\Omega}} [(\|a_3\|^{1/(p-1)}_{L^{\frac{p}{p-1}, \nu}(B(x,r))} + \|b_3 + b_0 c_3\|_{\mathcal{M}^{p-\varepsilon}(B(x,r))})^{1/(p-1)} + (r^{\varepsilon} \|c_3\|_{\mathcal{M}^{p-\varepsilon}(B(x,r))})^{1/p}]$$

if $p = \nu$.

We assume in the following

$$\int_0^{R_0} (a(r) + b(r) + \kappa(r)) \frac{dr}{r} < +\infty$$

We remark that the above assumptions are not the sharp and we refer to [M,Z] for some possible generalizations, we observe that our assumptions holds at least in the case where a_i, b_i, c_i, $i = 1, 2, 3$, are constants.

We say that $u \in H^{1,p}_{loc}(\Omega, X)$ is a local solution (subsolution) in Ω of the problem

$$\sum_{i=1}^{m} X_i^* A(x, v, Xv) + B(x, v, Xv) = \mu\,, \tag{1.5}$$

where μ is a Radon measure such that μ^+ and μ^- are in $H^{-1,p'}(\Omega, X)$, iff

$$\int_\Omega A(x, u, Xu).\, X\phi\, dx + \int_\Omega B(x, u, Xu)\, \phi\, dx = (\leq) \int_\Omega \phi \mu(dx)$$

for every ϕ (positive) bounded and in $H^{1,p}(\Omega, X)$ with compact support in Ω.

In the following we assume:

$$\int_0^{R_0} \left(\frac{\mu(B(x,r))}{|B(x,r)|} r^p \right)^{1/(p-1)} \frac{dr}{r} < +\infty, \tag{1.6}$$

for every $x \in \overline{\Omega}$, where the measure μ is extended by 0 to R^N.

The main result of our paper is the following:

THEOREM 1.1: *Let μ be positive and u be a positive local subsolution of (1.5) in Ω. Assume u is bounded above. Then*

$$u(x_0) \leqslant C\left[\left(\frac{1}{|B(x_0, r_0)|}\int_{(B(x_0,r_0)\cap\Omega\cap\{u>0\})} u^\gamma\,dx\right)^{1/\gamma} + \right.$$

$$+\int_0^{r_0}\left(\frac{\mu(B(x_0, r))}{|B(x_0, r)|}r^p\right)^{1/(p-1)}\frac{dr}{r} + \int_0^{r_0}\kappa(r)\frac{dr}{r} +$$

$$\left. + (\kappa(r_0) + \|u\|_{L^\infty(\Omega)})\int_0^{2r_0}\left(\frac{p-\operatorname{cap}(B(x_0, r)\cap\Omega^c, B(x_0, 2r))}{p-\operatorname{cap}(B(x_0, r), B(x_0, 2r))}\right)^{1/(p-1)}\frac{dr}{r}\right]$$

quasi every $x_0 \in \overline{\Omega}$ (in capacity) and $0 < r_0 \leqslant R_0$. The measure μ is extended by 0 to R^N and the constant C depends on ν, p, γ, R_0, the upper bound of $b_0 u$ and on the structure of the operator.

As consequence of Theorem 1.1. we have a result on the necessity part of the Wiener criterion for boundary points.

THEOREM 1.2: *Assume that for any $u_0 \in C_0^\infty(R^N)$, there exits a solution $u \in H^{1,p}(\Omega, X)$ of (1.5) relative to $\mu = 0$ with $(u - u_0) \in H_0^{1,p}(\Omega, X)$ such that*

$$\|u\|^p_{H^{1,p}(\Omega, X)} \leqslant C(\|u\|^{p-1}_{H^{1,p}(\Omega, X)} + 1)\|u_0\|H^{1,p}(\Omega, X)$$

$$\|u\|_{L^\infty(\Omega)} \leqslant C(\|u_0\|_{L^\infty(\Omega)} + 1).$$

If $x_0 \in \partial\Omega$ and

$$\int_0^1\left(\frac{p-\operatorname{cap}(B(x_0, r)\cap\Omega^c, B(x_0, 2r))}{p-\operatorname{cap}(B(x_0, r), B(x_0, 2r))}\right)^{1/(p-1)}\frac{dr}{r} < +\infty;$$

then x_0 is an irregular point of $\partial\Omega$.

We observe that the assumptions in Theotrem 1.1. hold if

(1.7) $\qquad B = 0, \quad a_1, c_3$ are constants, $\quad a_2 = c_2 = 0$

(1.8) $\qquad (A(x, \zeta, \xi_1) - A(x, \zeta, \xi_2), \xi_1 - \xi_2) \geqslant c_4|\xi_1 - \xi_2|^p$

for every $\xi_1, \xi_2 \in R^m$, where c_4 is a constant idependent of ζ. Under the assumptions (1.7), (1.8) we obtain the following result on the local continuity of a solution of (1.5).

THEOREM 1.3: *Let u be a local solution of (1.5) in Ω. Assume that (1.7), (1.8) hold* *and that*

$$\lim_{r\to 0}\sup_{x\in\Omega}\int_0^r\left(\frac{|\mu|(B(x,s))}{|B(x,s)|}s^p\right)\frac{ds}{s}=0, \tag{1.9}$$

where the measure μ is extended by 0 to R^N. Then u is locally continuous in Ω.

Moreover if $\sup_{x\in\Omega}|\mu|(B(x,r))\leqslant Cr^{\alpha}|B(x,r)|$, then u is locally Hölder continuous in Ω.

In section 2 we prove some result that will be used in the proof of Theorem 1.1. In section 3 we prove the Theorem 1.1; the proof follow an adaptation to our framework of the methods used in [17], [18] in the elliptic case (roughly speaking we use estimation and iteration techniques). In section 4 we give a proof by contradiction of Theorem 1.2. and finally in section 5 we prove Theorem 1.3; we use here a localization method and an estimate of the norm of the measure μ in $H^{-1,p'}(\Omega, X)$ involving the condition (1.9).

2. - Preliminary results

In section 2 we prove some results that will be used in the proof of Theorem 1.1.

Lemma 2.1: *Let μ be positive and u be a positive subsolution of (1.5) in $\Omega\subseteq B(x_0, r)$ and $l\geqslant 0$. Assume that u is bounded above.*

Let Φ be a nonegative bounded Borel measurable function on $(0,+\infty)$, which vanishes on $(0,l)$ and λ be the L^1-norm of Φ. Let ω be in $H^{1,p}(\Omega, X)\cap L^{\infty}(\Omega)$ with compact support in Ω and $0\leqslant\omega\leqslant 1$. Then

$$\int_L \Phi(u)(c_1|Xu|^p-c_2u^p-c_3)\,\omega^p\,dx\leqslant \tag{2.1}$$

$$\leqslant\lambda\int_L(b_0|Xu|^p+b_1|Xu|^{p-1}+b_2u^{p-1}+b_3)\,\omega^p\,dx+$$

$$+p\lambda\int_L(a_1|Xu|^{p-1}+a_2u^{p-1}+a_3)\,\omega^{p-1}|X\omega|\,dx+\lambda\mu(\{\omega>0\})$$

where $L=\{x\in\Omega: u(x)>l\}$.

Proof: We write

$$\Psi(t)=\int_0^t\Phi(s)\,ds\,.$$

We observe That Ψ is a bounded function and $\Psi(t)\leqslant\lambda$.

