Nonlinear Subelliptic Problems with Measure Data

SUMMARY. — We consider nonlinear subelliptic problems with measure data and we prove a generalization of an estimate given by Malý in the elliptic case. As consequence we give a proof of the necessity of Wiener criterion for the regularity of boundary points and a generalization to our subelliptic framework on a result given by Ziemer concerning the continuity of local solution of a nonlinear elliptic problem with measure data.

Problemi subellittici nonlineari con dato misura

SUMMARY. — Si considerano problemi sottoellittici non lineari con dato misura e si prova una generalizzazione di una stima data da Malý nel caso ellittico. Da tale risultato si deduce una prova della parte necessaria del criterio di Wiener per punti del bordo e una generalizzazione al caso subellittico di un precedente risultato di Ziemer concernente la continuità di soluzioni locali di un problema ellittico non lineare con dato misura.

1. INTRODUCTION

The necessity part of the Wiener criterion for the regularity of boundary points in the case of nonlinear elliptic problems has been proved in a recent paper by Malý, [17][18], using an estimate on positive subsolutions of the problem. The goal of the present paper is to prove a generalization of the previous estimate to the subelliptic case. In the subelliptic case the sufficient part of the Wiener criterion for the regularity of boundary points has been proved in [9] for operators that are homogeneous of or-

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(***) Memoria presentata il 2 marzo 1999 da Marco Biroli, socio dell’Accademia.
der \((p-1)\), in the general case we can also obtain a proof using the same methods considered in [6] in the case of relaxed Dirichlet problem for the subelliptic \(p\)-Laplacian (then we omit this part in the present paper); in the following we give a proof of the necessity part by the methods of Malý, [17], [18]. Moreover we deduce from our estimate also a criterion for continuity or Hölder continuity of a local solution to a nonlinear subelliptic problem with measure data; this last result generalizes a previous one in [20] concerning the Hölder continuity of the local solution to a nonlinear elliptic problem with measure data (see also [2] for the related notion of Kato measure in the nonlinear case).

We state now the result which is the main goal of the paper.

Let \(\Omega\) be a bounded open set of \(\mathbb{R}^N\) and \(X_i, i = 1, \ldots, m\), be vector fields defined in \(\mathbb{R}^N\) with \(C^\infty\) coefficients satisfying an Hörmander condition (i.e. the vector fields \(X_i\) and their commutators up to a fixed order span at every point \(\mathbb{R}^N\)); we denote by \(X\) the vector \((X_1, \ldots, X_m)\).

We recall that we have a distance associated to the vector fields that may be defined as

\[
d(x, y) = \sup \{ \phi(x) - \phi(y); \phi \in C^\infty_0(\mathbb{R}^N), |X\phi| \leq 1 \text{ a.e.}\}.
\]

The distance \(d\) defines a topology equivalent to the euclidean one. We denote by \(B(x, r)\) the set \(\{y; d(x, y) < r\}\), [9], [19], [21].

For \(x \in \Omega\) there exists \(R_0\) such that for \(s < r < R_0\) we have

\[
|B(x, s)| \geq c_0 \left( \frac{s}{r} \right)^\nu |B(x, r)|
\]

(1.1)

where \(c_0\) is a constant depending on \(\Omega\), but independent of \(s < r < R_0\), [9], [10], [19], [21] (here and in the following we denote, for a Lebesgue measurable set \(E\), by \(|E|\) the Lebesgue measure of \(E\); we say that \(\nu > 0\) is the (or an estimate of) intrinsic dimension of our problem.

We denote by \(H^{1, p}(\Omega, X), p \in (1, + \infty)\) the closure of \(C^\infty(\Omega)\) for the norm

\[
\|u\|_{H^{1, p}}^2 = \int_\Omega |Xu|^p dx + \int_\Omega |u|^p dx
\]

where, here and in the following, \(dx\) denotes the Lebesgue measure on \(\mathbb{R}^N\). By \(H^{1, p}_{\text{loc}}(\Omega, X)\) we denote the space of functions in \(H^{1, p}(O, X)\) for every relatively compact open set \(O\) with closure in \(\Omega\). The space \(H^{1, p}_0(\Omega, X), p \in (1, + \infty)\), is the closure of \(C^\infty_0(\Omega)\) in \(H^{1, p}(\Omega, X)\) and \(H^{-1, p'}(\Omega, X), 1/p + 1/p' = 1\), denotes the dual space of \(H^{1, p}(\Omega, X)\). Finally \(H^{-1, p'}_{\text{loc}}(\Omega, X)\) denotes the space of all distributions on \(\Omega\) that are in \(H^{-1, p'}(O, X)\) for every relatively compact open set \(O\) with closure in \(\Omega\).
We denote by $\mathcal{M}^s(R^N)$ the space of the function $f \in L^s(R^N)$ such that

$$\|f\|_{\mathcal{M}^s} = \sup_{B(x, r)} \frac{r^s}{|B(x, r)|} \int_{B(x, r)} |f| \, dx < +\infty,$$

and by $\mathcal{M}^s(\Omega)$ the space of the functions whose extension by 0 to $R^N$ is in $\mathcal{M}^s(R^N)$; $\|f\|_{\mathcal{M}^s(\Omega)}$ will be the norm in $\mathcal{M}^s(R^N)$ of the extension of $f$ by 0 to $R^N$.

For $x \in \Omega$ there exists $\bar{R}_0$ such that for $r < \bar{R}_0$ the following scaled Poincaré inequality holds, [15] [16],

$$\int_{B(x, r)} |u - u_x|^p \, dx \leq c_1 r^p \int_{B(x, r)} |Xu|^p \, dx$$

(1.2)

where $c_1$ is a constant depending on $\Omega$, but independent of $r < \bar{R}_0$. As a consequence we have the Sobolev inequalities for the intrinsic dimension $v$, [5], [6], [12].

We recall that a variational $p$-capacity of a set $E$ relative to the open set $\Omega$, $E \subset \subset \Omega$ may be defined (generalizing the usual Newtonian definition) as

$$p - \text{cap} (E, \Omega) = \inf_{\{v \in C^\infty_0(\Omega), v = 1 \text{ in a neighbourhood of } E\}} \left\{ \int_{\Omega} |Xv|^p \right\}$$

and we refer for the properties of such a type of capacity to [8]. We denote $p - \text{cap} (E) = p - \text{cap} (E, R^N)$ and we recall that, if $E \subset \subset \Omega$, $p - \text{cap} (E, \Omega) = 0$ iff $p - \text{cap} (E) = 0$.

Using the same methods as in [8] Theorem 4.1 and the estimates on the cut-off functions between balls, see [1], we obtain easily

$$p - \text{cap} (B(x_0, r), B(x_0, 2r)) = \frac{r^p}{|B(x_0, r)|}$$

for $B(x_0, 4r) \subset \Omega$, where $\Omega$ is a bounded open set and the constants may depend on $\Omega$.

A function $u$ defined in an open set $O$ with $O \subset \subset \Omega$ is quasi-continuous if for every $\varepsilon > 0$ there exists an open set $V$ with $p - \text{cap} (V, \Omega) \leq \varepsilon$ such that $u$ is continuous on $E \setminus V$; a function $u$ in $H^{1, p}(O, X)$ has a quasi-continuous q.e. representative (i.e. there exists a quasi-continuous function $\tilde{u}$ such that $\tilde{u} = u$ up to sets of zero $p$-capacity); the proof follows the same methods as in [13] for the case $p = 2$. In the following we identify $u$ with its quasi-continuous representative.

Let now $A: \Omega \times R \times R^m \rightarrow R^m$ and $B: \Omega \times R \times R^m \rightarrow R$ be (Lebesgue) mea-
surable functions such that the following structure conditions hold:

\[
\begin{align*}
|A(x, \xi, \eta)| & \leq a_1 |\xi|^{p-1} + a_2 |\xi|^{p-1} + a_3 \\
|B(x, \xi, \eta)| & \leq b_0 |\xi|^p + b_1 |\xi|^{p-1} + b_2 |\xi|^{p-1} + b_3 \\
A(x, \xi, \eta), \xi, \xi, \eta & \geq c_1 |\xi|^p - c_2 |\xi|^{p-1} - c_3,
\end{align*}
\]

a.e. in \( \Omega \) for every \( \xi \in \mathbb{R}, \eta \in \mathbb{R}^n \) and \( p \in (1, \nu) \).

