IRENE CRIMALDI - LUCA PRATELLI (*)

Paul Lévy Type Inequalities 
for Symmetric Random Variables

SUMMARY. — We prove some inequalities for a jointly symmetric system of \( n \) random variables with values in a measurable group. These inequalities include, as a particular case, the classical inequalities of Paul Lévy.

Diseguaglianze del tipo di Paul Lévy 
per variabili aleatorie simmetriche

SUNTO. — Si dimostrano alcune diseguaglianze per una \( n \)-upla globalmente simmetrica di variabili aleatorie a valori in un gruppo misurabile. Queste diseguaglianze contengono come caso particolare le diseguaglianze classiche di Paul Lévy.

1. - THE CASE OF MEASURABLE GROUP

Let \( G \) be a measurable abelian group, that is an abelian group (for which we shall use the additive notation) endowed with a \( \sigma \)-field \( \mathcal{G} \) with respect to which the operation \((x, y) \mapsto y - x\) is measurable as a mapping from the measurable space \((G \times G, \mathcal{G} \otimes \mathcal{G})\) into the measurable space \((G, \mathcal{G})\).

Let \((X_{j})_{1 \leq j \leq n}\) be a finite sequence of random variables defined on a probability space \((\Omega, \mathcal{A}, P)\) and taking values in \(G\). We shall say that \((X_{j})_{1 \leq j \leq n}\) is jointly symmet-


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ric if, for each sequence \((\varepsilon_j)_{1 \leq j \leq n}\) of elements of \(\{-1, 1\}\), the two random vectors

\[
[X_j]_{1 \leq j \leq n}, \quad [\varepsilon_j X_j]_{1 \leq j \leq n}
\]

have the same distribution. (In particular, this condition holds when the random variables \(X_j\) are independent and symmetric.)

A random variable \(T\) defined on the probability space \((\Omega, \mathcal{A}, P)\) and taking its values in \(\{1, \ldots, n, \infty\}\), is called an optional time with respect to the natural filtration associated with \((X_j)_{1 \leq j \leq n}\) if, for \(1 \leq j \leq n\), we have

\[
I_{\{T=j\}} = I_{A_j}(X_1, \ldots, X_j),
\]

where \(A_j\) is a suitable measurable set. If, moreover, it is possible to take the sets \(A_j\) such that (for each \(j\))

\[
I_{\{T=j\}} = I_{A_j}(X_1, \ldots, X_j) = I_{A_j}(-X_1, \ldots, -X_{j-1}, X_j),
\]

then we shall say that the optional time \(T\) is symmetric.

We observe that, if \(X, Y\) are two random variables with values in \(G\), then it is the same also for \(X + Y\).

**Theorem 1.1:** Let \((X_j)_{1 \leq j \leq n}\) be a jointly symmetric finite sequence of random variables on a probability space \((\Omega, \mathcal{A}, P)\) with values in \(G\), and \(T\) be an optional time with respect to the natural filtration associated with \((X_j)_{1 \leq j \leq n}\).

For \(1 \leq j \leq n\), let \(S_j = X_1 + \ldots + X_j\). Further let \(S_T = \sum_{j=1}^{n} I_{\{T=j\}} S_j\). Finally let \(f\) be any positive (or bounded) measurable function defined on \((G, \mathcal{G})\) such that

\[
f(x) \leq f(x + y) + f(x - y)
\]

for each pair \(x, y\) of elements in \(G\). Then

\[
\int_{\{T < \infty\}} f(S_T) \, dP \leq 2 \int_{\{T < \infty\}} f(S_n) \, dP.
\]

**Proof:** For \(1 \leq j \leq n\), let

\[
Z_j = S_n - S_j = X_{j+1} + \ldots + X_n.
\]

Then the two random vectors \([I_{\{T=j\}}, S_j, Z_j]\), \([I_{\{T=j\}}, S_j, -Z_j]\) have the same distribution. (This follows from the hypothesis of joint symmetry of \((X_j)_{1 \leq j \leq n}\) by the prop-
Therefore we have
\[ \int_{\{T=j\}} f(S_n) \, dP = \int_{\{T=j\}} f(S_j + Z_j) \, dP = \int_{\{T=j\}} f(S_j - Z_j) \, dP. \]