We remark that the function

$$\varphi(x) = \Psi(u(x))\,\omega^p(x)$$

is in $H^{1,p}(\Omega, X) \cap L^\infty(\Omega)$ and has compact support in Ω; then we use φ as test function in (1.5) and we obtain

$$\int_L A(x, u, Xu).\,Xu\,\Phi(u)\,\omega^p\,dx + p\int_L A(x, u, Xu).\,X\omega\,\omega^{p-1}\,\Psi(u)\,dx + \tag{2.2}$$

$$+ \int_L B(x, u, Xu)\,\Psi(u)\,\omega^p\,dx \le \int_L \Psi(u)\,\omega^p\,d\mu\,.$$

Taking into account the structure assumptions we obtain

$$\int_L A(x, u, Xu).\,Xu\,\Phi(u)\,\omega^p\,dx \ge \int_L \Phi(u)(c_1\,|Xu|^p - c_2 u^p - c_3)\,\omega^p\,dx - \tag{2.3}$$

$$-\int_L A(x, u, Xu).\,X\omega\,\omega^{p-1}\,\Psi(u)\,dx \le \int_L \Psi(u)(a_1\,|Xu|^{p-1} + a_2 u^{p-1} + a_3)\,\omega^{p-1}\,|X\omega|\,dx$$

and

$$-\int_L B(x, u, Xu)\,\Psi(u)\,\omega^p\,dx \le \tag{2.5}$$

$$\le \int_L \Psi(u)(b_0\,|Xu|^p + b_1\,|Xu|^{p-1} + b_2 u^{p-1} + b_3)\,\omega^p\,dx\,.$$

From (2.2), (2.3), (2.4), (2.5) we obtain

$$\int_L \Phi(u)(c_1\,|Xu|^p - c_2 u^p - c_3)\,\omega^p\,dx \le \tag{2.6}$$

$$\le \int_L \Psi(u)(b_0\,|Xu|^p + b_1\,|Xu|^{p-1} + b_2 u^{p-1} + b_3)\,\omega^p\,dx +$$

$$+ p\int_L \Psi(u)(a_1\,|Xu|^{p-1} + a_2 u^{p-1} + a_3)\,\omega^{p-1}\,|X\omega|\,dx + \int_L \Psi(u)\,\omega^p\,d\mu\,.$$

Since $\omega \le 1$ and $\Psi \le \lambda$, from (2.6) we obtain the result. ■

LEMMA 2.2: *Let u be a subsolution of (1.5) in Ω. Assume that u is bounded above. Let $B = B(x_0, r)$ be an open ball in R^N. Let η, φ, ψ be functions in $H^{1,p}(B, X)$ with values in $[0, 1]$ and $\eta\psi \in H_0^{1,p}(B \cap \Omega, X)$, $(1-\varphi)(1-\psi) = 0$, $|X\eta| \le 10/r$.*

Assume $l \geq 0$. *We write*

$$\omega = \eta\psi, \qquad \sigma = \omega\varphi$$

$$M = \|u\|_{L^\infty(\Omega)} + \kappa(r)$$

$$L = B \cap \Omega \cap \{u > l\}$$

$$E = L \cap \{\varphi < 1\}, \qquad F = L \cap \{\varphi = 1\}$$

and we assume

$$\int_B |X\omega|^p \, dx \leq C \frac{r^p}{|B|}$$

where $|D|$ *denote the (Lebesgue) measure of* D.

(i) *There are positive constant* C, K *depending only on* p, γ, *the upper bound of* $b_0 u$ *and on structure constants, such that, if* $\delta > 0$ *and*

$$\frac{1}{|B|} \int_E (1+v)^\gamma \, dx \leq K$$

then

$$\int_L |Xw|^p \omega^p \, dx \leq \tag{2.7}$$

$$\leq C \frac{1}{|B|} \left[\left(\frac{b(r)(l+\delta) + \kappa(r)}{\delta} \right)^p + \left(\frac{b(r)(l+\delta) + \kappa(r)}{\delta} \right)^{p-1} + \left(\frac{a(r)\, l}{\delta} \right)^{p-1} \right] +$$

$$+ C r^{-p} \int_E (1+v)^\gamma \, dx + C\delta^{1-p} M^{p-1} \int_B |X\sigma|^p \, dx + C\delta^{1-p} \mu(B),$$

where

$$w = w_\delta = (1+v)^{\gamma/q} - 1$$

$$v = v_\delta = \frac{(u-l)^+}{\delta}.$$

(ii) *There are positive constants C and k depending only on p, γ, R_0, the upper bound of $b_0 u$ and on structure constants, such that*

$$(2.8)\quad \left(\frac{1}{|B|}\int_L (u-l)^\gamma \omega^q\, dx\right)^{(p-1)/\gamma} \leq C[b(r)(l+\delta)+\kappa(r)+a(r)\,l]^{p-1}+$$

$$+C\frac{r^p}{|B|}\mu(B)+CM^{p-1}\frac{r^p}{|B|}\int_B |X\sigma|^p\, dx$$

provided that

$$(2.9)\quad k|E| \leq |E| \leq k|2B|$$

and

$$\frac{1}{|2B|}\int_E (u-l)^\gamma\, dx \leq \frac{2^\gamma}{|B|}\int_L (u-l)^\gamma \omega^q\, dx\,.$$

PROOF: First we observe that u is such that

$$\int_\Omega (A'(x,u,Xu).\, X\varphi + B'(x,u,Xu)\,\varphi)\, dx \leq 0$$

for all $\varphi \in H^{1,p}(\Omega, X)\cap L^\infty(\Omega)$ with $\varphi \geq 0$ and with $\{\varphi>0\}\subsetneq\{u>0\}$, where A' and B' satisfy to the same structure condition of A and B with constants a_i', $b_0'=0$, c_i', $i=1,2,3$, depending on the structure constants of A and B and on $M=\sup u$ (the proof is the same as in Lemma 3.4 pag. 164 [18]; then as long as we use test functions φ such that $\{\varphi>0\}\subsetneq\{u>0\}$, we assume, without loss of generality $b_0=0$.

We begin by proving (i).

We observe that

$$Xw=\frac{\gamma}{q}(1+v)^{-\varepsilon/p}Xv\,.$$

Since

$$(2.10)\quad w^q \leq (1+v)^\gamma \leq C(1+w^q)$$

$$(2.11)\quad w^p \leq C\min\{v^{p-\varepsilon}, v^p\} \leq C\min\{v^{p-1},(1+v)^\gamma\}$$

$$(2.12)\quad v^{p-1} \leq w^{p-1} \leq C(1+w^p)$$

$$(2.13)\quad v^p(1+v)^{-\varepsilon} \leq Cw^p$$

$$(2.14)\quad v^{p-1} \leq \delta^{1-p}u^{p-1} \leq \delta^{1-p}M^{p-1}$$

where $M = \sup_{\Omega} u$

$$\omega = \eta \quad \text{on } E \tag{2.15}$$

$$\omega = \sigma \quad \text{on } F \tag{2.16}$$

It follows that

$$\int_L |X(u\omega)|\, dx \le C\left(\int_E (1+v)^\gamma |X\eta|^p dx + \delta^{1-p} M^{p-1} \int_F |X\sigma|^p dx\right) + \\ + \delta^{-p} \int_L (1+v)^{-\tau} |Xu|^p \omega^p dx \tag{2.17}$$

where C depends on $(\gamma/q)^p$.

