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Homogenization of the p-Laplacian Associated with the Heisenberg Group (***)

ABSTRACT. — We consider the asymptotic behavior of the solutions of a quasilinear problem associated with the Heisenberg group in a periodically perforated domain. We give the critical size to obtain a shift of the spectrum using the energy method and explicit correctors.

Omogeneizzazione per il p-laplaciano associato al gruppo di Heisenberg

SUNTO. — Si considera il comportamento asintotico delle soluzioni di un problema quasilineare associato al gruppo di Heisenberg in un dominio con buchi periodici. Si ottiene la taglia critica che comporta uno spostamento dello spettro usando il metodo dell'energia e i correttori espliciti.

1. - INTRODUCTION

We are interested in studying the effects in homogenization with holes of the coupled degeneracy of the Heisenberg p-Laplacian (the one due to the Heisenberg group, the other one due to the structure of the operator).

Let us recall the definition of the left invariant vector fields associated with the Heisenberg group in $\mathbb{R}^3$:

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z},$$

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$$Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial z} .$$

We introduce the gradient ∇_H by

$$\nabla_H = (X, Y)^T = \sigma \nabla ,$$

where ∇ is the standard gradient: $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)^T$ and σ the following matrix

$$\sigma = \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \end{pmatrix}$$

Let us recall the following condition on commutator:

$$[X, Y] = XY - YX = -4 \frac{\partial}{\partial z} ;$$

that condition implies the subellipticity of the vector fields [Fo 1], [Ho] and a scaled Poincaré's ineqality of order 2 [Je].

Let us introduce now Δ_H^p, the p-Laplacian associated with the Heisenberg group in $\mathbb{R}^3$ for $p \in [2, \nu[$:

$$\Delta_H^p u = X(|\nabla_H u|^{p-2} Xu) + Y(|\nabla_H u|^{p-2} Yu) ,$$

where $\nu = 4$ is the homogeneous dimension of the Heisenberg group in $\mathbb{R}^3$ (for the Heisenberg group in $\mathbb{R}^{2n+1}$, $\nu = 2n+2$). To study this operator we use the Poincaré's and Sobolev's inequalities of order p, which are a consequence of the Poincaré's inequality of order 2 as proved in [BM].

Let us recall the results obtained in [LP], in the euclidian framework, for the p-Laplacian in $\mathbb{R}^N$: $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. We consider ε-periodically (for the euclidian group) distributed holes $\mathcal{T}^\varepsilon$ of size $r_\varepsilon = \varepsilon^{n/(n-p)}$. Let Ω be a bounded open domain in $\mathbb{R}^N$ and $\Omega^\varepsilon = \Omega - \mathcal{T}^\varepsilon$; they consider the problem

$$\text{(1.1)} \qquad \begin{cases} -\Delta_p u^\varepsilon = f> & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega \cup \partial\mathcal{T}^\varepsilon \end{cases}$$

and they proved that the zero extension on the holes $\mathcal{T}^\varepsilon$ of u^ε, denoted again by u^ε, converges to the solution u of the problem

$$\text{(1.2)} \qquad \begin{cases} -\Delta_p u + Cu = f> & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega \cup \partial\mathcal{T}^\varepsilon \end{cases}$$

where $C \neq 0$ is an explicit constant related to the p-capacity (cf. [LP], [MP] also concerning some results on convergence rate).

Thanks to the results obtained for the pavage in Heisenberg group, [BMT], we can repeat here the construction of the periodic homogenization problem for the Heisenberg p-Laplacian Δ_H^p. To obtain the shift of the spectrum, the critical size changes and we have to consider a size $r_\varepsilon = \varepsilon^{\nu/\nu - p}$, where ν is the homogeneous dimension of the Heisenberg group (we emphasize that ν acts as the effective dimension of the problem).

We then consider the problem of the construction of correctors in terms of the Folland distance (that is the distance associated to the Heisenberg Laplacian, [Fo1], [KV]).

2. - Preliminaries

Properties of the operator Δ_H^p.

Let us recall the definition and some properties of the Heisenberg group in any odd dimension $N = 2n + 1$, where $n \geqslant 1$.

Let $\xi = (x^1, x^2, \dots x^n, y^1, y^2, \dots y^n, z) = (x, y, z)$, where $x = (x^1, x^2, \dots x^n)$ and $y = (y^1, y^2, \dots y^n)$.

Let us consider the operators: for $j = 1 \dots n$ X^j and Y^j:

$$X^j = \frac{\partial}{\partial x^j} + 2y^j \frac{\partial}{\partial z},$$

$$Y^j = \frac{\partial}{\partial y^j} - 2x^j \frac{\partial}{\partial z}.$$

We denote by σ the $2n \times 2n + 1$-matrix such that

$$\sigma\nabla = (X^1, X^2, \dots X^n, Y^1, Y^2, \dots Y^n)^T = \nabla_H.$$

The vector fields X^j, Y^j for $j = 1 \dots n$, satisfy an Hörmander condition of order 1.