Let now \( Z_T = \sum_{j=1}^n I_{\{T=j\}} Z_j \). Adding the above equalities, we find
\[ \int_{\{T < \infty\}} f(S_n) \, dP = \int_{\{T < \infty\}} f(S_T + Z_T) \, dP = \int_{\{T < \infty\}} f(S_T - Z_T) \, dP. \]

Hence, using the property (1.3), we finally get
\[ \int_{\{T < \infty\}} f(S_T) \, dP \leq 2 \int_{\{T < \infty\}} f(S_n) \, dP. \]

In the same way we can prove the following theorem:

**Theorem 1.2**: Under the same hypotheses as that of the preceding theorem, let us suppose that the optional time \( T \) is symmetric, and let \( X_T = \sum_{j=1}^n I_{\{T=j\}} X_j \). Then
\[ \int_{\{T < \infty\}} f(X_T) \, dP \leq 2 \int_{\{T < \infty\}} f(S_n) \, dP. \]

**Proof**: For \( 1 \leq j \leq n \), let
\[ Y_j = S_n - X_j = \sum_{1 \leq i \leq n, i \neq j} X_i. \]

Then the two random vectors \([I_{\{T=j\}}, X_j, Y_j],[I_{\{T=j\}}, X_j, -Y_j]\) have the same distribution. (This follows from the hypothesis of joint symmetry of \((X_j)_{1 \leq j \leq n}\) by the property (1.2).) Therefore we have
\[ \int_{\{T=j\}} f(S_n) \, dP = \int_{\{T=j\}} f(X_j + Y_j) \, dP = \int_{\{T=j\}} f(X_j - Y_j) \, dP. \]

Let now \( Y_T = \sum_{j=1}^n I_{\{T=j\}} Y_j \). Adding the above equalities, we find
\[ \int_{\{T < \infty\}} f(S_n) \, dP = \int_{\{T < \infty\}} f(X_T + Y_T) \, dP = \int_{\{T < \infty\}} f(X_T - Y_T) \, dP. \]

Hence, using the property (1.3), we finally get
\[ \int_{\{T < \infty\}} f(X_T) \, dP \leq 2 \int_{\{T < \infty\}} f(S_n) \, dP. \]
2. - The case of a normed space

Now we shall specialize the hypotheses of the previous section supposing, in addition, that there exists a positive real function \( x \mapsto |x| \), measurable on \((G, \mathcal{G})\), such that

\begin{equation}
(2.1) \quad | - x | = |x|, \quad |x + y| \leq |x| + |y|, \quad |x| \leq |x + y| \vee |x - y|
\end{equation}

for each pair \( x, y \) of elements in \( G \).

These particular hypotheses are clearly satisfied when \( G \) is the underlying additive group of a separable normed space and \( \mathcal{G} \) coincides with the Borel \( \sigma \)-field of \( G \).

Remark 2.1: More generally, the hypotheses of the present section are satisfied when \( G \) is the underlying additive group of a (not necessarily separable) normed space for which there exists a countable set \( D \) of continuous linear forms generating the \( \sigma \)-field \( \mathcal{G} \) and such that the norm of any element \( x \) of \( G \) coincides with \( \sup_{l \in D} l(x) \). This situation includes, for instance, the important case of the space \( L^\infty \).

Let us remark that the last of the properties (2.1) implies that every function \( f \) such that \( f(x) = g(|x|) \), with \( g \) positive and increasing on \( \mathbb{R}_+ \), has the property (1.3).

If \( X \) is a random variable taking its values in \( G \), we shall denote by \( |X| \) the real random variable \( \omega \mapsto |X(\omega)| \).

Theorems (1.1), (1.2) imply the following corollary:

Corollary 2.2: Let \( (X_j)_{1 \leq j \leq n} \) be a jointly symmetric finite sequence of random variables on a probability space \((\Omega, \mathcal{A}, P)\) with values in \( G \).

Let \( S_j = X_1 + \ldots + X_j \). Then, for each positive real number \( s \), we have:

\begin{equation}
(2.2) \quad P \{ \sup_{1 \leq j \leq n} |S_j| > s\} \leq 2P\{ |S_n| > s\},
\end{equation}

\begin{equation}
(2.3) \quad P \{ \sup_{1 \leq j \leq n} |X_j| > s\} \leq 2P\{ |S_n| > s\}.
\end{equation}

Proof: In order to prove inequality (2.2), it is sufficient to apply Theorem (1.1) with

\[ T(\omega) = \inf \{ j : |S_j(\omega)| > s\} \quad \text{and} \quad f(x) = I_{|x|}(\omega)|.\]