Here \( c_1 \) is a positive constant, \( a_1 \) and \( b_0 \) are nonnegative constants and

\[(1.4) \quad a_2, a_3 \in L^{\nu/(\nu-1)}(\mathbb{R}^N)\]

if \( p \in (1, \nu) \) or

\[(1.4') \quad a_2, a_3 \in L^{(\nu + \sigma)/(\nu - 1)}(\mathbb{R}^N)\]

where \( \sigma > 0 \), if \( p = \nu \),

\[
b_1, b_2, b_3, c_2, c_3 \in W^{\frac{p}{p-q}}(\mathbb{R}^N)\]

\[
b_1, b_2, c_2 \in L^{\frac{p(q)}{p-q+\nu p}}(\mathbb{R}^N)\]

where \( q > p \) is a fixed exponent such that \( q < \nu p/(\nu - p) \) and \( \varepsilon \) is defined by the relation

\[
\varepsilon = (\nu - p) \left( \frac{q}{p} - 1 \right).
\]

We also define \( \tau \) and \( \gamma \) by

\[
\frac{1}{\tau} = \frac{p-1}{q} + \frac{1}{p}, \quad \gamma = (p-1) \tau.
\]

We assume

\[
\sup_{x \in \mathcal{D}} \|r^* [b_1^r + b_2 + c_2 \|W^0_x(r^*)\|] \leq C
\]

Let \( a(r), b(r), c(r) \) be defined as

\[
a(r) = \sup_{x \in \mathcal{D}} \|a_2\|_{L^{\nu/(\nu-1)}(B(x, r))}
\]

if \( p \in (1, \nu) \) or

\[
a(r) = \sup_{x \in \mathcal{D}} \|a_2\|_{L^{(\nu + \sigma)/(\nu - 1)}(B(x, r))}
\]
if \( p = \nu \),
\[
b(r) = \sup_{x \in D} \left\{ (r^\nu b_1^\nu + b_2 \|u\|_{W^{\nu-1,1}(B(x, r))}^{1/(\nu-1)})^{1/\nu} + (r^\nu c_2 \|u\|_{W^{\nu-1,1}(B(x, r))}^{1/\nu}) \right\}
\]
\[
\kappa(r) = \sup_{x \in D} \left\{ (\|u\|_{1/(\nu-1)} \|u\|_{W^{\nu-1,1}(B(x, r))}^{1/(\nu-1)} + (r^\nu c_3 \|u\|_{W^{\nu-1,1}(B(x, r))}^{1/\nu}) \right\}
\]
if \( p \in (1, \nu) \) or
\[
\kappa(r) = \sup_{x \in D} \left\{ (\|u\|_{1/(\nu-1)} \|u\|_{W^{\nu-1,1}(B(x, r))}^{1/(\nu-1)} + (r^\nu c_3 \|u\|_{W^{\nu-1,1}(B(x, r))}^{1/\nu}) \right\}
\]
if \( p = \nu \).

We assume in the following
\[
\int_0^{R_0} \left( \alpha(r) + b(r) + \kappa(r) \right) \frac{dr}{r} < +\infty
\]

We remark that the above assumptions are not the sharp and we refer to \([M, Z]\) for some possible generalizations, we observe that our assumptions holds at least in the case where \( a_i, b_i, c_i, i = 1, 2, 3 \), are constants.

We say that \( u \in H^{1, p}_0(\Omega, X) \) is a local solution (subsolution) in \( \Omega \) of the problem

\[
(1.5) \quad \sum_{i=1}^n X_i^* A(x, \nu, Xu) + B(x, \nu, Xu) = \mu,
\]

where \( \mu \) is a Radon measure such that \( \mu^+ \) and \( \mu^- \) are in \( H^{-1, p'}(\Omega, X) \), iff

\[
\int_{\Omega} A(x, u, Xu) \cdot X\phi \, dx + \int_{\Omega} B(x, u, Xu) \phi \, dx = (\leq) \int_{\Omega} \phi \mu(\, dx)
\]

for every \( \phi \) (positive) bounded and in \( H^{1, p}(\Omega, X) \) with compact support in \( \Omega \).

In the following we assume:

\[
(1.6) \quad \int_0^{R_0} \left( \frac{\mu(B(x, r))}{\|B(x, r)\|^p} \right)^{1/(p-1)} \frac{dr}{r} < +\infty,
\]

for every \( x \in \Omega \), where the measure \( \mu \) is extended by 0 to \( R^N \).

The main result of our paper is the following:
THEOREM 1.1: Let $\mu$ be positive and $u$ be a positive local subsolution of (1.5) in $\Omega$. Assume $u$ is bounded above. Then

$$
\begin{aligned}
u(x_0) & \leq C \left( \frac{1}{|B(x_0, r_0)|} \int_{B(x_0, r_0) \cap \Omega \cap \{u > 0\}} u^\gamma \, dx \right)^{1/\gamma} + \\
& + \int_0^{r_0} \left( \frac{\mu(B(x_0, r))}{|B(x_0, r)|} \right)^{1/(\gamma + 1)} \frac{dr}{r} + \int_0^{r_0} \kappa(r) \frac{dr}{r} + \\
& + \left( \kappa(r_0) + \|\mu\|_{L^\infty(\Omega)} \right) \left[ \int_0^{2r_0} \left( \frac{p - \text{cap} (B(x_0, r) \cap \Omega^c, B(x_0, 2r))}{p - \text{cap} (B(x_0, r), B(x_0, 2r))} \right)^{1/(p - 1)} \frac{dr}{r} \right]
\end{aligned}
$$

quasi every $x_0 \in \Omega$ (in capacity) and $0 < r_0 \leq R_0$. The measure $\mu$ is extended by 0 to $\mathbb{R}^N$ and the constant $C$ depends on $\nu$, $p$, $\gamma$, $R_0$, the upper bound of $b_0 u$ and on the structure of the operator.

As a consequence of Theorem 1.1, we have a result on the necessity part of the Wiener criterion for boundary points.

THEOREM 1.2: Assume that for any $u_0 \in C_0^\infty(\mathbb{R}^N)$, there exists a solution $u \in H^{1, \frac{p}{2}}(\Omega, X)$ of (1.5) relative to $\mu = 0$ with $(u - u_0) \in H^{1, \frac{p}{2}}(\Omega, X)$ such that

$$
\|\mu\|_{H^{1, \frac{p}{2}}(\Omega, X)} \leq C(\|\mu\|_{H^{1, \frac{p}{2}}(\Omega, X)} + 1) \|u_0\|_{H^{1, \frac{p}{2}}(\Omega, X)}
$$

$$
\|\mu\|_{L^\infty(\Omega)} \leq C(\|u_0\|_{L^\infty(\Omega)} + 1).
$$

If $x_0 \in \partial \Omega$ and

$$
\left( \frac{p - \text{cap} (B(x_0, r) \cap \Omega^c, B(x_0, 2r))}{p - \text{cap} (B(x_0, r), B(x_0, 2r))} \right) \frac{1}{(p - 1)} \frac{dr}{r} < + \infty;
$$

then $x_0$ is an irregular point of $\partial \Omega$.

We observe that the assumptions in Theorem 1.1 hold if

(1.7) $B = 0, \quad a_1, c_1$ are constants, $a_2 = c_2 = 0$

(1.8) $(A(x, \xi, \xi_1) - A(x, \xi, \xi_2), \xi_1 - \xi_2) \geq c_4 |\xi_1 - \xi_2|^p$

for every $\xi_1, \xi_2 \in \mathbb{R}^N$, where $c_4$ is a constant independent of $\xi$. Under the assumptions (1.7), (1.8) we obtain the following result on the local continuity of a solution of (1.5).

THEOREM 1.3: Let $u$ be a local solution of (1.5) in $\Omega$. Assume that (1.7), (1.8) hold
and that

$$
\lim_{r \to 0} \sup_{x \in \Omega} \int_0^r \left( \frac{\mu(B(x, s))}{|B(x, s)|} \right)^{\frac{1}{p}} ds < 0,
$$

where the measure \( \mu \) is extended by 0 to \( \mathbb{R}^N \). Then \( u \) is locally continuous in \( \Omega \).

Moreover if \( \sup_{x \in \Omega} \mu(B(x, r)) \leq C r^n |B(x, r)| \), then \( u \) is locally Hölder continuous in \( \Omega \).