For two vectors $\xi = (x, y, z)$ and $\xi' = (x', y', z')$, we define the ξ' right-translation of ξ:

$$\xi + \xi' = (x + x', y + y', z + z' + 2(x' \cdot y - x \cdot y')),$$

where $\cdot$ stands for the standard scalar product in $\mathbb{R}^k$, (here $k = n$), and the homotheties

$$\alpha \circ \xi = (\alpha x, \alpha y, \alpha^2 z).$$

where α is a positive real number, and αx is the standard homothety in $\mathbb{R}^n$.

Notice that $\xi + \xi' \neq \xi' + \xi$; we shall call $\xi' + \xi$ the ξ' left-translation of ξ, let us remark that the Lebesgue measure is invariant with respect to these right or left translations.

Let us recall that the operator ∇_H is invariant with respect to left translations, i.e.

for fixed ξ',

$$\nabla_H(u(\xi'+\xi)) = (\nabla_H(u))(\xi'+\xi).$$

One can check that $\alpha\circ(\xi+\xi') = \alpha\circ\xi + \alpha\circ\xi'$ and

$$\nabla_H(u(\alpha\circ\xi)) = \alpha(\nabla_H u)(\alpha\circ\xi).$$

We call $\varrho^*(\xi,\xi')$ the intrinsic distance between ξ and ξ' by:

$$\varrho^*(\xi,\xi') = \sup\left\{\phi(\xi)-\phi(\xi');\ \phi\in C_0^1(\mathbb{R}^N),\ \sum_{j=1}^{n}(X^j(\phi))^2+(Y^j(\phi))^2\leq 1\right\}.$$

Let us also introduce the distance:

$$\varrho(\xi,\xi') = (((x-x')^2+(y-y')^2)^2+(z-z'-2(x'\cdot y-x\cdot y'))^2)^{1/4},$$

The following equalities are satisfied by ϱ:

$$\varrho(\alpha\circ\xi,\alpha\circ\xi') = \alpha\varrho(\xi,\xi'),$$

$$\varrho(\eta+\xi,\eta+\xi') = \varrho(\xi,\xi').$$

These identities are also true for ϱ^*.

Using these properties, it is possible to check that these distances are equivalent and then we will use ϱ because of its explicit form.

For any $\xi\in\mathbb{R}^N$, we define the «radius» ϱ:

$$\varrho = \varrho(\xi,0).$$

Using some computations as in [CDG2] and [BMT] we can prove the following identities:

$$X^j(\varrho) = \frac{1}{\varrho^3}\left(x^j\sum_{i=1}^{n}((x^i)^2+(y^i)^2)+y^jz\right),\quad Y^j(\varrho) = \frac{1}{\varrho^3}\left(y^j\sum_{i=1}^{n}((x^i)^2+(y^i)^2)-x^jz\right),$$

$$\|\nabla_H(\varrho)\|^2 = \frac{\sum_{i=1}^{n}((x^i)^2+(y^i)^2)}{\varrho^2}$$

and

$$\sum_{i=1}^{n}(X^j)^2(\varrho^4)+(Y^j)^2(\varrho^4) = 4(2n+4)\sum_{i=1}^{n}((x^i)^2+(y^i)^2),$$

$$\sum_{i=1}^{n}(X^j)^2(\varrho)+(Y^j)^2(\varrho) = \frac{(2n+1)}{\varrho}\|\nabla_H(\varrho)\|^2.$$

Let us introduce Δ_H^p the p-laplacian associated with the Heisenberg group for $p \in [2, \infty)$:

$$\Delta_H^p u = \sum_{j=1}^{n} X^j(|\nabla_H u|^{p-2} X^j(u)) + Y^j(|\nabla_H u|^{p-2} Y^j(u))\, dx$$

and the Sobolev spaces relative to ∇_H, let Ω be an open bounded domain in $\mathbb{R}^N$, the space $W_H^{1,p}(\Omega)$ is defined as the completion of the space $C^\infty(\overline{\Omega})$ for the norm:

$$\|u\|_{W_H^{1,p}} = (\|u\|_{L^p(\Omega)}^p + \|\nabla_H u\|_{L^p(\Omega)}^p)^{1/p}.$$

The space $W_{H,0}^{1,p}(\Omega)$ is defined as the closure of $C_0^\infty(\overline{\Omega})$ in $W_H^{1,p}(\Omega)$.

For regular «radial» functions, that is functions depending only on ϱ, using the preceding computations (see [CDG2]):

$$\Delta_H^p f(\varrho) = (p-1)\,|\nabla_H \varrho|^p\, |f'(\varrho)|^{p-2}\left[f''(\varrho) + \frac{(2n+1)}{p-1}\,\frac{f'(\varrho)}{\varrho}\right].$$

From now on, the Lebesgue measure which appears in the integral will be omitted.

We recall (see [BT]) that there exists a constant C such that if $u \in W_{H,0}^{1,p}(\Omega)$ then

$$\int_\Omega |u|^p \leq C \int_\Omega |\nabla_H u|^p.$$

We will say that a function u is a (weak) solution of

$$\begin{cases} -\Delta_H^p u = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $f \in W_H^{-1,p'}(\Omega)$ (the dual space of $W_{H,0}^{1,p}(\Omega)$, $1/p + 1/p' = 1$) if $u \in W_{H,0}^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla_H u|^{p-2} \nabla_H u \cdot \nabla_H v = \langle f, v\rangle \quad \text{for any } v \in W_{H,0}^{1,p}(\Omega)\,.$$

PAVAGE: Let us recall the construction of a periodic pavage associated with the Heisenberg group defined in [BMT].