In the same way, in order to prove inequality (2.3), it is sufficient to apply Theorem (1.2) with

\[ T(\omega) = \inf \{ j : |X_j(\omega)| > s\} \quad \text{and} \quad f(x) = I_{|x|}(\omega)|.\]

Inequalities (2.2), (2.3) are known as inequalities of Paul Lévy.
Now we shall prove the following theorem:

**Theorem 2.3:** Let \((X_j)_{1 \leq j \leq n}\) be a finite sequence of independent symmetric random variables on a probability space \((\Omega, \mathcal{F}, P)\) with values in \(G\), and let \(T\) be an optional time with respect to the natural filtration associated with \((X_j)_{1 \leq j \leq n}\).

For \(1 \leq j \leq n\), let \(S_j = X_1 + \ldots + X_j\). Further let \(S_T = \sum_{j=1}^{n} I_{\{T=j\}} S_j, s_T = \sup_{\omega \in \Omega} |S_T(\omega)|\). Then:

(a) For each positive increasing function \(g\) defined on \([0, + \infty]\), we have

\[
\int_{\{T < \infty\}} g(|S_n|) \, dP \leq 2 P\{T < \infty\} \int g(s_T + |S_n|) \, dP.
\]

(b) For each positive real number \(s\), we have

\[
P\{|S_n| > s, T < \infty\} \leq 2 P\{T < \infty\} P\{|S_n| > s - s_T\}.
\]

**Proof:** Let \(Z_j = S_n - S_j = X_{j+1} + \ldots + X_n\) and \(g\) be a positive increasing function defined on \([0, + \infty]\). Then

\[
\int_{\{T=j\}} g(|S_n|) \, dP \leq \int_{\{T=j\}} g(|S_j| + |Z_j|) \, dP \leq \int_{\{T=j\}} g(s_T + |Z_j|) \, dP = P\{T=j\} \int g(s_T + |Z_j|) \, dP.
\]

(The final equality follows from the fact that \(Z_j\) and \(I_{\{T=j\}}\) are independent.) Hence, adding these relations, we get

\[
\int_{\{T < \infty\}} g(|S_n|) \, dP \leq \sum_{j=1}^{n} P\{T=j\} \int g(s_T + |Z_j|) \, dP.
\]

Let us now define the function \(f\) by setting \(f(x) = g(s_T + |x|)\). Applying to this function Theorem (1.1) (in which we replace the sequence \((X_1, \ldots, X_n)\) by the sequence \((X_n, \ldots, X_1)\) and take for optional time the constant \(n - j\)), we find:

\[
\int g(s_T + |Z_j|) \, dP \leq 2 \int g(s_T + |S_n|) \, dP.
\]

Introducing this inequality into (2.5), we get finally the relation (2.4) and thus statement (a) is proved.

Statement (b) is a particular case of (a), obtained by taking for \(g\) the indicator function \(I_{[s, \infty)}\).
Using the theorem just proved, we can easily deduce the following corollary:

**Corollary 2.4:** Under the same hypotheses as that of the preceding theorem, let us suppose also that there exists a constant \( c \) such that \( |X_j| \leq c \), for each \( j \). Then we have

\[
(1 - 2 P\{\sup_{1 \leq j \leq n} |S_j| > s\} \ e^{\lambda(s+c)}) \int \exp (\lambda |S_n|) \ dP \leq P\{\sup_{1 \leq j \leq n} |S_j| \leq s\} \ e^{\lambda s}
\]

for each pair \( s, \lambda \) of positive real numbers.

The proof follows on applying Theorem (2.3) with

\[
T(\omega) = \inf\{ j: |S_j(\omega)| > s \} \quad \text{and} \quad g(x) = e^{\lambda x}.
\]

**Remark 2.5:** Let \((X_j)_{j \geq 1}\) be an infinite sequence of independent symmetric random variables on a probability space \((\Omega, \mathcal{A}, P)\) with values in \( G \), such that \( |X_j| \leq c \) for each \( j \).

Let \( S_n = X_1 + \ldots + X_n \) and let us suppose that \( (|S_n|)_{n \geq 1} \) converges almost surely to a positive real random variable \( Z \).