Consider now the function

$$\Phi(t) = \left(1 + \frac{(t-l)^+}{\delta}\right)^{-\tau}, \quad t > l \qquad \Phi(t) = 0\,, \quad t \le l\,.$$

We observe that the L^1-norm of Φ is bounded by $(\tau - 1)\,\delta$. We apply the result in Lemma 2.1. and we obtain

$$\int_L (1+v)^{-\tau} |Xu|^p \omega^p dx \le \\ \le C \int_L [(c_2 u^p + c_3)(1+v)^{-\tau} + \delta(b_1 |Xu|^{p-1} + b_2 u^{p-1} + b_3)]\, \omega^p dx + \\ + C\delta \int_L (a_1 |Xu|^{p-1} + a_2 u^{p-1} + a_3)\, \omega^{p-1} |X\omega|\, dx + C\delta\mu(\{\omega > 0\})\,. \tag{2.18}$$

Then, taking into account that $u = l + \delta v$ on L,

$$\int_L (1+v)^{-\tau} |Xu|^p \omega^p dx \le C \int_L [c_2 l^p + \delta b_2 l^{p-1} + \delta b_3 + c_3]\, \omega^p dx + \\ + C\delta \int_L [a_2 l^{p-1} + a_3]\, \omega^{p-1} |X\omega|\, dx + C\delta \int_L (b_1 |Xu|^{p-1} \omega^p + a_1 |Xu|^{p-1} \omega^{p-1} |X\omega|)\, dx + \\ + C \int_L [\delta^p a_2 v^{p-1} \omega^p]\, dx + C\delta\mu(B)\,. \tag{2.19}$$

We have

$$\int_E a^{p-1}\omega^{p-1}|X\omega|\,dx \le C\left(\int_B |X\omega^p|\,dx\right)^{1/p}\left(\int_B a^{p'}\omega^p\,dx\right)^{1-1/p} \le \tag{2.20}$$

$$\le \|a^{1/(p-1)}\|_{L^{\nu+\sigma}(\Omega)}\int_B |X\omega^p|\,dx \le C\frac{r^p}{|B|}\|a^{1/(p-1)}\|_{L^{\nu+\sigma}(\Omega)},$$

where $\sigma = 0$ if $p < \nu$, $\sigma > 0$ if $p = \nu$.

Moreover we have for $g \in \mathcal{M}^{p-\varepsilon}$

$$\int_B g\,dx \le \|g\|_{\mathcal{M}^{p-\varepsilon}}\frac{|B|}{r^{p-\varepsilon}}.$$

We denote

$$d(r) = \frac{|B|}{r^p}\left[\left(\frac{b(r)(l+\delta)+\kappa(r)}{\delta}\right)^p + \left(\frac{b(r)(l+\delta)+\kappa(r)}{\delta}\right)^{p-1} + \left(\frac{a(r)\,l}{\delta}\right)^{p-1}\right].$$

We estimate the the terms in the right-handside of (2.19); we split the integrals into the two integrals on the domains E and F.

Consider at first the integrals on E.

We obtain

$$\delta\int_E (b_1|Xu|^{p-1}\omega^p + a_1|Xu|^{p-1}\omega^{p-1}|X\omega|)\,dx \le \tag{2.21}$$

$$\le \varepsilon_1\int_L (1+v)^{-\tau}|Xu|^p\omega^p\,dx + C\varepsilon_1^{\xi-1}\delta^p\int_E (b_1^p\eta^p + a_1^p|X\eta|^p)(1+v)^\gamma\,dx \le$$

$$\le \varepsilon_1\int_L (1+v)^{-\tau}|Xu|^p\omega^p\,dx + C\varepsilon_1^{\xi-1}\delta^p\left(\frac{r^p}{|B|}(b(r)+1)^{p-1} + \int_E b_1^p w^q\omega^p\,dx\right).$$

We observe that the following inequality holds

$$v^{p-1} \le \frac{1}{p'}v^p(1+v)^{-\tau} + \frac{1}{p}(1+v)^\gamma.$$

Then

$$\int_E (b_2 v^{p-1} + c_2 v^p (1+v)^{-t})\,\omega^p\,dx \le C \int_E b_2 (1 + w^q)\,\omega^p\,dx \le$$

$$\le C \frac{|B|}{r^p} b(r)^p + C \int_E b_2 w^q \omega^p\,dx .$$

From the integrability assumptions on $(b_1^p + b_2)^{q/p}$ we obtain if $p < \nu$

$$\int_E (b_1^p + b_2)\, w^q \omega^p\,dx \le C \left(|B|^{-1} \int_E (1+v)^\gamma\,dx \right)^{1-p/q} \left(\int_B (w\omega)^{\nu p/(\nu-p)}\,dx \right)^{(\nu-p)/\nu} |B|^{(q-p)/p}$$

and if $p = \nu$

$$\int_E (b_1^p + b_2)\, w^q \omega^p\,dx \le C \left(|B|^{-1} \int_E (1+v)^\gamma\,dx \right)^{1-p/q} \left(\int_B (w\omega)^{p/\varepsilon_0}\,dx \right)^{\varepsilon_0} |B|^{(q-p)/p}.$$

Then

$$(2.22) \qquad \int_E (b_1^p + b_2)\, w^q \omega^p\,dx \le C \left(|B|^{-1} \int_E (1+v)^\gamma\,dx \right)^{1-p/q} r^\chi \int_B |X(w\omega)|^p\,dx \le$$

$$\le C r^\chi K^{1-p/q} \int_B |X(w\omega)|^p\,dx$$

where the constant C depends on R_0 and χ is a positive exponent such that $r^\chi \ge c|B|^{(q-p)/p}$ (such a χ exists and depends only on the open bounded set Ω).

Using the integrability assumptions on a_2 we obtain

$$(2.23) \qquad \int_E a_2 v^{p-1} \eta^{p-1} |X\eta|^p\,dx \le \varepsilon_1 \int_E a_2^{p'} (w\omega)^p\,dx + C\varepsilon_1^{1-p} \int_E (1+v)^\gamma |X\eta|^p\,dx \le$$

$$\le \varepsilon_1 \int_B |X(w\omega)|^p\,dx + C\varepsilon_1^{1-p} \int_E (1+v)^\gamma |X\eta|^p\,dx .$$

We estimate the integral over E of the other terms in the right-handside of (2.19) by (2.20) and (2.21) and we get, that the the integrals over E of the other terms in the right-handside of (2.19) may be estimated by

$$(2.24) \qquad C\delta^p (\varepsilon_1 + \varepsilon_1^{1-p} r^\chi K^{1-p/q}) \int_B |X(w\omega)|^p\,dx +$$

$$+ C\delta^p (1 + \varepsilon_1^{1-p}) \left(r^{-p} \int_E (1+v)^\gamma\,dx + \delta^p d(r) \right).$$

Now we will estimate the integrals over F. We use again the Lemma 2.1 with Φ, that is the characteristic function of the interval $[l, M]$ and with σ instead of ω (we recall that $\omega = \sigma$ on F).