We shall denote by $Q = [-1, +1)^N$ the usual cube in $\mathbb{R}^N$ with edge of length 2 centered in the origin. Let $k^j \in \mathbb{Z}$ for any $j = 1, \dots, N$ and $k = (2k^1, 2k^2, \dots, 2k^n, 2k^{n+1}, 2k^{n+2}, \dots, 2k^{2n}, 2k^{2n+1})$. Let $Q_k = k + Q$, the cube Q left-translated by k with respect to translations of the Heisenberg group. Is has been proved in [BMT] that for any $\xi' \in \mathbb{R}^N$, there exists a unique $k \in \mathbb{Z}^N$ such that

$\xi' \in Q_k$. Consider now this pavage dilated by a small parameter $\varepsilon > 0$:

$$Q_k^\varepsilon = \varepsilon \circ Q_k = \varepsilon \circ (k + Q) = (\varepsilon \circ k) + (\varepsilon \circ Q).$$

It is easy to remark that this is a pavage, indeed the translations commute with the homotheties. The number of Q_k^ε such that $Q_k^\varepsilon \subset \Omega$ is of order $|\Omega|/\varepsilon^\nu$, indeed $|\Omega| \approx \approx |Q_k^\varepsilon| \#\{Q_k^\varepsilon \subset \Omega\} = |Q^\varepsilon| \#\{Q_k^\varepsilon \subset \Omega\}$, where we have denoted by $|A|$ the Lebesgue measure of A, by Q^ε the set $\varepsilon \circ Q$ and by $\#\{Q_k^\varepsilon \subset \Omega\}$ the number of subsets Q_k^ε contained in Ω.

Let us denote by B the subset of Q: $B = \{\xi : \varrho(\xi, 0) < 1\}$ (Korány's ball); we now introduce the notations:

$$B^a = \{\xi : \varrho(\xi, 0) < a\} = a \circ B,$$

$$B_k^{a,\varepsilon} = (\varepsilon \circ k) + B^a,$$

$$T^{a,\varepsilon} = \cup_k \overline{B}_k^{a,\varepsilon}.$$

Let us remark that $B_k^{a,\varepsilon} \subset Q_k^\varepsilon$, if $a \leqslant \varepsilon$.

For a given function v_0 defined in Q^ε, we can define a function denoted by v_0^ε as the periodic extension of v_0, with respect to the intrinsic pavage Q_k^ε: indeed let $\xi' \in \mathbb{R}^N$ then there exists a unique $k \in \mathbb{Z}^N$ (piecewise constant with respect to ξ'), and a unique $\xi \in Q^\varepsilon$ such that

$$\xi' = (\varepsilon \circ k) + \xi$$

and then, because the inverse of the left translation of a vector ξ by a vector $\varepsilon \circ k$ is the left translation by the vector $\varepsilon \circ \ominus k$, we define

$$v^\varepsilon(\xi') = v_0^\varepsilon(\xi) = v_0^\varepsilon((\varepsilon \circ (\ominus k)) + \xi'). \tag{2.1}$$

3. - Homogenization of Δ_H^p

Let $a(\cdot)$ be a positive real function $a(\cdot): (0, 1] \to (0, 1)$ such that $a(t) < t$. Let ε be a sequence $\varepsilon \in (0, 1]$ converging to zero and set $T^{a(\varepsilon)} = T^{a(\varepsilon), \varepsilon}$.

We wish to study the limit as $\varepsilon \to 0$ of the weak solutions u^ε of the following problems:

$$\begin{cases} -\Delta_H^p u^\varepsilon = f> & \text{in } \Omega \backslash T^{a(\varepsilon)}, \\ u^\varepsilon = 0 & \text{on } \partial\Omega \cup \partial T^{a(\varepsilon)}, \end{cases} \tag{3.1}$$

that is of the minimization problem

$$(P^\varepsilon) \qquad \inf\left\{\int_{\Omega\setminus T^{a(\varepsilon)}} |\nabla_H v|^p\, d\xi - p\int_{\Omega\setminus T^{a(\varepsilon)}} fv\, d\xi;\ v\in W^{1,p}_{H,0}(\Omega\setminus T^{a(\varepsilon)})\right\}.$$

The main result is the following:

THEOREM 3.1: *Let Ω be a bounded open domain of $\mathbb{R}^N$, $N=2n+1$, $n\geq 1$, $\nu=N+1$, $\nu>p\geq 2$ and $f\in L^\infty(\Omega)$.*

Assume that

$$(3.2) \qquad \lim_{\varepsilon\to 0}\frac{a(\varepsilon)^{\nu-p}}{2^N\varepsilon^\nu}=\alpha\in[0,\infty[.$$

Let $\overline{u}^\varepsilon$ be the extension of u^ε in Ω:

$$\overline{u}^\varepsilon=\begin{cases} u^\varepsilon & \text{in } \Omega\setminus T^{a(\varepsilon)}\\ 0 & \text{in } T^{a(\varepsilon)}.\end{cases}$$