In section 2 we prove some result that will be used in the proof of Theorem 1.1. In section 3 we prove the Theorem 1.1; the proof follow an adaptation to our framework of the methods used in [17], [18] in the elliptic case (roughly speaking we use estimation and iteration techniques). In section 4 we give a proof by contradiction of Theorem 1.2. and finally in section 5 we prove Theorem 1.3; we use here a localization method and an estimate of the norm of the measure \( \mu \) in \( H^{-1, p}(\Omega, X) \) involving the condition (1.9).

2. - Preliminary Results

In section 2 we prove some results that will be used in the proof of Theorem 1.1.

**Lemma 2.1:** Let \( \mu \) be positive and \( u \) be a positive subsolution of (1.5) in \( \Omega \subset B(x_0, r) \) and \( l \geq 0 \). Assume that \( u \) is bounded above.

Let \( \Phi \) be a nonnegative bounded Borel measurable function on \((0, +\infty)\), which vanishes on \((0, l)\) and \( \lambda \) be the \( L^1 \)-norm of \( \Phi \). Let \( \omega \) be in \( H^{1, p}(\Omega, X) \cap L^\infty(\Omega) \) with compact support in \( \Omega \) and \( 0 \leq \omega \leq 1 \). Then

$$
\int_L \Phi(u)(c_1 |Xu|^p - c_2 u^p - c_3) \omega^p dx \leq \\
\leq \lambda \int_L (b_0 |Xu|^p + b_1 |Xu|^{p-1} + b_2 u^{p-1} + b_3) \omega^p dx + \\
+p\lambda \int_L (a_1 |Xu|^{p-1} + a_2 u^{p-1} + a_3) \omega^{p-1} |Xu| dx + \lambda \mu(\{ \omega > 0 \})
$$

where \( L = \{ x \in \Omega : u(x) > l \} \).

**Proof:** We write

$$
\Psi(t) = \int_0^t \Phi(s) ds.
$$

We observe that \( \Psi \) is a bounded function and \( \Psi(t) \leq \lambda \).
We remark that the function

\[ \varphi(x) = \Psi(u(x)) \omega^p(x) \]

is in \( H^{1,p}(\Omega, X) \cap L^\infty(\Omega) \) and has compact support in \( \Omega \); then we use \( \varphi \) as test function in (1.5) and we obtain

\[
\begin{align*}
\int_{\Omega} A(x, u, Xu). Xu \Phi(u) \omega^p \, dx &+ p \int_{\Omega} A(x, u, Xu). X_\omega \omega^{p-1} \Psi(u) \, dx + \\
&+ \int_{\Omega} B(x, u, Xu) \Psi(u) \omega^p \, dx \leq \int_{\Omega} \Psi(u) \omega^p \, d\mu.
\end{align*}
\]

Taking into account the structure assumptions we obtain

\[
\begin{align*}
\int_{\Omega} A(x, u, Xu). Xu \Phi(u) \omega^p \, dx &\geq \int_{\Omega} \Phi(u)(c_1 |Xu|^p - c_2 u^p - c_3) \omega^p \, dx - \\
&- \int_{\Omega} A(x, u, Xu). X_\omega \omega^{p-1} \Psi(u) \, dx \leq \int_{\Omega} \Psi(u)(a_1 |Xu|^{p-1} + a_2 u^{p-1} + a_3) \omega^{p-1} |X_\omega| \, dx
\end{align*}
\]

and

\[
\begin{align*}
- \int_{\Omega} B(x, u, Xu) \Psi(u) \omega^p \, dx &\leq \\
&\leq \int_{\Omega} \Psi(u)(b_0 |Xu|^p + b_1 |Xu|^{p-1} + b_2 u^{p-1} + b_3) \omega^p \, dx.
\end{align*}
\]

From (2.2), (2.3), (2.4), (2.5) we obtain

\[
\begin{align*}
\int_{\Omega} \Phi(u)(c_1 |Xu|^p - c_2 u^p - c_3) \omega^p \, dx &\leq \\
&\leq \int_{\Omega} \Psi(u)(b_0 |Xu|^p + b_1 |Xu|^{p-1} + b_2 u^{p-1} + b_3) \omega^p \, dx + \\
&+ p \int_{\Omega} \Psi(u)(a_1 |Xu|^{p-1} + a_2 u^{p-1} + a_3) \omega^{p-1} |X_\omega| \, dx + \int_{\Omega} \Psi(u) \omega^p \, d\mu.
\end{align*}
\]

Since \( \omega \leq 1 \) and \( \Psi \leq \lambda \), from (2.6) we obtain the result. ■

**Lemma 2.2:** Let \( u \) be a subsolution of (1.5) in \( \Omega \). Assume that \( u \) is bounded above. Let \( B = B(x_0, r) \) be an open ball in \( \mathbb{R}^N \). Let \( \eta, \varphi, \psi \) be functions in \( H^{1,p}(B, X) \) with values in \([0, 1]\) and \( \eta \psi \in H^{1,p}_0(B \cap \Omega, X) \), \( (1 - \varphi)(1 - \psi) = 0 \), \( |X\eta| \leq 10/r \).
Assume $l \geq 0$. We write
\[
\omega = \eta \psi, \quad \sigma = \omega \psi
\]
\[
M = \|u\|_{L^p(\Omega)} + \kappa(r)
\]
\[
L = B \cap \Omega \cap \{u > l\}
\]
\[
E = L \cap \{\varphi < 1\}, \quad F = L \cap \{\varphi = 1\}
\]
and we assume
\[
\int_B |X\omega|^\rho \, dx \leq C \frac{r^p}{|B|}
\]
where $|D|$ denote the (Lebesgue) measure of $D$.

(i) There are positive constants $C, K$ depending only on $p, \gamma$, the upper bound of $b_0 u$ and on structure constants, such that, if $\delta > 0$ and
\[
\frac{1}{|B|} \int_E (1 + v)^\gamma \, dx \leq K
\]
them
\[
(2.7) \quad \int_L |X\omega|^\rho \omega \rho \, dx \leq
\]
\[
\leq C \frac{1}{|B|} \left[ \left( \frac{b(r)(l + \delta) + \kappa(r)}{\delta} \right)^p + \left( \frac{b(r)(l + \delta) + \kappa(r)}{\delta} \right)^{p-1} + \left( \frac{a(r)}{\delta} \right)^{p-1} \right] +
\]
\[
+ C r^{-p} \int_E (1 + v)^\gamma \, dx + C \delta^{1-p} M^{p-1} \int_B |X\sigma|^p \, dx + C \delta^{1-p} \mu(B),
\]
where
\[
\omega = \omega_\delta = (1 + v)^{\gamma/\delta} - 1
\]
\[
\nu = \nu_\delta = \frac{(u - l)^+}{\delta},
\]
(ii) There are positive constants $C$ and $k$ depending only on $p$, $\gamma$, $R_0$, the upper bound of $b_0 u$ and on structure constants, such that

\[
(2.8) \quad \left( \frac{1}{|B|} \int_{L} (u - 1)^{\gamma} \omega^q \, dx \right)^{(p-1)/p} \leq C[b(r)(i + \delta) + \kappa(r) + \alpha(r) l]^{p-1} + C \frac{r^p}{|B|} \mu(B) + CM^{-1} \frac{r^p}{|B|} \int |X\sigma|^p \, dx
\]

provided that

\[
(2.9) \quad k|E| \leq |E| \leq k|2B|
\]

and

\[
\frac{1}{|2B|} \int_{E} (u - 1)^{\gamma} \omega^q \, dx \leq \frac{2\gamma}{|B|} \int_{E} (u - 1)^{\gamma} \omega^q \, dx.
\]

**Proof:** First we observe that $u$ is such that

\[
\int_{\Omega} (A'(x, u, Xu), X\varphi + B'(x, u, Xu) \varphi) \, dx \leq 0
\]

for all $\varphi \in H^{1,p}(\Omega, X) \cap L^\infty(\Omega)$ with $\varphi \geq 0$ and with $\{\varphi > 0\} \subset \{u > 0\}$, where $A'$ and $B'$ satisfy the same structure condition of $A$ and $B$ with constants $a_i', b_i' = 0$, $c_i'$, $i = 1, 2, 3$, depending on the structure constants of $A$ and $B$ and on $M = \sup u$ (the proof is the same as in Lemma 3.4 pag. 164 [18]; then as long as we use test functions $\varphi$ such that $\{\varphi > 0\} \subset \{u > 0\}$, we assume, without loss of generality $b_0 = 0$.