The L^1-norm of Φ is bounded by M and we get

$$\int_L |Xu|^p \sigma^p \, dx \leq CM \int_L (a_1 |Xu|^{p-1} \sigma^{p-1} |X\sigma| + b_1 |Xu|^{p-1} \sigma^p) \, dx +$$

$$+ C \int_L [(c_2 u^p + c_3) + M(b_2 u^{p-1} + b_3)] \, \sigma^p \, dx +$$

$$+ CM \int_L (a_2 u^{p-1} + a_3) \, \sigma^{p-1} |X\sigma| \, dx + CM\mu(\{\sigma > 0\}).$$

We obtain

$$(2.25) \qquad \int_L |Xu|^p \sigma^p \, dx \leq$$

$$\leq C \int_B (M^p b_2 + M b_3 + c_3) \, \sigma^p \, dx + CM \int_B (a_2 M^{p-1} + a_3) \, \sigma^{p-1} |X\sigma| \, dx +$$

$$+ CM \int_L (a_1 |Xu|^{p-1} \sigma^{p-1} |X\sigma| + b_1 |Xu|^{p-1} \sigma^p) \, dx + CM\mu(B).$$

We have for $\varepsilon_2 > 0$

$$(2.26) \qquad |Xu|^{p-1} \sigma^{p-1} (b_1 \sigma + a_1 |X\sigma|) \leq \frac{\varepsilon_2}{M} |Xu|^p \sigma^p + \left(\frac{\varepsilon_2}{M}\right)^{1-p} (b_1^p \sigma^p + a_1^p |X\sigma|^p).$$

By the integrability assumptions on a_2 and $(b_1^p + b_2)$ we obtain

$$(2.27) \qquad \int_B [a_2 \sigma^{p-1} |X\sigma| + (b_1^p + b_2) \, \sigma^p \, dx \leq C \left(\int_B a_2^{p'} \sigma^p \, dx \right)^{1-1/p} \left(\int_B |X\sigma|^p \, dx \right)^{1/p} +$$

$$+ C|B|^{(q-p)/q} \left(\int_B (b_1^p + b_2)^{q/p} \sigma^q \, dx \right)^{p/q} \leq C \int_B |X\sigma|^p \, dx.$$

From (2.20), (2.25), (2.26), (2.27) we obtain

$$(2.28) \qquad \delta^p \int_F [(b_2 v^p (1+v) - \tau + b_2 v^{p-1}) \, \omega^p + a_2 v^{p-1} \omega |X\omega|] \, dx +$$

$$+ \delta \int_L (b_1 \omega + a_1 |X\omega|) |Xu|^{p-1} \omega^{p-1} \, dx \leq C\delta M^{p-1} \int_B (b_2 \sigma^p + a_2 \sigma^{p-1} |X\sigma|) \, dx +$$

$$+c\frac{\delta\varepsilon_2}{M}\int_F |Xu|^p\sigma^p\,dx + C\delta\varepsilon_2^{1-p}M^{p-1}\int_F (b_1^p\sigma^p + a_1^p|X\sigma|^p)\,dx \leq$$

$$\leq C\varepsilon_2\delta\int_L (a_1|Xu|^{p-1}\sigma^{p-1}|X\sigma| + b_1|Xu|^{p-1}\sigma^p)\,dx +$$

$$+C(1+\varepsilon_2^{1-p})\left(M^{p-1}\delta\int_B |X\sigma|^p\right)dx + \delta^p d(r) + C\delta\mu(B).$$

By an appropriate choice of ε_2 we get

$$\delta^p\int_F [(b_2 v^p(1+v) - \tau + b_2 v^{p-1})\omega^p + a_2 v^{p-1}\omega|X\omega|]\,dx + \tag{2.29}$$

$$+\delta\int_L (b_1\omega + a_1|X\omega|)|Xu|^{p-1}\omega^{p-1}\,dx \leq CM^{p-1}\delta\int_B |X\sigma|^p\,dx + C\delta^p d(r) + C\delta\mu(B).$$

From (2.20), (2.24), (2.29) we obtain

$$\int_L (1+v)^{-\tau}|Xu|^p\omega^p\,dx \leq C\delta^p(\varepsilon_1 + \varepsilon_1^{1-p}r^\chi K^{1-p/q})\int_B |X(w\omega)|^p\,dx + \tag{2.30}$$

$$+C(1+\varepsilon_1^{1-p})\left[\delta^p r^{-p}\int_E (1+v)^\gamma\,dx + CM^{p-1}\delta\int_B |X\sigma|^p\,dx + C\delta^p d(r) + \delta\mu(B)\right].$$

Using (2.18) and an appropriate choice of ε_1, K, R_0 we obtain

$$\int_L |X(w\omega)|^p\,dx \leq \tag{2.31}$$

$$\leq Cr^{-p}\int_E (1+v)^\gamma\,dx + cd(r) + C\delta^{1-p}M^{p-1}\int_B |X\sigma|^p\,dx + C\delta^{1-p}\mu(B),$$

where we take into account that $\omega = 0$ on $B\setminus\Omega$. From (2.31) we get easily (i).

Now we prove (ii). Fix $\kappa > 0$ to be specified later. We use the part (i) choosing

$$\delta = \left(\frac{1}{\kappa|B|}\int_L (u-l)^\gamma\omega^q\,dx\right)^{1/\gamma}$$

$$K = 2^{2\gamma-1}\frac{|2B|}{|B|}\kappa.$$

The part (i) give the first restriction on the choice of κ. We observe that

$$\kappa = |B|^{-1} \int_L v^\gamma \omega^q \, dx . \tag{2.32}$$

By (2.9) we obtain

$$\int_E (1+v)^\gamma \, dx \leq 2^{\gamma-1} \left(|E| + \int_E v^\gamma \, dx \right) \leq$$

$$\leq 2^{\gamma-1} |2B| k + 2^{2\gamma-1} \frac{|2B|}{|B|} \int_L v^\gamma \omega^q \, dx \leq 2^{2\gamma-1} |2B| k .$$

From (2.9), (2.32) we get

$$2k|B| \leq 2 \int_L v^\gamma \omega^q \, dx \leq \chi \int_L \omega^q \, dx + \int_{L \cap \{v^\gamma \geq \chi/2\}} v^\gamma \omega^q \, dx \leq$$

$$\leq \chi \left(|E| + \int_F \sigma^q \, dx \right) + \int_{L \cap \{v^\gamma \geq \chi/2\}} v^\gamma \omega^q \, dx \leq \kappa |B| + \int_{L \cap \{v^\gamma \geq \chi 2\}} v^\gamma \omega^q \, dx + \int_B \sigma^q \, dx ,$$

where $\chi = |B|/|2B|$.