Then, following an argument similar to that of Ledoux-Talagrand in [3], we find, as a consequence of corollary (2.4), that \( \int \exp (\lambda Z) \ dP < \infty \) for some \( \lambda > 0 \). Indeed, if in corollary (2.4) we choose \( s \) such that \( P\{\sup_n |S_n| > s\} \leq (4e)^{-1} \), and let \( \lambda = (s+c)^{-1} \), we get, for each strictly positive integer \( n \),

\[
\int \exp (\lambda |S_n|) \ dP \leq 2e^{\lambda s},
\]

and hence, by Fatou's Lemma, we obtain

\[
\int \exp (\lambda Z) \ dP \leq 2e^{\lambda s}.
\]

Further, from the above inequality, we can easily deduce that \( Z \) has finite moments of any order.

Applying once more Theorem (2.3), we can also deduce the following result, due to J. Hoffmann-Jørgensen [1,2]:

**Theorem 2.6:** Under the same hypotheses as that of the preceding theorem, we have

\[
P\{|S_n| > 2s + t\} \leq 4(P\{|S_n| > s\})^2 + P\{\sup_{1 \leq j \leq n} |X_j| > t\}
\]

for each pair \( s, t \) of positive real numbers.
PROOF: Let us consider the two optional times $U, V$ defined by:

$$U(\omega) = \inf \{ j \in \mathbb{N}: 1 \leq j \leq n, |S_j(\omega)| > s \},$$

$$V(\omega) = \inf \{ j \in \mathbb{N}: 1 \leq j \leq n, |X_j(\omega)| > t \}.$$

Then $\{ V < \infty \} = \{ \sup_{1 \leq j \leq n} |X_j| > t \}$. Moreover, the obvious inclusion

$$\{ |S_n| > 2s + t \} \subset \{ U < \infty \}$$

implies

$$P\{ |S_n| > 2s + t \} \leq P\{ |S_n| > 2s + t, U < V \} + P\{ V < \infty \}.$$

Therefore, it is enough to prove

$$(2.6) \quad P\{ |S_n| > 2s + t, U < V \} \leq 4(P\{ |S_n| > s \})^2.$$ 

To this end, let us denote by $T$ the optional time which coincides with $U$ on $\{ U < V \}$ and with $+ \infty$ on $\{ U \geq V \}$. Let $\omega$ be an element in $\{ T < \infty \}$ and $j = T(\omega) = U(\omega)$. Then $j < V(\omega)$ implies $|X_j(\omega)| \leq t$ and consequently we have

$$|S_j(\omega)| \leq |S_j(\omega) - X_j(\omega)| + |X_j(\omega)| \leq s + t.$$ 

Hence, with the notations of Theorem (2.3), we have $s_T \leq s + t$. Using this inequality and applying statement (b) of Theorem (2.3), we get:

$$(2.7) \quad P\{ |S_n| > 2s + t, U < V \} = P\{ |S_n| > 2s + t, T < \infty \} \leq$$

$$\leq 2P\{ T < \infty \} \cdot P\{ |S_n| > 2s + t - s_T \} \leq 2P\{ T < \infty \} \cdot P\{ |S_n| > s \}.$$ 

On the other hand, applying (2.2), we find

$$P\{ T < \infty \} \leq P\{ U < \infty \} = P\{ \sup_{1 \leq j \leq n} |S_j| > s \} \leq 2P\{ |S_n| > s \}.$$

Therefore, in order to prove (2.6), it is sufficient to insert the last inequality into (2.7).

3. - THE REAL CASE

Let us now consider the particular case in which $G$ is the additive group $(\mathbb{R}, +)$ of real numbers and $\mathcal{S}$ the Borel $\sigma$-field of $\mathbb{R}$. In this situation, the elementary inequality $x \leq (x + y) \vee (x - y)$, which holds for each pair $x, y$ of real numbers, shows that any positive increasing function $f$ defined on $\mathbb{R}$ has the property (1.3). Therefore, if in Theorem (1.1) we choose

$$T(\omega) = \inf \{ j: S_j(\omega) > s \} \quad \text{and} \quad f(x) = I_{1s, \infty}(x),$$

we get the following corollary:
**Corollary 3.1:** Let \((X_j)_{1 \leq j \leq n}\) be a jointly symmetric finite sequence of real random variables on a probability space \((\Omega, \mathcal{A}, P)\).

Let \(S_j = X_1 + \ldots + X_j\). Then, for each real number \(s\), we have

\[
P\{\sup_{1 \leq j \leq n} S_j > s\} \leq 2 P\{S_n > s\}.
\]

**Bibliography**