Then $(\overline{u}^\varepsilon)$ converges strongly in $L^p(\Omega)$ to u defined as the weak solution of the following problem:

$$(3.3) \qquad \begin{cases} -\Delta^p_H u+\alpha C|u|^{p-2}u=f> & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where the constant C is the capacity related to the operator Δ^p_H given by

$$(3.4) \qquad C=\inf\left\{\int_{\mathbb{R}^N}|\nabla_H v|^p\, d\xi\,;\ v\in C_0^\infty(\mathbb{R}^N),\ v\geq 1 \text{ on } B\right\}.$$

Actually, u is the solution of the minimization problem

$$(P) \qquad \inf\left\{\int_\Omega |\nabla_H v|^p\, d\xi+\alpha C\int_\Omega |v|^p\, d\xi-p\int_\Omega fv\, d\xi;\ v\in W^{1,p}_{H,0}(\Omega)\right\}.$$

Moreover

$$(3.5) \qquad \lim_{\varepsilon\to 0}\int_\Omega |\nabla_H\overline{u}^\varepsilon|^p\, d\xi=\int_\Omega|\nabla_H u|^p\, d\xi+\alpha C\int_\Omega |u|^p\, d\xi\,.$$

PROOF OF THEOREM 3.1:

STEP 1: Compactness of $(\overline{u}^\varepsilon)$.

Proposition 3.1: *The sequence* $(\overline{u}^\varepsilon)$ *is bounded in* $W^{1,p}_{H,0}(\Omega)$ *and strongly relatively compact in* $L^p(\Omega)$.

Proof: Since u^ε verifies

$$\int_{\Omega\setminus T^{a(\varepsilon)}} |\nabla_H u^\varepsilon|^{p-2}\nabla_H u^\varepsilon \nabla_H v\, d\xi = \int_{\Omega\setminus T^{a(\varepsilon)}} fv\, d\xi$$

for any $v \in W^{1,p}_{H,0}(\Omega\setminus T^{a(\varepsilon)})$, we have in particular

$$\int_{\Omega\setminus T^{a(\varepsilon)}} |\nabla_H u^\varepsilon|^p\, d\xi = \int_{\Omega\setminus T^{a(\varepsilon)}} fu^\varepsilon\, d\xi\,.$$

Consider now the extension $\overline{u}^\varepsilon$ of u^ε. The following estimate is true

$$\int_\Omega |\nabla_H \overline{u}^\varepsilon|^p\, d\xi \leqslant \|\overline{u}^\varepsilon\|_{L^p(\Omega)}\|f\|_{L^{p'}(\Omega)}\,.$$

From Poincaré's inequality, we obtain

$$\int_\Omega |\nabla_H \overline{u}^\varepsilon|^p\, d\xi \leqslant C\,.$$

Hence $\overline{u}^\varepsilon$ is bounded in $W^{1,p}_{H,0}(\Omega)$ and strongly relatively compact in $L^p(\Omega)$ from the compact imbedding: $W^{1,p}_{H,0}(\Omega) \hookrightarrow L^p(\Omega)$. ■

Consequence: Let u^ε be the solution of (P^ε). There exist $u \in W^{1,p}_{H,0}(\Omega)$ and a subsequence $(\overline{u}^{\varepsilon_k})_{k\in\mathbb{N}}$ of $(\overline{u}^\varepsilon)$ such that,

$$\text{(3.6)}\qquad \begin{cases} \overline{u}^{\varepsilon_k} \rightharpoonup u \text{ weakly in } W^{1,p}_{H,0}(\Omega) \text{ as } k \to \infty\,, \\ \overline{u}^{\varepsilon_k} \to u \text{ strongly in } L^p(\Omega) \text{ as } k \to \infty\,. \end{cases}$$

Our goal is to identify this cluster point u.

Step 2: Test functions.

In order to characterize the limit function u, we will use a sequence of tests functions w^ε defined as follows. Let us consider the sequence of capacitary potentials w_0^ε related to the nonlinear Heisenberg Laplacian, defined in Q^ε by:

$$\text{(3.7)}\qquad \begin{cases} -\Delta_H^p w_0^\varepsilon = 0 & \text{in } B^\varepsilon\setminus B^{a(\varepsilon)}\,, \\ w_0^\varepsilon = 0 & \text{in } B^{a(\varepsilon)}\,, \\ w_0^\varepsilon = 1 & \text{in } Q^\varepsilon\setminus B^\varepsilon\,. \end{cases}$$

Let us define w^ε as the periodic extension of w_0^ε to the whole space $\mathbb{R}^N$ (see (2.1)). Due to of the invariance of the operator with respect to translations and homotheties, the sequence (w^ε) verifies

$$
(3.8)\qquad \begin{cases} -\Delta_H^p w^\varepsilon = 0 & \text{in } B_k^\varepsilon \backslash B_k^{a(\varepsilon)}, \\ w^\varepsilon = 0 & \text{in } B_k^{a(\varepsilon)}, \\ w_k^\varepsilon = 1 & \text{in } Q_k^\varepsilon \backslash B_k^\varepsilon. \end{cases}
$$

LEMMA 3.1: *The sequence* (w^ε) *is such that:*

$$
(3.9)\qquad w^\varepsilon \to 1 \text{ strongly in } L^p(\Omega),
$$

$$
(3.10)\qquad \nabla_H w^\varepsilon \rightharpoonup 0 \text{ weakly in } L^p(\Omega),
$$

$$
(3.11)\qquad |\nabla_H w^\varepsilon|^p \, d\xi \to \alpha C \, d\xi \ \star\text{-weakly in } \mathcal{M}(\overline{\Omega}),
$$

$$
(3.12)\qquad \int_\Omega |\nabla_H w^\varepsilon|^q \, d\xi \to 0 \text{ for every } 0 < q < p .
$$

PROOF.