Then

$$k|B| \leq \int_{L \cap \{v^\gamma \geq \chi/2\}} v^\gamma \omega^q \, dx + \int_B \sigma^q \, dx \leq C \left(\int_L w^q \omega^q \, dx + \int_B \sigma^q \, dx \right).$$

From the Sobolev inequality we obtain

$$\left(\frac{1}{|B|} \int_L w^q \omega^q \, dx \right)^{p/q} = \left(\frac{1}{|B|} \int_B w^q \omega^q \, dx \right)^{p/q} \leq \tag{2.33}$$

$$\leq C \frac{r^p}{|B|} \int_B |X(w\omega)|^p \, dx = C \frac{r^p}{|B|} \int_L |X(w\omega)|^p \, dx$$

and

$$\left(\frac{1}{|B|} \int_B \sigma^q \, dx \right)^{p/q} \leq C \frac{r^p}{|B|} \int_B |X\sigma|^p \, dx . \tag{2.34}$$

Hence

$$k^{p/q} \leq C \frac{r^p}{|B|} \left(\int_L |X(w\omega)|^p \, dx + \int_B |X\sigma|^p \, dx \right). \tag{2.35}$$

From (2.31) we obtain

$$k^{p/q} \leq C_1 k + C_2 \left[\left(\frac{b(r)(l+\delta)\,\kappa(r)}{\delta} \right)^p + \right.$$

$$\left. + \left(\frac{b(r)(l+\delta) + \kappa(r)}{\delta} \right)^{p-1} + \left(\frac{a(r)\,l}{\delta} \right)^{p-1} + M^{p-1} \int_B |X\sigma|^p\,dx + \mu(B) \right].$$

We choose k so small that $C_3 = k^{p/q} - C_1 k > 0$ and we obtain

$$1 \leq C \left[\left(\frac{b(r)(l+\delta)\,\kappa(r)}{\delta} \right)^p + \right.$$

$$\left. + \left(\frac{b(r)(l+\delta) + \kappa(r)}{\delta} \right)^{p-1} + \left(\frac{a(r)\,l}{\delta} \right)^{p-1} + M^{p-1} \int_B |X\sigma|^p\,dx + \mu(B) \right].$$

It follows that either

$$\frac{1}{2} \leq C \left(\frac{b(r)(l+\delta)\,\kappa(r)}{\delta} \right)^p$$

or

$$\frac{1}{2} \leq C \left[\left(\frac{b(r)(l+\delta) + \kappa(r)}{\delta} \right)^{p-1} + \left(\frac{a(r)\,l}{\delta} \right)^{p-1} + M^{p-1} \int_B |X\sigma|^p\,dx + \mu(B) \right].$$

In any case we deduce that the result in the part (ii) of Lemma 2.2 holds. ∎

3. - Proof of Theorem 1.1

We are now in position to give the proof of Theorem 1.1.

We denote $M = \kappa(r_0) + \|\mu\|_{L^\infty(B(x_0, r_0))}$ and set $k \in (0, 1)$ be the constant in Lemma 2.2. We write $r_j = 2^{-j} r_0$ and we denote by η_j the cut-off function between the balls $B(x_0, r_{j+1})$ and $B(x_0, r_j)$ and by g_j the potential of the set $B(x_0, r_j) \setminus \Omega$ in $B(x_0, 2r_{j-1})$. We have

$$\int (r_j^{-p} g_j^p + |Xg_j|^p)\,dx \leq p - \operatorname{cap}(B(x_0, r_j) \setminus \Omega, B(x_0, 2r_j)). \tag{3.1}$$

We denote

$$\psi_j = \min(1, (2-3g_j)^+)$$

$$\varphi_j = \min(1, 3g_j + 3g_{j-1})$$

$$B_j = B(x_0, r_j)$$

$$L_j = B_j \cap \Omega \cap \{u \geq l_j\}$$

$$E_j = L_j \cap \{\varphi_j < 1\}$$

$$F_j = L_j \cap \{\varphi_j = 1\}$$

$$a_j = a(r_j), \qquad b_j = b(r_j), \qquad \kappa_j = \kappa(r_j),$$

where l_j are constants that will be chosen later. By (3.1) we have

$$\text{(3.2)} \qquad \int (r_j^{-p} \varphi_j^p + |X\varphi_j|^p)\, dx \leq$$

$$\leq C(p-\operatorname{cap}(B(x_0, r_{j-1}) \setminus \Omega, B(x_0, 2r_{j-2})) + p-\operatorname{cap}(B(x_0, r_j) \setminus \Omega, B(x_0, 2r_{j-1})))$$

$$\int |X\psi_j|^p)\, dx \leq p-\operatorname{cap}(B(x_0, r_j) \setminus \Omega, B(x_0, 2r_{j-1})).$$

We define recursively $l_0 = 0$ and

$$l_{j+1} = l_j + \left(\frac{1}{k|B_j|} \int_{L_j} (u-l_j)^\gamma \psi_j^q \eta_j^q \, dx \right)^{1/\gamma}.$$

We define

$$\delta_j = l_{j+1} - l_j.$$

We prove now that for $j \geq 1$ we have

$$\text{(3.3)} \qquad \delta_j \leq \frac{1}{2}\delta_{j-1} + C\left[(a_j + b_j)\, l_{j+1} + \kappa_j + \left(\frac{\mu(B_j)}{|B_j|} r^p \right)^{1/(p-1)} + \right.$$

$$\left. + M\left(\frac{C(p-\operatorname{cap}(B(x_0, r_{j-1}) \setminus \Omega, B(x_0, 2r_{j-2})) + p-\operatorname{cap}(B(x_0, r_j) \setminus \Omega, B(x_0, 2r_{j-1})))}{p-\operatorname{cap}(B(x_0, r_j), B(x_0, r_{j-1}))} \right)^{1/(p-1)} \right].$$

The result holds when $\delta_j \leq (1/2)\, \delta_{j-1}$, so we assume $\delta_{j-1} \leq 2\delta_j$.

Since $\psi_{j-1}\eta_{j-1} = 1$ on E_j, we have

$$\text{(3.4)} \qquad |E_j|\,\delta_{j-1}^\gamma \leq \int_{E_j} (l_j - l_{j-1})\, \psi_{j-1}^q \eta_{j-1}^q \, dx \leq \int_{L_{j-1}} (u - l_{j-1})\, \psi_{j-1}^q \eta_{j-1}^q \, dx.$$

Then

$$|E_j| \leq k|B_{j-1}| = k|2B_j|. \tag{3.5}$$

Moreover

$$\int_{E_j} (u-l_j)^\gamma\, dx \leq \int_{L_{j-1}} (u-l_{j-1})^\gamma \psi_{j-1}^q \eta_{j-1}^q\, dx = \delta_{j-1}^\gamma |B_{j-1}| = 2^\gamma k|2B_j| =$$

$$= \frac{|B_{j-1}|}{|B_j|} \int_{L_j} (u-l_j)^\gamma \psi_j^q \eta_j^q\, dx .$$

From Lemma 2.2 we obtain

$$\delta_j \leq C\left[(a_j+b_j)\, l_{j+1} + \kappa_j + \left(\frac{\mu(B_j)}{|B_j|} r^p\right)^{1/(p-1)} + \right. \tag{3.6}$$

$$\left. + M\left(\frac{C(p-\operatorname{cap}(B(x_0,r_{j-1})\setminus\Omega, B(x_0,2r_{j-2})) + p-\operatorname{cap}(B(x_0,r_j)\setminus\Omega, B(x_0,2r_{j-1})))}{p-\operatorname{cap}(B(x_0,r_j), B(x_0,2r_{j-2}))}\right)^{1/(p-1)}\right].$$

Using now the same methods as in [18] Theorem 4.27 we prove by (3.6) that

$$\lim_{j\to+\infty} l_j \leq C\left[\left(\frac{1}{|B(x_0,r_0)|} \int_{\{B(x_0,r_0)\cap\Omega\cap\{u>0\}\}} u^\gamma\, dx\right)^{1/\gamma} + \right. \tag{3.7}$$

$$+ \int_0^{r_0} \left(\frac{\mu(B(x_0,r))}{|B(x_0,r)|} r^p\right)^{1/(p-1)} \frac{dr}{r} + \int_0^{r_0} \kappa(r)\frac{dr}{r} +$$

$$\left. + M \int_0^{2r_0} \left(\frac{p-\operatorname{cap}(B(x_0,r)\cap\Omega^c, B(x_0,2r))}{p-\operatorname{cap}(B(x_0,r), B(x_0,2r))}\right)^{1/(p-1)} \frac{dr}{r}\right]$$

where $r_0 \leq R_0$.