We begin by proving (i).

We observe that

\[
Xu = \frac{\gamma}{q} (1 + v)^{-\gamma} Xv.
\]

Since

\[
(2.10) \quad \omega^q \leq (1 + v)^q \leq C(1 + w^q)
\]

\[
(2.11) \quad w^p \leq C \min \{\varphi^p, \varphi\} \leq C \min \{v^p, (1 + v)^p\}
\]

\[
(2.12) \quad v^p \leq w^p \leq (1 + v)^p
\]

\[
(2.13) \quad v^p (1 + v)^{-\gamma} \leq C w^p
\]

\[
(2.14) \quad v^p \leq \delta^{1-p} w^p \leq \delta^{1-p} M^p
\]
where $M = \sup_{\Omega} u$

(2.15) $\omega = \eta$ on $E$

(2.16) $\omega = \sigma$ on $F$

It follows that

(2.17) $\int_{E} |X(\omega \omega)| \, dx \leq C \left( \int_{E} (1 + v)^{r} |X\eta|^{p} \, dx + \delta^{1-p} M^{p-1} \right) \int_{F} |X\sigma|^{p} \, dx + \delta^{-p} \int_{E} (1 + v)^{-r} |Xu|^{p} \omega^{p} \, dx$

where $C$ depends on $(\gamma/q)^{p}$.

Consider now the function

$$
\Phi(t) = \left(1 + \frac{(t-I)^{+}}{\delta} \right)^{-r}, \quad t > I, \quad \Phi(t) = 0, \quad t \leq I.
$$

We observe that the $L^{1}$-norm of $\Phi$ is bounded by $(\tau - 1) \delta$. We apply the result in Lemma 2.1 and we obtain

(2.18) $\int_{E} (1 + v)^{-r} |Xu|^{p} \omega^{p} \, dx \leq$

$$
\leq C \int_{E} \left[ (c_{2} u^{p} + c_{3})(1 + v)^{-r} + \delta (b_{1} |Xu|^{p-1} + b_{2} u^{p-1} + a_{3}) \omega^{p} \, dx + \right. \\
\left. + C \delta \int_{E} (a_{1} |Xu|^{p-1} + a_{2} u^{p-1} + a_{3}) \omega^{p-1} |X\omega| \, dx + C \delta \mu(\{\omega > 0\}). \right]
$$

Then, taking into account that $u = I + \delta v$ on $E$,

(2.19) $\int_{E} (1 + v)^{-r} |Xu|^{p} \omega^{p} \, dx \leq C \int_{E} \left[ c_{2} I^{p} + \delta b_{2} I^{p-1} + \delta b_{3} + c_{3} \right] \omega^{p} \, dx +$

$$
+ C \delta \int_{E} \left[ a_{2} I^{p-1} + a_{3} \right] \omega^{p-1} |X\omega| \, dx + C \delta \int_{E} \left( b_{1} |Xu|^{p-1} + a_{1} |Xu|^{p-1} \omega^{p-1} |X\omega| \right) \, dx +$

$$
+ C \int_{E} \left[ \delta \omega^{p-1} \omega^{p} \right] \, dx + C \delta \mu(B). \right]
We have

\[ (2.20) \quad \int_E a^{p-1} \omega^{p-1} |X\omega| \, dx \leq C \left( \int_B |X\omega|^p \, dx \right)^{\frac{1}{p}} \left( \int_B a^{p} \omega^{p} \, dx \right)^{\frac{1}{p} - 1} \leq \]

\[ \leq \|a^{1/(p-1)}\|_{L^{\infty}(\Omega)} \int_B |X\omega|^p \, dx \leq C \frac{r^p}{|B|} \|a^{1/(p-1)}\|_{L^{\infty}(\Omega)}, \]

where \( \sigma = 0 \) if \( p < \nu \), \( \sigma > 0 \) if \( p = \nu \).
Moreover we have for \( g \in \mathcal{M}^{\nu - 2}_p \)

\[ \int_B g \, dx \leq \|g\|_{\nu - 2} \frac{|B|}{r^p}. \]

We denote

\[ d(r) = \frac{|B|}{r^p} \left[ \left( \frac{b(r)(l + \delta) + \kappa(r)}{\delta} \right)^p + \left( \frac{b(r)(l + \delta) + \kappa(r)}{\delta} \right)^{p-1} + \left( \frac{a(r) l}{\delta} \right)^{p-1} \right]. \]

We estimate the terms in the right-hand side of (2.19); we split the integrals into the two integrals on the domains \( E \) and \( F \).
Consider at first the integrals on \( E \).
We obtain

\[ (2.21) \quad \delta \int_E (b_1 |Xu|^p \omega \omega + a_1 |Xu|^{p-1} \omega^{p-1} |X\omega|) \, dx \leq \]

\[ \leq \varepsilon_1 \int_L (1 + \nu)^{-\tau} |Xu|^p \omega^\nu \, dx + C \varepsilon_1^{-1} \delta^{\tau} \int_E (b_1^p \eta^p + a_1^p |X\eta|^p)(1 + \nu)^\nu \, dx \leq \]

\[ \leq \varepsilon_1 \int_L (1 + \nu)^{-\tau} |Xu|^p \omega^\nu \, dx + C \varepsilon_1^{-1} \delta^{\tau} \left( \frac{r^p}{|B|} (b(r) + 1)^{\tau-1} \int_E b_1^p \omega^\nu \omega^\nu \, dx \right). \]

We observe that the following inequality holds

\[ \nu^{p-1} \leq \frac{1}{p} \nu^p (1 + \nu)^{-\tau} + \frac{1}{p} (1 + \nu)^\nu. \]
Then
\[ \int_E (b_1 v_{p-1} + c v_p (1 + v)^{-1}) \omega^p \, dx \leq C \int_E b_2 (1 + \omega^p) \omega^p \, dx \leq C \frac{|B|}{r_p} b(r)^p + C \int_E b_2 \omega^p \omega^p \, dx. \]

From the integrability assumptions on \((b_1^p + b_2) v^p\) we obtain if \(p < \nu\)
\[ \int_E (b_1^p + b_2) \omega^p \omega^p \, dx \leq C \left( |B|^{-1} \int_E (1 + v)^p \, dx \right)^{1 - \frac{p}{\nu}} \left( \int_B (\omega \omega)^{p/(\nu - p)} \, dx \right)^{\nu - p}/\nu |B|^{(\nu - p)/p}. \]

and if \(p = \nu\)
\[ \int_E (b_1^p + b_2) \omega^p \omega^p \, dx \leq C \left( |B|^{-1} \int_E (1 + v)^p \, dx \right)^{1 - \frac{p}{\nu}} \left( \int_B (\omega \omega)^{p/p} \, dx \right)^{p} |B|^{(p - p)/p}. \]

Then
\[ (2.22) \int_E (b_1^p + b_2) \omega^p \omega^p \, dx \leq C \left( |B|^{-1} \int_E (1 + v)^p \, dx \right)^{1 - \frac{p}{\nu}} \int_B |X(\omega \omega)|^p \, dx \leq C r^\chi K^{1 - p/\nu} \int_B |X(\omega \omega)|^p \, dx \]

where the constant \(C\) depends on \(R_0\) and \(\chi\) is a positive exponent such that \(r^\chi \geq \epsilon |B|^{(p - p)/p}\) (such a \(\chi\) exists and depends only on the open bounded set \(\Omega\)).

Using the integrability assumptions on \(a_2\) we obtain
\[ (2.23) \int_E a_2 v^{p-1} \eta^{p-1} |X\eta|^p \, dx \leq \epsilon_1 \int_E a_2 v (\omega \omega)^p \, dx + C \epsilon_1^{-p} \int_E (1 + v)^p |X\eta|^p \, dx \leq \epsilon_1 \int_B |X(\omega \omega)|^p \, dx + C \epsilon_1^{-p} \int_E (1 + v)^p |X\eta|^p \, dx. \]

We estimate the integral over \(E\) of the other terms in the right-hand-side of (2.19) by (2.20) and (2.21) and we get, that the integrals over \(E\) of the other terms in the right-hand-side of (2.19) may be estimated by
\[ (2.24) \ C\delta^p (\epsilon_1 + \epsilon_1^{-p} r^\chi K^{1 - p/\nu}) \int_B |X(\omega \omega)|^p \, dx + \\
+ C\delta^p (1 + \epsilon_1^{-p}) \left( r^{-p} \int_E (1 + v)^p \, dx + \delta^p d(r) \right). \]
Now we will estimate the integrals over $F$. We use again the Lemma 2.1 with $\Phi$, that is the characteristic function of the interval $[l, M]$ and with $\sigma$ instead of $\omega$ (we recall that $\omega = \sigma$ on $F$).