PROOF OF (3.11): Let $\omega \subset \Omega$ and denote by $K(\omega)$ the set of $k \in \mathbb{Z}^N$ such that $Q_k^\varepsilon \subset \omega$. Using the definition of the extension w^ε of w_0^ε, we get

$$
\int_\omega |\nabla_H w^\varepsilon|^p \, d\xi \approx \sum_{k \in K(\omega)} \int_{B_k^\varepsilon \backslash B_k^{a(\varepsilon)}} |\nabla_H w^\varepsilon|^p \, d\xi =
$$

$$
= \#\{B_k^\varepsilon \subset \omega\} \int_{B^\varepsilon \backslash B^{a(\varepsilon)}} |\nabla_H w_0^\varepsilon|^p \, d\xi = \frac{|\omega|}{2^N \varepsilon^\nu} a_\varepsilon^{\nu - p} \int_{B^{\varepsilon/a(\varepsilon)} \backslash B} |\nabla_H w_0^\varepsilon|^p \, d\xi .
$$

By (3.2) and since $\int_{B^{\varepsilon/a(\varepsilon)} \backslash B} |\nabla_H w_0^\varepsilon|^p \, d\xi$ converges to C, defined in (3.4), as $\varepsilon \to 0$, we obtain

$$
\int_\omega |\nabla_H w^\varepsilon|^p \, d\xi \to \alpha C |\omega| .
$$

This implies (3.11).

PROOF OF (3.9) AND (3.10): From (3.11), $\nabla_H w^\varepsilon$ is bounded in $L^p(\Omega)$. Moreover, from Poincaré's inequality, $w^\varepsilon - 1$ is bounded in $L^p(\Omega)$. Hence, using the compactness of the imbedding $W_{H,0}^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, there exists $w \in W_{H,0}^{1,p}(\Omega)$ such that (w^ε) (up to a subsequence) converges to w weakly in $W_{H,0}^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$.

Actually $w = 1$ and thus all the sequence (w^ε) converge. For that, let us denote by

χ^ε the characteristic function $\sum_k \chi_{Q_k^\varepsilon \setminus B_k^\varepsilon}$, then

$$\chi^\varepsilon = \chi^\varepsilon w^\varepsilon .$$

We have (cf. Lemma 1 of [BMT]),

$$\chi^\varepsilon \longrightarrow \frac{1}{|Q|} |Q \setminus B| \quad \text{in } L^\infty\text{-weak } * .$$

For any $\phi \in C^\infty(\Omega)$,

$$\int_\Omega \chi^\varepsilon w^\varepsilon \phi = \int_\Omega \chi^\varepsilon \phi .$$

Passing to the limit for $\varepsilon \to 0$,

$$\frac{1}{|Q|} \int_\Omega |Q \setminus B| w \phi = \frac{1}{|Q|} \int_\Omega |Q \setminus B| \phi ,$$

and therefore

$$\int_\Omega w\phi = \int_\Omega \phi \quad \text{for any } \phi \in C^\infty(\Omega) ,$$

which implies that $w \equiv 1$. Hence (3.9) and (3.10) are proved.

PROOF OF (3.12): Let $0 < q < p$. It is possible to write w_0^ε explicitely:

$$w_0^\varepsilon = \begin{cases} \dfrac{\varrho^{(p-\nu)/(p-1)} - a(\varepsilon)^{(p-\nu)/(p-1)}}{\varepsilon^{(p-\nu)/(p-1)} - a(\varepsilon)^{(p-\nu)/(p-1)}} & \text{in } B^\varepsilon \setminus B^{a(\varepsilon)}, \\ 0 & \text{in } B^{a(\varepsilon)}, \\ 1 & \text{in } Q^\varepsilon \setminus B^\varepsilon, \end{cases} \tag{3.13}$$

and thus

$$\frac{\partial w_0^\varepsilon}{\partial \varrho} = \frac{(p-\nu)}{(p-1)} \frac{\varrho^{(p-\nu)/(p-1)-1}}{\varepsilon^{(p-\nu)/(p-1)} - a(\varepsilon)^{(p-\nu)/(p-1)}} . \tag{3.14}$$

As in the proof of (3.11), we have

$$\int_\Omega |\nabla_H w^\varepsilon|^q \, d\xi \approx \sum_{k \in K(\Omega)} \int_{B_k^\varepsilon \setminus B_k^{a(\varepsilon)}} |\nabla_H w^\varepsilon|^q \, d\xi = \frac{|\Omega|}{2^N \varepsilon^\nu} \int_{B^\varepsilon \setminus B^{a(\varepsilon)}} |\nabla_H w_0^\varepsilon|^q \, d\xi .$$