It remains to prove that

$$u(x_0) \leq \lim_{j\to+\infty} l_j \tag{3.8}$$

for q.e. x_0 and at every point x_0 of continuity of u.

We choose $\varepsilon > 0$ and we write $l = \lim_{j\to+\infty} l_j$. We denote

$$w_j = (2^{\gamma/q}-1)^{-1}\left(\left(1+\frac{(u-l-\varepsilon)}{\varepsilon}\right)^{\gamma/q} - 1\right).$$

Then $w_j\psi_j\eta_j \in H_0^{1,p}(B_j, X)$ and $w_j\psi_j\eta_j + \varphi_j\eta_j \geqslant 1$ on $B_{j+1}\cap\Omega\cap\{u > l+2\varepsilon\}$. Thus

$$p-\operatorname{cap}(B_{j+1}\cap\Omega\cap\{u>l+2\varepsilon\}, B_j) \leqslant \int_{B_j\cap\Omega} |X(w_j\psi_j\eta_j)|^p\,dx + \int_{B_j} |X(\varphi_j\eta_j)|^p\,dx .$$

Then, using the result of Lemma 2.2, we obtain

$$(3.9)\quad p-\operatorname{cap}(B_{j+1}\cap\Omega\cap\{u>l+2\varepsilon\}, B_j) \leqslant$$

$$\leqslant C\Big[p-\operatorname{cap}(B_{j+1}, B_j)(\varepsilon^{-p}(b_j(l+\varepsilon)+k_j)^p + \varepsilon 1 - p(b_j(l+\varepsilon)+k_j+a_j l)^{p-1}) +$$

$$+ r_j^{-p}\int_{E_j}\left(1+\frac{(u-l-\varepsilon)}{\varepsilon}\right)^{\gamma} dx + \varepsilon 1 - p\mu(B_j) +$$

$$+ \varepsilon 1 - p(1+M)^{p-1}\int_{B_j}(r_j^p\varphi^p + |X\varphi|^p + |X\psi|^p)\,dx\Big].$$

where

$$E_j = B_{j+1}\cap\Omega\cap\{u>l+\varepsilon\}\cap\{\varphi_j<1\}$$

and the application of Lemma 2.2 take into account the following inequality

$$(3.10)\quad \sum_{j=1}^{\infty}\left(\frac{1}{|B_j|}\int_{E_j}\left(1+\frac{(u-l-\varepsilon)}{\varepsilon}\right)^{\gamma} dx\right)^{1/(p-1)} \leqslant$$

$$\leqslant C\sum_{j=1}^{\infty}\left(\frac{1}{|B_j|}\int_{E_j'}\varepsilon^{-\gamma}(u-l_{j-1})^{\gamma}\,dx\right)^{1/(p-1)} \leqslant$$

$$\leqslant C\sum_{j=1}^{\infty}\left(\frac{1}{|B_j|}\int_{L_{j-1}}\varepsilon^{-\gamma}(u-l_{j-1})^{\gamma}\eta_{j-1}\psi j-1\,dx\right)^{1/(p-1)} \leqslant$$

$$\leqslant C\sum_{j=1}^{\infty}(k\varepsilon^{-\gamma}\delta_{j-1}^{\gamma})^{1/(p-1)} < +\infty .$$

From (3.9) we have

$$(3.11)\qquad \sum_{j=1}^{\infty}\left(\frac{p-\operatorname{cap}(B_{j+1}\cap\Omega\cap\{u>l+2\varepsilon\},B_j)}{p-\operatorname{cap}(B_{j+1},B_j)}\right)^{1/(p-1)}\leqslant$$

$$\leqslant C\Bigg[\sum_{j=1}^{\infty}\varepsilon^{-p'}(b_j(l+\varepsilon)+k_j)^{p'}+\sum_{j=1}^{\infty}\varepsilon^{-1}(b_j(l+\varepsilon)+k_j+a_j l)+$$

$$+\sum_{j=1}^{\infty}\left(\frac{1}{|B_j|}\int_{E_j'}\left(1+\frac{(u-l-\varepsilon)}{\varepsilon}\right)^{\gamma}dx\right)^{1/(p-1)}+\varepsilon^{-1}\sum_{j=1}^{\infty}\left(\frac{\mu(B_j)}{|B_j|}\right)^{1/(p-1)}+$$

$$+\varepsilon^{-1}\sum_{j=1}^{\infty}(1+M)\left(\frac{p-\operatorname{cap}(B(x_0,r_{j-1})\setminus\Omega,B(x_0,2r_{j-2}))}{p-\operatorname{cap}(B(x_0,r_{j-1}),B(x_0,2r_{j-2}))}\right)^{1/(p-1)}+$$

$$+\varepsilon^{-1}\sum_{j=1}^{\infty}(1+M)\left(\frac{p-\operatorname{cap}(B(x_0,r_j)\setminus\Omega,B(x_0,2r_{j-1}))}{p-\operatorname{cap}(B(x_0,r_j),B(x_0,2r_{j-1}))}\right)^{1/(p-1)}<+\infty\,.$$

From (3.11) the result follows.

4. - Proof of Theorem 1.2

Assume that x_0 is a regular point of $\partial\Omega$.

Choose $\sigma>0$ and $\varrho\in(0,1)$ to be specified later.

We observe that the set $\{x_0\}$ has zero p-capacity. Then there exists a nonegative function $u_0\in C^1(R^N)$ with support in $B(x_0,4\varrho)$ such that $\|u_0\|_{1,p}\leqslant\sigma$ and $u(x_0)=1$. Let $u\in H^{1,p}(\Omega,X)$ be the solution of (1.5) in Ω with $(u-u_0)\in H_0^{1,p}(\Omega,X)$; then

$$\int_{\Omega}u^p\,dx\leqslant C\sigma^p.$$

We have that u is continuous at x_0 and from Theorem 1.1

$$(4.1)\qquad u(x_0)\leqslant C_1\left(\frac{1}{|B(x_0,\varrho)|}\int_{B(x_0,\varrho)\cap\Omega}u^p\,dx\right)^{1/p}+$$

$$+C_2(\|u\|_{L^\infty(\Omega)}+\kappa(r_0))\int_0^{\varrho}\frac{p-\operatorname{cap}(B(x_0,s)\cap\Omega^c,B(x_0,2s))}{p-\operatorname{cap}(B(x_0,r),B(x_0,2s))}\,\frac{ds}{s}+C_3\int_0^{\varrho}\kappa(s)\,\frac{ds}{s}\,.$$

We can find ϱ such that

$$(4.2)\qquad C_2(\|u\|_{L^\infty(\Omega)}+\kappa(r_0))\int_0^{\varrho}\frac{p-\operatorname{cap}(B(x_0,s)\cap\Omega^c,B(x_0,2s))}{p-\operatorname{cap}(B(x_0,r),B(x_0,2s))}\frac{ds}{s}+ +C_3\int_0^{\varrho}\kappa(s)\frac{ds}{s}\le\frac{1}{3}.$$

Moreover choosing $\sigma\le(1/3)(|B(x_0,\varrho)|)^{1/p}$ we have

$$(4.3)\qquad C_1\left(\frac{1}{|B(x_0,\varrho)|}\int_{B(x_0,\varrho)\cap\Omega}u^p\,dx\right)^{1/p}\le\left(\frac{\sigma^p}{|B(x_0,\varrho)|}\right)^{1/p}\le\frac{1}{3}.$$

From (4.1), (4.2), (4.3) we obtain

$$u(x_0)\le\frac{2}{3}.$$

From the regularity of the point x_0 we obtain $u(x_0)=u_0(x_0)=1$; so we have a contradiction and the result follows.