The $L^1$-norm of $\Phi$ is bounded by $M$ and we get
\[
\int |Xu|^p \sigma^p \, dx \leq CM \int (a_1 |Xu|^{p-1} \sigma^{p-1} |X\sigma| + b_1 |Xu|^{p-1} \sigma^p) \, dx +
\]
\[
+ C \int (a_2 |Xu|^{p-1} \sigma^{p-1} |X\sigma| + b_1 |Xu|^{p-1} \sigma^p) \, dx +
\]
\[
+ CM \int (a_2 |Xu|^{p-1} \sigma^{p-1} |X\sigma| + a_1 |Xu|^{p-1} \sigma^p) \, dx + CM\mu(\{\sigma > 0\}) ;
\]

We obtain
\[
(2.25) \quad \int |Xu|^p \sigma^p \, dx \leq \n\]
\[
\leq C \int (M^p b_2 + Mb_1 + c_1) \sigma^p \, dx + CM \int (a_2 M^{p-1} + a_1) \sigma^{p-1} |X\sigma| \, dx +
\]
\[
+ CM \int (a_1 |Xu|^{p-1} \sigma^{p-1} |X\sigma| + b_1 |Xu|^{p-1} \sigma^p) \, dx + CM\mu(B) .
\]

We have for $\varepsilon_2 > 0$
\[
(2.26) \quad |Xu|^{p-1} \sigma^{p-1} (b_1 \sigma + a_1 |X\sigma|) \leq \frac{\varepsilon_2}{M} |Xu|^p \sigma^p + \left( \frac{\varepsilon_2}{M} \right)^{1-p} (b_1^p \sigma^p + a_1^p |X\sigma|^p) .
\]

By the integrability assumptions on $a_2$ and $(b_1^p + b_2)$ we obtain
\[
(2.27) \quad \int |a_2 \sigma^{p-1} |X\sigma| + (b_1^p + b_2) \sigma^p \, dx \leq C \left( \int_{b} |a_2 \sigma^p \, dx \right)^{1-1/p} \left( \int |X\sigma|^p \, dx \right)^{1/p} +
\]
\[
+ C |B|^{(q-p)/q} \left( \int_{b} (b_1^p + b_2)^{q/p} \sigma^q \, dx \right)^{q/q} \leq C \int_{b} |X\sigma|^p \, dx .
\]

From (2.20), (2.25), (2.26), (2.27) we obtain
\[
(2.28) \quad \delta^p \int [(b_1 v^p (1+v) - r + b_2 v^{p-1}) \omega^p + a_2 v^{p-1} \omega |X\omega|] \, dx +
\]
\[
+ \delta \int (b_1 \omega + a_1 |X\omega|) |Xu|^{p-1} \omega^{p-1} \, dx \leq C \delta M^{p-1} \int (b_2 \sigma^p + a_2 \sigma^{p-1} |X\sigma|) \, dx +
\]
\[ \epsilon \frac{d \nu}{M} \int |Xu|^p \sigma^p \, dx + C \delta \frac{1}{p M^p} \int (b_1 \sigma^p + a_1 |X| \sigma^p) \, dx \leq \]

\[ \leq C \epsilon \delta \int (a_1 |Xu|^{-1} \sigma^{-1} |X\sigma| + b_1 |Xu|^{-1} \sigma^p) \, dx + \]

\[ + C (1 + \epsilon \delta^{-p}) \left( M^{p-1} \delta \int |X\sigma|^p \right) \, dx + \delta \, d(r) + C \delta \mu(B). \]

By an appropriate choice of \( \epsilon \), we get

\[ \delta^p \int \left[ (b_1 \sigma^p (1 + \nu) - \tau + b_2 \nu^{-1}) \omega^p + a_2 \nu^{-1} \omega |X\omega| \right] \, dx + \]

\[ + \delta \int \left( b_1 \omega + a_1 |X\omega| \right) |Xu|^{-1} \omega^{-1} \, dx \leq C M^{p-1} \delta \int |X\sigma|^p \, dx + C \delta \, d(r) + C \delta \mu(B). \]

From (2.20), (2.24), (2.29) we obtain

\[ (1 + \nu)^{-1} |Xu|^p \omega^p \, dx \leq C \delta^p (\epsilon_1 + \epsilon \delta^{-p} \nu^{-p} K^1 \nu^{-p}) \int |X(u\omega)|^p \, dx + \]

\[ + C (1 + \epsilon \delta^{-p}) \left( \delta^p \nu^{-p} \int (1 + \nu)^{-1} \, dx + C M^{p-1} \delta \int |X\sigma|^p \, dx + C \delta \, d(r) + \delta \mu(B) \right). \]

Using (2.18) and an appropriate choice of \( \epsilon_1, K, R_0 \) we obtain

\[ (u - l)^{\eta} \omega^p \, dx \leq \]

\[ \leq C r^{-\eta} \int (1 + \nu) \, dx + C \delta (r) + C M^{1-p} M^{p-1} \int |X\sigma|^p \, dx + C \delta \mu(B), \]

where we take into account that \( \omega = 0 \) on \( B \setminus \Omega \). From (2.31) we get easily (i).

Now we prove (ii). Fix \( \kappa > 0 \) to be specified later. We use the part (i) choosing

\[ \delta = \left( \frac{1}{\kappa |B|} \right)^{1/2} \int (u - l)^{\eta} \omega^p \, dx \]

\[ \kappa = 2^{\gamma - 1} \left( \frac{2 |B|}{|B|} \right)^{1/2}. \]
The part (i) give the first restriction on the choice of \( \kappa \). We observe that

\[
\kappa = |B|^{-1} \int_L v^\gamma \omega^\delta \, dx.
\]

By (2.9) we obtain

\[
\int_E (1 + v)^{\gamma} \, dx \leq 2^{\gamma - 1} \left( \int_E |E| + \int_E v^\gamma \, dx \right) \leq 2^{\gamma - 1} |2B| k + 2^{\gamma - 1} \frac{|2B|}{|B|} \int_L v^\gamma \omega^\delta \, dx \leq 2^{\gamma - 1} |2B| k.
\]

From (2.9), (2.32) we get

\[
2k|B| \leq 2 \int_L v^\gamma \omega^\delta \, dx \leq \chi \int_L \omega^\delta \, dx + \int_{L \cap \{v^\gamma \geq \chi/2\}} v^\gamma \omega^\delta \, dx \leq \chi \left( \int |E| + \int_{F} \sigma^\delta \, dx \right) + \int_{L \cap \{v^\gamma \geq \chi/2\}} v^\gamma \omega^\delta \, dx + \int_{L \cap \{v^\gamma \geq \chi/2\}} \sigma^\delta \, dx,
\]

where \( \chi = |B|/|2B| \).