But $\nabla_H w_0^\varepsilon = \sigma \nabla w_0^\varepsilon = \sigma(\partial w_0^\varepsilon / \partial \varrho) \nabla \varrho = (\partial w_0^\varepsilon / \partial \varrho) \nabla_H \varrho$. Hence (in the following, c will denote any constant)

$$\int_{B^\varepsilon \setminus B^{a(\varepsilon)}} |\nabla_H w_0^\varepsilon|^q \, d\xi = \int_{B^\varepsilon \setminus B^{a(\varepsilon)}} \left| \frac{\partial w_0^\varepsilon}{\partial \varrho} \right|^q |\nabla_H \varrho|^q \, d\xi \leq$$

$$\leq c \int_{B^\varepsilon \setminus B^{a(\varepsilon)}} \left| \frac{\partial w_0^\varepsilon}{\partial \varrho} \right|^q d\xi \leq c \int_{a(\varepsilon)}^{\varepsilon} \frac{\varrho^{(1-\nu)q/(p-1)}}{(a(\varepsilon)^{(p-\nu)/(p-1)} - \varepsilon^{(p-\nu)/(p-1)})^q} \varrho^{\nu-1} \, d\varrho \leq$$

$$\leq \frac{c}{a(\varepsilon)^{(p-\nu)q/(p-1)}} [\varrho^{((1-\nu)q/(p-1))+\nu}]_{a(\varepsilon)}^{\varepsilon} .$$

We then easily deduce that

$$\int_\Omega |\nabla_H w^\varepsilon|^q \, d\xi \to 0 . \qquad \blacksquare$$

STEP 3: Γ-convergence.

The two following propositions are closely related to the Γ-convergence for the strong topology of $L^p(\Omega)$ of the functionals

$$F_\varepsilon(v) = \begin{cases} \displaystyle\int_{\Omega \setminus T^{a(\varepsilon)}} |\nabla_H v|^p \, d\xi - p \int_{\Omega \setminus T^{a(\varepsilon)}} fv \, d\xi, & \text{if } v \in W_{H,0}^{1,p}(\Omega), \ v = 0 \text{ on } T^{a(\varepsilon)}, \\ +\infty \text{ otherwise} \end{cases}$$

to the functional

$$F(v) = \begin{cases} \displaystyle\int_\Omega |\nabla_H v|^p \, d\xi + \alpha C \int_\Omega |v|^p \, d\xi - p \int_\Omega fv \, d\xi, & \text{if } v \in W_{H,0}^{1,p}(\Omega), \\ +\infty \text{ otherwise} . \end{cases}$$

They will easily imply the convergence of the minimization problems (P_ε) to (P) (see step 4).

PROPOSITION 3.2: *For all $v \in C_0^\infty(\Omega)$ there exists $v^\varepsilon \in W_{H,0}^{1,p}(\Omega)$ such that $v^\varepsilon = 0$ on $T^{a(\varepsilon)}$, (v^ε) converges to v in $L^p(\Omega)$ and*

$$\lim_{\varepsilon \to 0} \int_\Omega |\nabla_H v^\varepsilon|^p \, d\xi = \int_\Omega |\nabla_H v|^p \, d\xi + \alpha C \int_\Omega |v|^p \, d\xi .$$

PROOF: Let us choose $v^\varepsilon = v w^\varepsilon$. From lemma 3.1, we have $v^\varepsilon = 0$ on $T^{a(\varepsilon)}$, (v^ε) converges to v in $L^p(\Omega)$, $|\nabla_H v^\varepsilon|$ is bounded in $L^p(\Omega)$ and $\nabla_H v^\varepsilon$ converges to $\nabla_H v$

a.e. on Ω. Applying Theorem 1 of [BL], we get

$$\lim_{\varepsilon\to 0}\left(\int_\Omega |\nabla_H v^\varepsilon|^p\, d\xi - \int_\Omega |\nabla_H v^\varepsilon - \nabla_H v|^p\, d\xi\right) = \int_\Omega |\nabla_H v|^p\, d\xi .$$

But $|\nabla_H v^\varepsilon - \nabla_H v| = |(w^\varepsilon - 1)\nabla_H v + v\nabla_H w^\varepsilon|$. Hence, since w^ε converges to 1 in $L^p(\Omega)$ and $|\nabla_H w^\varepsilon|^p\, d\xi$ converges to $\alpha C d\xi$ weakly $\star$ in measure (cf. Lemma 3.1), we get

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla_H v^\varepsilon - \nabla_H v|^p\, d\xi = \lim_{\varepsilon\to 0}\int_\Omega |v\nabla_H w^\varepsilon|^p\, d\xi = \alpha C\int_\Omega |v|^p\, d\xi .$$

Consequently

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla_H v^\varepsilon|^p\, d\xi = \int_\Omega |\nabla_H v|^p\, d\xi + \alpha C\int_\Omega |v|^p\, d\xi . \qquad \blacksquare$$