5. - Proof of Theorem 1.3

First we prove the following result:

Lemma 5.1: *Let μ be a positive measure in $H^{-1,q}(B(x_0,R),X)$. We denote again by μ the extension of μ to R^N by 0. Assume that the following conditions holds*

$$\sup_{x\in B(x_0,R)}\int_0^{2R}\left(\frac{\mu(B(x,\varrho))}{|B(x,\varrho)|}\varrho^p\right)^{1/(p-1)}\frac{d\varrho}{\varrho}<+\infty.$$

Then

$$\|\mu\|_{H^{-1,q}(B(x_0,R),X)}\le C(\mu(B(x_0,R)))\sup_{x\in B(x_0,R)}\left[\int_0^{2R}\left(\frac{\mu(B(x,\varrho))}{|B(x,\varrho)|}\varrho^p\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\right]^{(p-1)/p}$$

Proof: Let w be the solution of the problem

$$(5.1)\qquad \int_{B(x_0,R)}|Xw|^{p-2}XwXv\,dx=\int_{B(x_0,R)}v\mu(dx)$$

for every $v\in H_0^{1,p}(B(x_0,R),X)$, where $w\in H_0^{1,p}(B(x_0,R),X)$. We observe that w is positive and that w (extended by 0) is a subsolution of the subelliptic p-Laplace operator relative to λ in R^N. Then from Theorem 1.1 we have

$$\sup_{B(x_0,R)} w \leq$$

$$\leq C\left[\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} |w|^p\,dx\right)^{1/p} + \sup_{x\in B(x_0,R)}\int_0^{2R}\left(\frac{\mu(B(x,\varrho))}{|B(x,\varrho)|}\varrho^p\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\right] \leq$$

$$\leq C\left[\left(\frac{R^p}{|B(x_0,R)|}\int_{B(x_0,R)} |Xw|^p\,dx\right)^{1/p} + \sup_{x\in B(x_0,R)}\int_0^{2R}\left(\frac{\mu(B(x,\varrho))}{|B(x,\varrho)|}\varrho^p\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\right]$$

where C denotes possibly different constants independent of R; then

$$\int_{B(x_0,R)} |Xw|^p\,dx \leq C\mu(B(x_0,R))\left(\frac{R^p}{|B(x_0,R)|}\int_{B(x_0,R)} |Xw|^p\,dx\right)^{1/p} +$$

$$+ C\mu(B(x_0,R))\sup_{x\in B(x_0,R)}\left[\int_0^{2R}\left(\frac{\mu(B(x_0,\varrho))}{|B(x_0,\varrho)|}\varrho^p\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\right].$$

We obtain

$$\int_{B(x_0,R)} |Xw|^p \leq C\mu(B(x_0,R))\sup_{x\in B(x_0,R)}\left[\int_0^{2R}\left(\frac{\mu(B(x_0,\varrho))}{|B(x_0,\varrho)|}\varrho^p\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\right].$$

From (3.14) we obtain that

$$\|\mu\|_{H^{-1,q}(B(x_0,R),X)} \leq \left(\int_{B(x_0,R)} |Xw|^p\,dx\right)^{(p-1)/p}$$

so we have the result. ■

We are now in position to prove Theorem 1.3.

We assume (without loss of generality) $a_3 = c_3 = 0$ and $A(x,\zeta,\xi) = A(x,\xi)$. We assume also (without loss of generality) $\text{ess-}\lim\sup_{x\to x_0} u(x) \geq 0 \geq \text{ess-}\lim\inf_{x\to x_0} u(x)$. Let now $\tilde{u}$ be the solution of the problem

$$\text{(5.1)}\qquad \int_{B(x_0,4R)} A(x, X\tilde{u})\,Xv\,dx = \int_{B(x_0,4R)} v|\mu|(dx)$$

for every $v \in H^{1,p}(\Omega, X)$ with $\mathrm{supp}(v) \subseteq \Omega$, where $\tilde{u} \in H^{1,p}_{loc}(\Omega, X)$ with $\tilde{u} = u^+$ on $\partial B(x_0, 4R)$, $B(x_0,4R) \subseteq \overline{B(x_0,8R)} \subseteq \Omega$.

We observe that from Theorem 1.1 we obtain easily

$$\sup_{B(x_0,4R)} \tilde{u} \leq C\left[\sup_{B(x_0,4R)} u + \sup_{x\in B(x_0,4R)} \int_0^{4R} \left(\frac{|\mu|(B(x,\varrho))}{|B(x,\varrho)|}\varrho^p\right)^{1/(p-1)} \frac{d\varrho}{\varrho}\right].$$

Let $B(x_0, 64r) \subsetneq B(x_0, R) \subsetneq \overline{B(x_0, 8R)} \subsetneq \Omega$ and denote by w the solution of the problem

$$\int_{B(x_0,32r)} A(x, Xw)\, Xv\, dx = 0 \qquad \forall v \in H_0^{1,p}(B(x_0, 32r), X)$$

where $w \in H^{1,p}(B(x_0, 32r), X)$ and $\tilde{u} = w$ on $\partial B(x_0, 32r)$.

Finally we denote by χ the function $(\tilde{u} - w) \in H_0^{1,p}(B(x_0, 32r), X)$.

Let $k \geq \sup_{B(x_0,2r)} w = M_w(2r)$ we recall that, since $A(x, 0) = 0$, we have

$$\int_{B(x_0,32r)} A(x, X(\tilde{u}-k)^+)\, Xv\, dx \leq \int_{B(x_0,32r)} v|\mu|(dx) \tag{5.3}$$

for every $v \in H^{1,p}(B(x_0, 32r), X)$ with $\operatorname{supp}(v) \subsetneq B(x_0, 32r)$ and $v \geq 0$. From (5.3), using the results in Theorem 1.1, we obtain

$$\begin{aligned}
\sup_{B(x_0,r)} (\tilde{u}-k)^+ &\leq C\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,2r)} ((\tilde{u}-k)^+)^p\, dx\right)^{\frac{1}{p}} + \\
&+ C \sup_{x\in B(x_0,r)}\left[\int_0^{2r}\left(\frac{|\mu|(B(x,\varrho))}{|B(x,\varrho)|}\varrho^p\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\right] \leq C\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,32r)} \chi^p\, dx\right)^{1/p} + \\
&+ C \sup_{x\in B(x_0,r)}\left[\int_0^{2r}\left(\frac{|\mu|(B(x,\varrho))}{|B(x,\varrho)|}\varrho^p\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\right] \leq C\left(\frac{r^p}{|B(x_0,r)|}\int_{B(x_0,32r)} |X(\chi)|^p\, dx\right)^{1/p} + \\
&+ C\left[\sup_{x\in B(x_0,32r)}\int_0^{32r}\left(\frac{|\mu|(B(x,\varrho))}{|B(x,\varrho)|}\varrho^p\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\right].
\end{aligned} \tag{5.4}$$