Then

\[
k|B| \leq 2 \int_{L \cap \{v^\gamma \geq \chi/2\}} v^\gamma \omega^\delta \, dx + \int_{B} \sigma^\delta \, dx \leq C \left( \int_{L} w^\gamma \omega^\delta \, dx + \int_{B} \sigma^\delta \, dx \right).
\]

From the Sobolev inequality we obtain

\[
(2.33) \quad \left( \frac{1}{|B|} \int_L w^\gamma \omega^\delta \, dx \right)^{\frac{r^p}{r^q}} = \left( \frac{1}{|B|} \int_B w^\gamma \omega^\delta \, dx \right)^{\frac{r^p}{r^q}} \leq C \frac{r^p}{|B|} \int_B |X(w \omega)|^p \, dx = C \frac{r^p}{|B|} \int_L |X(\omega \omega)|^p \, dx
\]

and

\[
(2.34) \quad \left( \frac{1}{|B|} \int_B \sigma^\delta \, dx \right)^{\frac{r^p}{r^q}} \leq C \frac{r^p}{|B|} \int_B |X\sigma|^p \, dx.
\]

Hence

\[
(2.35) \quad k^{\frac{r^p}{r^q}} \leq C \frac{r^p}{|B|} \left( \int_L |X(\omega \omega)|^p \, dx + \int_B |X\sigma|^p \, dx \right).
\]
From (2.31) we obtain
\[ k^{p\delta} \leq C_1 k + C_2 \left[ \left( \frac{b(r)(l + \delta) \kappa(r)}{\delta} \right)^p + \left( \frac{a(r) l}{\delta} \right)^{p-1} + M^{p-1} \int_B |X\sigma|^p \, dx + \mu(B) \right]. \]

We choose \( k \) so small that \( C_2 = k^{p\delta} - C_1 k > 0 \) and we obtain
\[ 1 \leq C \left[ \left( \frac{b(r)(l + \delta) \kappa(r)}{\delta} \right)^p + \left( \frac{a(r) l}{\delta} \right)^{p-1} + M^{p-1} \int_B |X\sigma|^p \, dx + \mu(B) \right]. \]

It follows that either
\[ \frac{1}{2} \leq C \left( \frac{b(r)(l + \delta) \kappa(r)}{\delta} \right)^p \]

or
\[ \frac{1}{2} \leq C \left[ \left( \frac{b(r)(l + \delta) + \kappa(r)}{\delta} \right)^{p-1} + \left( \frac{a(r) l}{\delta} \right)^{p-1} + M^{p-1} \int_B |X\sigma|^p \, dx + \mu(B) \right]. \]

In any case we deduce that the result in the part (ii) of Lemma 2.2 holds. □

3. Proof of Theorem 1.1

We are now in position to give the proof of Theorem 1.1.

We denote \( M = \kappa(r_0) + \|u\|_{L^\infty(B(x_0, r_0))} \) and set \( k \in (0, 1) \) the constant in Lemma 2.2. We write \( r_j = 2^{-j} r_0 \) and we denote by \( \eta_j \) the cut-off function between the balls \( B(x_0, r_{j+1}) \) and \( B(x_0, r_j) \) and by \( g_j \) the potential of the set \( B(x_0, r_j) \setminus \Omega \) in \( B(x_0, 2r_{j-1}) \).

We have
\[ \int (r_j^{-p} g_j^p + |Xg_j|^p) \, dx \leq p - \text{cap} \left( B(x_0, r_j) \setminus \Omega, B(x_0, 2r_j) \right). \]
We denote
\[ \psi_j = \min (1, (2 - 3g_j)^+) \]
\[ \phi_j = \min (1, 3g_j + 3g_{j-1}) \]
\[ B_j = B(x_0, r_j) \]
\[ L_j = B_j \cap \Omega \cap \{ u \geq l_j \} \]
\[ E_j = L_j \cap \{ \phi_j < 1 \} \]
\[ F_j = L_j \cap \{ \phi_j = 1 \} \]
\[ a_j = a(r_j), \quad b_j = b(r_j), \quad \kappa_j = \kappa(r_j), \]
where \( l_j \) are constants that will be chosen later. By (3.1) we have
\[
\int (r_j^{-q} \phi_j^q + |X \phi_j|^p) \, dx \leq
\]
\[
\leq C(p - \text{cap} (B(x_0, r_{j-1}) \setminus \Omega, B(x_0, 2r_{j-2}))) + p - \text{cap} (B(x_0, r_j) \setminus \Omega, B(x_0, 2r_{j-1})))
\]
\[
\int |X \psi_j|^p \, dx \leq p - \text{cap} (B(x_0, r_j) \setminus \Omega, B(x_0, 2r_{j-1})).
\]
We define recursively \( l_0 = 0 \) and
\[
l_{j+1} = l_j + \left( \frac{1}{k |B_j|} \int_{l_j}^{u - l_j} \psi_j^q \eta_j^p \, dx \right)^{1/q}
\]
We define
\[ \delta_j = l_{j+1} - l_j. \]
We prove now that for \( j \geq 1 \) we have
\[
\delta_j \leq \frac{1}{2} \delta_{j-1} + C \left[ (a_j + b_j) l_{j+1} + \kappa_j + \left( \frac{\mu(B_j)}{|B_j|^p} \right)^{1/(p-1)} \right.
\]
\[
+ \left. M \left( C(p - \text{cap} (B(x_0, r_j) \setminus \Omega, B(x_0, 2r_{j-2}))) + \frac{p - \text{cap} (B(x_0, r_j) \setminus \Omega, B(x_0, 2r_{j-1})))}{p - \text{cap} (B(x_0, r_j), B(x_0, r_{j-1}))} \right)^{1/(p-1)} \right].
\]
The result holds when \( \delta_j \leq (1/2) \delta_{j-1} \), so we assume \( \delta_{j-1} \leq 2 \delta_j \).
Since \( \psi_{j-1} \eta_{j-1} = 1 \) on \( E_j \), we have
\[
|E_j| \delta_{j-1}^p \leq \int_{E_j} (l_j - l_{j-1}) \psi_{j-1}^p \eta_{j-1}^p \, dx \leq \int_{l_{j-1}} (u - l_{j-1}) \psi_{j-1}^p \eta_{j-1}^p \, dx.
\]
Then
\begin{equation}
|E_j| \leq k|B_{j-1}| = k|2B_j|.
\end{equation}
Moreover
\begin{equation}
\int_{E_j} (u - l_j)^{\gamma} \, dx \leq \int_{E_{j-1}} \left((u - l_{j-1})^{\gamma} \psi_{j-1}^{j-1} \eta_{j-1}^{j-1}\right) \, dx = \delta_{j-1} |B_{j-1}| = 2^{\gamma} k |2B_j| = \frac{|B_{j-1}|}{|B_j|} \int_{E_j} (u - l_j)^{\gamma} \psi_j \eta_j \, dx.
\end{equation}

From Lemma 2.2, we obtain
\begin{equation}
\delta_j \leq C \left[ (a_j + b_j) l_{j+1} + k_j + \left( \frac{\mu(B_j)}{|B_j|} \right)^{1/(p-1)} + \left( \frac{C_p}{p - \text{cap}(B(x_0, r_j) \setminus B(x_0, 2r_j))} + p - \text{cap}(B(x_0, r_j) \setminus B(x_0, 2r_j)) \right)^{1/(p-1)} \right].
\end{equation}

Using now the same methods as in [18] Theorem 4.27, we prove by (3.6) that
\begin{equation}
\lim_{j \to +\infty} l_j \leq C \left[ \left( \frac{1}{|B(x_0, r_0)|} \int_{B(x_0, r_0) \cap \Omega} u^\gamma \, dx \right)^{1/\gamma} + \int_0^{r_0} \left( \frac{\mu(B(x_0, r))}{|B(x_0, r)|} \right)^{1/(p-1)} \frac{dr}{r} + \int_0^{r_0} \frac{dr}{r} + M \left( \frac{p - \text{cap}(B(x_0, r) \cap \Omega^c, B(x_0, 2r))}{p - \text{cap}(B(x_0, r), B(x_0, 2r))} \right)^{1/(p-1)} \frac{dr}{r} \right]
\end{equation}
where \( r_0 \leq R_0 \).

It remains to prove that
\begin{equation}
u(x_0) \leq \lim_{j \to +\infty} l_j
\end{equation}
for q.e. \( x_0 \) and at every point \( x_0 \) of continuity of \( u \).