Proposition 3.3: *Let u and $\overline{u}^{\varepsilon_k}$ as in* (3.6). *Then*

$$\liminf_{k\to\infty}\int_\Omega |\nabla_H \overline{u}^{\varepsilon_k}|^p\, d\xi \geqslant \int_\Omega |\nabla_H u|^p\, d\xi + \alpha C\int_\Omega |u|^p\, d\xi .$$

Proof: For simplicity, we denote in this proof u^ε instead of $\overline{u}^{\varepsilon_k}$. Let $\phi \in C_0^\infty(\Omega)$ and $\phi^\varepsilon = \phi w^\varepsilon$. Since the functional $v \mapsto \int_\Omega |\nabla_H v|^p\, d\xi$ is convex, we have

$$\int_\Omega |\nabla_H u^\varepsilon|^p\, d\xi \geqslant \int_\Omega |\nabla_H \phi^\varepsilon|^p\, d\xi + p\int_\Omega |\nabla_H \phi^\varepsilon|^{p-2}\nabla_H \phi^\varepsilon\cdot\nabla_H(u^\varepsilon - \phi^\varepsilon)\, d\xi ,$$

that is

$$\liminf_{\varepsilon\to 0}\int_\Omega |\nabla_H u^\varepsilon|^p\, d\xi \geqslant$$

$$\geqslant (1-p)\liminf_{\varepsilon\to 0}\int_\Omega |\nabla_H \phi^\varepsilon|^p\, d\xi + p\liminf_{\varepsilon\to 0}\int_\Omega |\nabla_H \phi^\varepsilon|^{p-2}\nabla_H \phi^\varepsilon\cdot\nabla_H u^\varepsilon\, d\xi .$$

By Proposition 3.2,

$$\int_\Omega |\nabla_H \phi^\varepsilon|^p\, d\xi \to \int_\Omega |\nabla_H \phi|^p\, d\xi + \alpha C\int_\Omega |\phi|^p\, d\xi .$$

It remains to find the limit behavior of

$$I_\varepsilon = \int_\Omega |\nabla_H \phi^\varepsilon|^{p-2}\nabla_H \phi^\varepsilon\cdot\nabla_H u^\varepsilon\, d\xi .$$

Set $I_\varepsilon = A_\varepsilon + B_\varepsilon$ with

$$A_\varepsilon = \int_\Omega |\nabla_H \phi^\varepsilon|^{p-2} \nabla_H \phi \cdot \nabla_H u^\varepsilon w^\varepsilon \, d\xi \quad \text{and} \quad B_\varepsilon = \int_\Omega |\nabla_H \phi^\varepsilon|^{p-2} \nabla_H w^\varepsilon \cdot \nabla_H u^\varepsilon \phi \, d\xi .$$

a) Let us first study the asymptotic behavior of A_ε. Using Holder's inequality and lemma 3.1, we have

$$\left| \int_\Omega |\nabla_H \phi^\varepsilon|^{p-2} \nabla_H \phi \cdot \nabla_H u^\varepsilon (w^\varepsilon - 1) \, d\xi \right| \leqslant$$

$$\leqslant \|\nabla_H \phi^\varepsilon\|^{p-2}_{L^p(\Omega)} \|\nabla_H \phi\|_{L^\infty(\Omega)} \|\nabla_H u^\varepsilon\|_{L^p(\Omega)} \| |w^\varepsilon - 1| \|_{L^p(\Omega)} \to 0 .$$

It follows that

$$\liminf_{\varepsilon\to 0} A_\varepsilon = \liminf_{\varepsilon\to 0} \int_\Omega |\nabla_H \phi^\varepsilon|^{p-2} \nabla_H \phi \cdot \nabla_H u^\varepsilon \, d\xi .$$

We claim that $|\nabla_H \phi^\varepsilon|^{p-2} \to |\nabla_H \phi|^{p-2}$ strongly in $L^{p'}(\Omega)$. We then deduce that

$$\liminf_{\varepsilon\to 0} A_\varepsilon = \int_\Omega |\nabla_H \phi|^{p-2} \nabla_H \phi \cdot \nabla_H u \, d\xi$$

since $\nabla_H u^\varepsilon$ converges to $\nabla_H u$ weakly in $L^p(\Omega)$.

To show the claim, we will use the following inequality which was used in the proof of [B L] : Let $\alpha \in \mathbb{R}$, $\alpha > 0$. For every $\delta > 0$, there exists $C_\delta > 1$ such that, for all $X, Y \in \mathbb{R}^N$,

$$(3.15) \qquad \big| |X - Y|^\alpha - |X|^\alpha \big| \leqslant \delta |X|^\alpha + C \delta |Y|^\alpha .$$

From this ineqality (with $\alpha = p - 2$), we have

$$\int_\Omega \big| |\nabla_H \phi^\varepsilon|^{p-2} - |\nabla_H \phi|^{p-2} \big|^{p'} d\xi =$$

$$= \int_\Omega \big| |\nabla_H \phi + (w_\varepsilon - 1) \nabla_H \phi + \phi \nabla_H w^\varepsilon|^{p-2} - |\nabla_H \phi|^{p-2} \big|^{p'} d\xi \leqslant$$

$$\leqslant \int_\Omega (\delta |\nabla_H \phi|^{p-2} + C_\delta |(w_\varepsilon - 1) \nabla_H \phi + \phi \nabla_H w^\varepsilon|^{p-2})^{p'} d\xi .$$