From (5.4) and taking into account the result in Lemma 5.1 we obtain for $r \leq \overline{R_0}$, with $\overline{R_0}$ suitable,

$$\sup_{B(x_0,r)} (\tilde{u}-k)^+ \leq C \sup_{x\in B(x_0,32r)} \int_0^{32r}\left(\frac{|\mu|(B(x,\varrho))}{|B(x,\varrho)|}\varrho^p\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\,. \tag{5.5}$$

Let now $k = M_w(2r)$ we obtain

$$\sup_{B(x_0,r)} \tilde{u}^+ \leq M_w(2r) + C \sup_{x \in B(x_0,32r)} \int_0^{32r} \left(\frac{|\mu|(B(x,\varrho))}{|B(x,\varrho)|} \varrho^p \right)^{1/(p-1)} \frac{d\varrho}{\varrho} \leq \tag{5.6}$$

$$\leq (1-\delta_0) \sup_{B(x_0,4r)} \tilde{u}^+ + C \sup_{x \in B(x_0,32r)} \int_0^{32r} \left(\frac{|\mu|(B(x,\varrho))}{|B(x,\varrho)|} \varrho^p \right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

where $0 < \delta_0 < 1$ Then, by iteration, we obtain for every $r \leq (R/64) \wedge \overline{R_0}$

$$\sup_{B(x_0,r)} u^+ \leq \sup_{B(x_0,r)} \tilde{u}^+ \leq \tag{5.7}$$

$$\leq C_1 \left(\frac{r}{R} \right)^{\alpha} \sup_{B(x_0,R)} \tilde{u}^+ + C \sup_{x \in B(x_0,R)} \int_0^{R} \left(\frac{|\mu|(B(x,\varrho))}{|B(x,\varrho)|} \varrho^p \right)^{1/(p-1)} \frac{d\varrho}{\varrho} \leq$$

$$\leq C_1 \left(\frac{r}{R} \right)^{\alpha} \sup_{B(x_0,4R)} u^+ + C \sup_{x \in B(x_0,4R)} \int_0^{4R} \left(\frac{|\mu|(B(x,\varrho))}{|B(x,\varrho)|} \varrho^p \right)^{1/(p-1)} \frac{d\varrho}{\varrho}.$$

We observe now that by the same methods may be applied to obtain the estimate (5.6) we finally we obtain

$$\sup_{B(x_0,r)} |u| \leq C_1 \left(\frac{r}{R} \right)^{\alpha} \frac{d\varrho}{\varrho} \sup_{B(x_0,4R)} |u| + \tag{3.22}$$

$$+ C \sup_{x \in B(x_0,4R)} \int_0^{4R} \left(\frac{|\mu|(B(x,\varrho))}{|B(x,\varrho)|} \varrho^p \right)^{1/(p-1)} \frac{d\varrho}{\varrho}.$$

From (3.21) easily follows that u assume at x_0 the value $g(x_0) = 0$ with continuity.

REFERENCES

[1] M. Biroli, *The Wiener test for Poincaré-Dirichlet forms, Classical and Modern Potential Theory and Applications*, GowriSankaran et al. (eds.), Kluwer Academic Publishers, Dordrecht-Boston-London (1994), 93-103.

[2] M. Biroli, *Nonlinear Kato measures and nonlinear subelliptic Schrödinger problems*, Rend. Acc. Naz. Sc. detta dei XL, Memorie di Matematica e Appl., 21 (1997), 235-252.

[3] M. Biroli - U. Mosco, *A Saint-Venant type principle for Dirichlet forms on discontinuous media*, Ann. Mat. Pura e Appl., CLXIX (1996), 125-181.

[4] M. Biroli - U. Mosco, *Sobolev and isoperimetric inequalities for Dirichlet forms on homogeneous spaces*, Rend. Acc. Naz. Lincei, Mat. e Appl., 6 (IX) 1 (1995), 33-44.

[5] M. Biroli - U. Mosco, *Sobolev inequalities on homogeneous spaces*, Pot. An., 4(4) (1995), 311-324.

[6] M. Biroli - N. Tchou, *Relaxed Dirichlet problem for the subelliptic p-Laplacian*, Ann. Mat. Pura e Appl., in print.

[7] L. Capogna - D. Danielli - N. Garofalo, *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations*, Commun. in P.D.E., 18 (1993), 1765-1794.

[8] L. Capogna - D. Danielli - N. Garofalo, *Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations*, Am. J. of Mathematics, 118 (1995), 1153-1196.

[9] D. Danielli, *Regularity at the boundary for solutions of nonlinear subelliptic equations*, Indiana Un. Math. J., 44(1) (1995), 269-286.

[10] C. L. Fefferman - D. Phong, *Subelliptic eigenvalue problems, Harmonic Analysis*, Wadsworth, Chicago (1981), 590-606.

[11] C. L. Fefferman - A. Sanchez Calle, *Fundamental solution for second order subelliptic operators*, Ann. of Math., 124 (1996), 247-272.

[12] B. Franchi - G. Lu - R. L. Wheeden, *Weighted Poincaré inequalities for Hörmander vector fields and local regularity for a class of degenerate elliptic equations*, Pot. An., 4(4) (1995), 361-376.

[13] M. Fukushima - Y. Oshima - M. Takeda, *Dirichlet forms and Markov processes*, W. de Gruyter & Co., Berlin-Heidelberg-New York, 1994.

[14] R. Gariepy - P. W. Ziemer, *A regularity condition at the boundary for solutions of quasilinear elliptic equations*, Arch. Rat. Mech. An., 67 (1997), 25-39.

[15] D. Jerison, *The Poincaré inequality for vector fields satisfying an Hörmander's condition*, Duke Math. J., 53 (1986), 502-523.

[16] D. Jerison - A. Sanchez Calle, *Subelliptic second order differential operators, Harmonic Analysis*, Lec. Notes in Math. 1277, Springer Verlag, Berlin-Heidelberg-New York (1987), 46-77.

[17] J. Malý, *Pointwise estimates of nonnegative subsolutions of quasilinear elliptic equations at irregular points*, Comm. Math. Univ. Carolinae, 37 (1996), 23-42.

[18] J. Malý - P. W. Ziemer, *Fine regularity of solutions of elliptic partial differential equations*, Mathematical Surveys and Monographs 51, American Mathematical Soc. (1997).

[19] A. Nagel - E. Stein - S. Weinger, *Balls and metrics defined by vector fields I: Basic properties*, Acta Math., 155 (1985), 102-147.

[20] J. M. Rakotoson - W. P. Ziemer, *Local behavior of solutions of quasilinear elliptic equations with general structure*, Trans. of A.M.S., 319 (1990), 747-764.

[21] A. Sanchez Calle, *Fundamental solutions and geometry of square of vector field*, Inv. Math., 78 (1984), 143-160.

Direttore responsabile: Prof. A. Ballio - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5/L - Bologna