We choose \( \epsilon > 0 \) and we write \( l = \lim_{j \to +\infty} l_j \). We denote
\begin{equation}u_\epsilon = (2^{\gamma} - 1)^{-1} \left( \left( \frac{u - l - \epsilon}{\epsilon} \right)^{\gamma/\epsilon} - 1 \right).
\end{equation}
Then \( w_j, \psi, \eta_j \in H^{1,p}(B_j, X) \) and \( w_j, \psi, \eta_j + \varphi, \eta_j \geq 1 \) on \( B_{j+1} \cap \Omega \cap \{ u > l + 2\varepsilon \} \). Thus

\[
p - \text{cap}(B_{j+1} \cap \Omega \cap \{ u > l + 2\varepsilon \}, B_j) \leq \int_{B_j} |X(w_j, \psi, \eta_j)|^p \, dx + \int_{B_j} |X(\varphi, \eta_j)|^p \, dx.
\]

Then, using the result of Lemma 2.2, we obtain

(3.9) \[
p - \text{cap}(B_{j+1} \cap \Omega \cap \{ u > l + 2\varepsilon \}, B_j) \leq \]

\[
\leq C \left[ p - \text{cap}(B_{j+1}, B_j)(\varepsilon^{-\gamma}(b_j(l+\varepsilon) + k_j)^p + \varepsilon 1 - p(b_j(l+\varepsilon) + k_j + a_j)\psi^{-1}) + \\
+ r_j^{-\gamma} \int_{E_j} \left( 1 + \frac{u - l - \varepsilon}{\varepsilon} \right)^\gamma \, dx + \varepsilon 1 - p\mu(B_j) + \\
+ \varepsilon 1 - p(1 + M)\psi^{-1} \int_{B_j} (r_j^p \varphi^p + |X\varphi|^p + |X\psi|^p) \, dx \right].
\]

where

\[ E_j = B_{j+1} \cap \Omega \cap \{ u > l + \varepsilon \} \cap \{ \varphi_j < 1 \} \]

and the application of Lemma 2.2 take into account the following inequality

(3.10) \[
\sum_{j=1}^n \left( \frac{1}{|B_j|} \int_{E_j} \left( 1 + \frac{u - l - \varepsilon}{\varepsilon} \right)^\gamma \, dx \right)^{1/(p-1)} \leq \]

\[
\leq C \sum_{j=1}^n \left( \frac{1}{|B_j|} \int_{E_j} e^{-\gamma(u - l_{j-1})} \, dx \right)^{1/(p-1)} \leq \]

\[
\leq C \sum_{j=1}^n \left( \frac{1}{|B_j|} \int_{L_{j-1}} e^{-\gamma(u - l_{j-1})} \eta_j \eta_j \eta_j - 1 \, dx \right)^{1/(p-1)} \leq \]

\[
\leq C \sum_{j=1}^n (ke^{-\gamma} \delta_j^2)^{1/(p-1)} < +\infty.
\]
From (3.9) we have

\[
(3.11) \quad \sum_{j=1}^{\infty} \left( \frac{p - \operatorname{cap}(B_{j+1} \cap \Omega \cap \{u > l + 2\varepsilon\}, B_j)}{p - \operatorname{cap}(B_{j+1}, B_j)} \right)^{1/(p-1)} \leq C \left( \sum_{j=1}^{\infty} e^{-\varepsilon^p} (b_j(l + \varepsilon) + k_j)^p + \sum_{j=1}^{\infty} e^{-\varepsilon^p} (b_j(l + \varepsilon) + k_j + a_j) + \right.
\]

\[
+ \sum_{j=1}^{\infty} \left( \frac{1}{|B_j|} \right) \left( 1 + \frac{(u - l - \varepsilon)}{\varepsilon} \right)^p \left( \int_{B_j} dx \right)^{1/(p-1)} + e^{-\varepsilon^p} \sum_{j=1}^{\infty} \left( \frac{\mu(B_j)}{|B_j|} \right)^{1/(p-1)} +
\]

\[
+ \varepsilon^{-1} \sum_{j=1}^{\infty} (1 + M) \left( \frac{p - \operatorname{cap}(B(x_0, r_{j-1}) \setminus \Omega, B(x_0, 2r_{j-2}))}{p - \operatorname{cap}(B(x_0, r_{j-1}), B(x_0, 2r_{j-2}))} \right)^{1/(p-1)} +
\]

\[
+ \varepsilon^{-1} \sum_{j=1}^{\infty} (1 + M) \left( \frac{p - \operatorname{cap}(B(x_0, r_j) \setminus \Omega, B(x_0, 2r_{j-1}))}{p - \operatorname{cap}(B(x_0, r_j), B(x_0, 2r_{j-1}))} \right)^{1/(p-1)} < + \infty.
\]

From (3.11) the result follows.

4. Proof of Theorem 1.2

Assume that \(x_0\) is a regular point of \(\partial \Omega\).

Choose \(\sigma > 0\) and \(\rho \in (0, 1)\) to be specified later.

We observe that the set \(\{x_0\}\) has zero \(p\)-capacity. Then there exists a nonnegative function \(u_0 \in C^1(R^N)\) with support in \(B(x_0, 4\rho)\) such that \(\|u_0\|_{L^p} \leq \sigma\) and \(u(x_0) = 1\). Let \(u \in H^{1,p}(\Omega, X)\) be the solution of (1.5) in \(\Omega\) with \((u - u_0) \in H^{1,p}(\Omega, X)\); then

\[
\int_{\Omega} u^p \, dx \leq C \sigma^p.
\]

We have that \(u\) is continuous at \(x_0\) and from Theorem 1.1

\[
(4.1) \quad u(x_0) \leq C_1 \left( \frac{1}{|B(x_0, \rho)|} \int_{B(x_0, \rho) \cap \Omega} u^p \, dx \right)^{1/p} +
\]

\[
+ C_2 \left( \|u\|_{L^\infty(\Omega)} + \kappa(r_0) \right) \int_{0}^{\rho} \frac{p - \operatorname{cap}(B(x_0, s) \cap \Omega, B(x_0, 2s))}{p - \operatorname{cap}(B(x_0, r), B(x_0, 2s))} \frac{ds}{s} + C_3 \int_{0}^{\rho} \kappa(s) \frac{ds}{s}.
\]
We can find \( q \) such that

\[
C_2 \left( \left\| u \right\|_{L^\infty(Q)} + \kappa(r_0) \right) \int_0^\rho \frac{p - \text{cap} \left( B(x_0, s) \cap \Omega', B(x_0, 2s) \right)}{p - \text{cap} \left( B(x_0, r), B(x_0, 2s) \right)} \frac{ds}{s} + C_3 \int_0^\rho \frac{\kappa(s) ds}{s} \leq \frac{1}{3}.
\]

Moreover, choosing \( \sigma \leq (1/3) \left( \left| B(x_0, q) \right| \right)^{1/p} \) we have

\[
C_1 \left( \frac{1}{\left| B(x_0, q) \right|} \right) \left( \int_{B(x_0, q)} u^p \, dx \right)^{1/p} \leq \left( \frac{\sigma^p}{\left| B(x_0, q) \right|} \right)^{1/p} \leq \frac{1}{3}.
\]

From (4.1), (4.2), (4.3) we obtain

\[
u(x_0) \leq \frac{2}{3}.
\]

From the regularity of the point \( x_0 \) we obtain \( \nu(x_0) = \nu_0(x_0) = 1 \); so we have a contradiction and the result follows.

5. - Proof of Theorem 1.3

First we prove the following result:

**Lemma 5.1:** Let \( \mu \) be a positive measure in \( H^{-1,-q}(B(x_0, R), X) \). We denote again by \( \mu \) the extension of \( \mu \) to \( \mathbb{R}^N \) by 0. Assume that the following conditions holds

\[
\sup_{x \in B(2q, R)} \left( \frac{\mu(B(x, q))}{|B(x, q)|} \right)^{1/p - 1} \frac{d\theta}{\theta} < +\infty.
\]

Then

\[
\| \mu \|_{H^{-1,-q}(B(2q, R), X)} \leq C(\mu(B(x_0, R))) \sup_{x \in B(2q, R)} \left[ \int_0^2 \left( \frac{\mu(B(x, q))}{|B(x, q)|} \right)^{1/p - 1} \frac{d\theta}{\theta} \right]^{(p - 1)/p}.
\]

**Proof:** Let \( w \) be the solution of the problem

\[
\begin{align*}
\int_{B(2q, R)} |Xw|^{p-2} Xw Xv \, dx &= \int_{B(2q, R)} w \mu(dx) \\
&+ \int_{B(x_0, R)} |v|^{p-2} v w \, dx
\end{align*}
\]

for every \( v \in H^{1-p}_0(B(x_0, R), X) \), where \( w \in H^{1-p}_0(B(x_0, R), X) \). We observe that \( w \) is
positive and that \( w \) (extended by 0) is a subsolution of the subelliptic \( p \)-Laplace operator relative to \( \lambda \) in \( \mathbb{R}^N \). Then from Theorem 1.1 we have

\[
\sup_{B(x_0, R)} w \leq C \left( \frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} |w|^p \, dx \right)^{1/p} + \sup_{x \in B(x_0, R)} \left( \frac{2R}{|B(x, \varrho)|} \int_0^R \left( \frac{\mu(B(x, \varrho))}{|B(x, \varrho)| \varrho^p} \right)^{1/(p-1)} \, d\varrho \right) \]

\[
\leq C \left[ \left( \frac{R^p}{|B(x_0, R)|} \int_{B(x_0, R)} |Xw|^p \, dx \right)^{1/p} + \sup_{x \in B(x_0, R)} \left( \frac{2R}{|B(x_0, \varrho)|} \int_0^{2R} \left( \frac{\mu(B(x_0, \varrho))}{|B(x_0, \varrho)| \varrho^p} \right)^{1/(p-1)} \, d\varrho \right) \right]
\]

where \( C \) denotes possibly different constants independent of \( R \); then

\[
\int_{B(x_0, R)} |Xw|^p \, dx \leq C \mu(B(x_0, R)) \left( \frac{R^p}{|B(x_0, R)|} \int_{B(x_0, R)} |Xw|^p \, dx \right)^{1/p} +
\]

\[
+ C \mu(B(x_0, R)) \sup_{x \in B(x_0, R)} \left[ \int_0^{2R} \left( \frac{\mu(B(x_0, \varrho))}{|B(x_0, \varrho)| \varrho^p} \right)^{1/(p-1)} \, d\varrho \right].
\]

We obtain

\[
\int_{B(x_0, R)} |Xw|^p \, dx \leq C \mu(B(x_0, R)) \sup_{x \in B(x_0, R)} \left[ \int_0^{2R} \left( \frac{\mu(B(x_0, \varrho))}{|B(x_0, \varrho)| \varrho^p} \right)^{1/(p-1)} \, d\varrho \right].
\]

From (3.14) we obtain that

\[
\|\mu\|_{H^{-1, p}(B(x_0, R))} \leq \left( \int_{B(x_0, R)} |Xw|^p \, dx \right)^{(p-1)/p}
\]

so we have the result. \( \blacksquare \)

We are now in position to prove Theorem 1.3.

We assume (without loss of generality) \( a = c = 0 \) and \( A(x, \xi, \xi) = A(x, \xi) \). We assume also (without loss of generality) ess-sup \( \sup_{x \to x_0} u(x) \geq 0 \) ess-sup \( \inf_{x \to x_0} u(x) \). Let now \( \tilde{u} \) be the solution of the problem

\[
(5.1) \quad \int_{B(x_0, 4R)} A(x, X\tilde{u}) \, Xv \, dx = \int_{B(x_0, 4R)} v \, |\mu| \, (dx)
\]

for every \( v \in H^{1, p}(\Omega, X) \) with \( \text{supp}(v) \subseteq \Omega \), where \( \tilde{u} \in H^{1, p}_{\text{loc}}(\Omega, X) \) with \( \tilde{u} = u^+ \) on \( \partial B(x_0, 4R), B(x_0, 4R) \subseteq B(x_0, 8R) \subseteq \Omega \).
We observe that from Theorem 1.1 we obtain easily
\[
\sup_{B(x_0, 4R)} \tilde{u} \leq C \left[ \sup_{B(x_0, 4R)} \mu + \sup_{x \in B(x_0, 4R)} \int_0^{4R} \left( \frac{|\mu (B(x, q))|}{|B(x, q)|} Q^p \right)^{1/(p-1)} \frac{d\rho}{Q} \right].
\]
Let \( B(x_0, 64r) \subseteq B(x_0, R) \subseteq B(x_0, 8R) \subseteq \Omega \) and denote by \( w \) the solution of the problem
\[
\int_{B(x_0, 32r)} A(x, x\omega) x\nu \, dx = 0 \quad \forall \nu \in H^1_0(B(x_0, 32r), X)
\]
where \( \nu \in H^1_0(B(x_0, 32r), X) \) and \( \bar{u} = \omega \) on \( \partial B(x_0, 32r) \).
Finally we denote by \( \chi \) the function \( (\bar{u} - \omega) \in H^1_0(B(x_0, 32r), X) \).
Let \( k \geq \sup_{B(x_0, 2r)} w = M_{w}(2r) \) we recall that, since \( A(x, 0) = 0 \), we have
\[
\int_{B(x_0, 32r)} A(x, x(\bar{u} - k)^+) x\nu \, dx \leq \int_{B(x_0, 32r)} \nu |\mu| \, dx
\]
for every \( \nu \in H^1_0(B(x_0, 32r), X) \) with \( \text{supp}(\nu) \subseteq B(x_0, 32r) \) and \( \nu \geq 0 \). From (5.3), using the results in Theorem 1.1, we obtain
\[
\sup_{B(x_0, r)} (\bar{u} - k)^+ \leq C \left( \frac{1}{|B(x_0, r)|} \int_{B(x_0, 2r)} ((\bar{u} - k)^+) \, dx \right)^{1/p} + \\
+ C \sup_{x \in B(x_0, r)} \left[ \int_0^{2r} \left( \frac{|\mu (B(x, q))|}{|B(x, q)|} Q^p \right)^{1/(p-1)} \frac{d\rho}{Q} \right] \leq C \left( \frac{1}{|B(x_0, r)|} \int_{B(x_0, 32r)} \chi^p \, dx \right)^{1/p} + \\
+ C \sup_{x \in B(x_0, r)} \left[ \int_0^{2r} \left( \frac{|\mu (B(x, q))|}{|B(x, q)|} Q^p \right)^{1/(p-1)} \frac{d\rho}{Q} \right] \leq C \left( \frac{r^p}{|B(x_0, r)|} \int_{B(x_0, 32r)} |X(\chi)|^p \, dx \right)^{1/p} + \\
+ C \left[ \sup_{x \in B(x_0, 32r)} \left[ \int_0^{32r} \left( \frac{|\mu (B(x, q))|}{|B(x, q)|} Q^p \right)^{1/(p-1)} \frac{d\rho}{Q} \right] \right].
\]
From (5.4) and taking into account the result in Lemma 5.1 we obtain for \( r \leq R_0 \), with \( R_0 \) suitable,
\[
\sup_{B(x_0, r)} (\bar{u} - k)^+ \leq C \sup_{x \in B(x_0, 32r)} \left[ \int_0^{32r} \left( \frac{|\mu (B(x, q))|}{|B(x, q)|} Q^p \right)^{1/(p-1)} \frac{d\rho}{Q} \right].
\]
Let now $k = M_w(2r)$ we obtain

\[ (5.6) \quad \sup_{B(x_0, r)} \tilde{u}^+ \leq M_w(2r) + C \sup_{x \in B(x_0, 32r)} \int_0^{32r} \left( \frac{|\mu|(B(x, \varrho))}{|B(x, \varrho)|} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} \]

\[ \leq (1 - \delta_0) \sup_{B(x_0, 4r)} \tilde{u}^+ + C \sup_{x \in B(x_0, 32r)} \int_0^{32r} \left( \frac{|\mu|(B(x, \varrho))}{|B(x, \varrho)|} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} \]

where $0 < \delta_0 < 1$. Then, by iteration, we obtain for every $r \leq (R/64) \wedge R_0$

\[ (5.7) \quad \sup_{B(x_0, r)} u^+ \leq \sup_{B(x_0, r)} \tilde{u}^+ \leq \]

\[ \leq C_1 \left( \frac{r}{R} \right)^a \sup_{B(x_0, R)} \tilde{u}^+ + C \sup_{x \in B(x_0, R)} \int_0^{4R} \left( \frac{|\mu|(B(x, \varrho))}{|B(x, \varrho)|} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} \]

\[ \leq C_1 \left( \frac{r}{R} \right)^a \sup_{B(x_0, 4R)} u^+ + C \sup_{x \in B(x_0, 4R)} \int_0^{4R} \left( \frac{|\mu|(B(x, \varrho))}{|B(x, \varrho)|} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} \]

We observe now that by the same methods may be applied to obtain the estimate (5.6) we finally we obtain

\[ (3.22) \quad \sup_{B(x_0, r)} |u| \leq C_1 \left( \frac{r}{R} \right)^a \sup_{B(x_0, 4R)} |u| + \]

\[ + C \sup_{x \in B(x_0, 4R)} \int_0^{4R} \left( \frac{|\mu|(B(x, \varrho))}{|B(x, \varrho)|} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} \]

From (3.21) easily follows that $u$ assume at $x_0$ the value $g(x_0) = 0$ with continuity.

REFERENCES