By lemma 3.1, $|(w_\varepsilon - 1) \nabla_H \phi + \phi \nabla_H w^\varepsilon| \to 0$ in $L^{(p-2)p'}(\Omega)$ (noting that $(p-2) p' < p$). It follows that

$$\limsup_{\varepsilon\to 0} \int_\Omega \big| |\nabla_H \phi^\varepsilon|^{p-2} - |\nabla_H \phi|^{p-2} \big|^{p'} d\xi \leqslant \delta^{p'} \int_\Omega |\nabla_H \phi|^{(p-2)p'} d\xi .$$

Since this inequality holds for every $\delta > 0$, we deduce

$$\lim_{\varepsilon\to 0} \int_\Omega \left| |\nabla_H \phi^\varepsilon|^{p-2} - |\nabla_H \phi|^{p-2} \right|^{p'} d\xi = 0$$

and the claim is proved.

b) Let us now study the asymptotic behavior of B_ε. Applying (3.15) again, we get

$$\left| B_\varepsilon - \int_\Omega |\phi \nabla_H w^\varepsilon|^{p-2} \nabla_H w^\varepsilon \cdot \nabla_H u^\varepsilon \phi \, d\xi \right| \leqslant$$

$$\leqslant \int_\Omega \left| |\nabla_H \phi^\varepsilon|^{p-2} - |\phi \nabla_H w^\varepsilon|^{p-2} \right| |\nabla_H w^\varepsilon| \, |\nabla_H u^\varepsilon| \, |\phi| \, d\xi \leqslant$$

$$\leqslant \delta \int_\Omega |\phi \nabla_H w^\varepsilon|^{p-1} |\nabla_H u^\varepsilon| \, d\xi + C_\delta \int_\Omega |\nabla_H \phi|^{p-2} |w^\varepsilon|^{p-2} |\nabla_H w^\varepsilon| \, |\nabla_H u^\varepsilon| \, d\xi \leqslant$$

$$\leqslant \delta c \|\nabla_H w^\varepsilon\|^{p/p'}_{L^p(\Omega)} \|\nabla_H u^\varepsilon\|_{L^p(\Omega)} + C_\delta c \|\nabla_H w^\varepsilon\|_{L^{p'}(\Omega)} \|\nabla_H u^\varepsilon\|_{L^p(\Omega)}$$

where c denote any constant. Since $\|\nabla_H w^\varepsilon\|_{L^{p'}(\Omega)} \to 0$, it follows that

$$\limsup_{\varepsilon\to 0} \left| B_\varepsilon - \int_\Omega |\phi \nabla_H w^\varepsilon|^{p-2} \nabla_H w^\varepsilon \cdot \nabla_H u^\varepsilon \phi \, d\xi \right| \leqslant \delta c \,,$$

for all $\delta > 0$. Therefore

$$\liminf_{\varepsilon\to 0} B_\varepsilon = \liminf_{\varepsilon\to 0} \int_\Omega |\phi \nabla_H w^\varepsilon|^{p-2} \nabla_H w^\varepsilon \cdot \nabla_H u^\varepsilon \phi \, d\xi =$$

$$= \liminf_{\varepsilon\to 0} \int_\Omega |\nabla_H w^\varepsilon|^{p-2} \nabla_H w^\varepsilon \cdot \nabla_H u^\varepsilon |\phi|^{p-2} \phi \, d\xi =$$

$$= \liminf_{\varepsilon\to 0} \left(\int_\Omega |\nabla_H w^\varepsilon|^{p-2} \nabla_H w^\varepsilon \cdot \nabla_H (u^\varepsilon |\phi|^{p-2} \phi) \, d\xi - \right.$$

$$\left. - \int_\Omega |\nabla_H w^\varepsilon|^{p-2} \nabla_H w^\varepsilon \cdot \nabla_H (|\phi|^{p-2} \phi) \, u^\varepsilon \, d\xi \right).$$

But, choosing $\beta > 0$ such that the imbedding $W_H^{1,p}(\Omega) \hookrightarrow L^{p+\beta}(\Omega)$ is continuous and applying Holder's inequality, we get

$$\left| \int_\Omega |\nabla_H w^\varepsilon|^{p-2} \nabla_H w^\varepsilon \cdot \nabla_H (|\phi|^{p-2}\phi)\, u^\varepsilon \, d\xi \right| \leqslant$$

$$\leqslant C \left(\int_\Omega |u^\varepsilon|^{p+\beta} d\xi \right)^{1/(p+\beta)} \cdot \left(\int_\Omega |\nabla_H w^\varepsilon|^{(p-1)(p+\beta)/(p+\beta-1)} d\xi \right)^{1-1/(p+\beta)} .$$

By lemma 3.1, $\int_\Omega |\nabla_H w^\varepsilon|^{(p-1)(p+\beta)/(p+\beta-1)} d\xi \to 0$ since $(p-1)(p+\beta)/(p+\beta-$ $-1) < p$. Consequently

$$\liminf_{\varepsilon\to 0} B_\varepsilon = \liminf_{\varepsilon \to 0} \int_\Omega |\nabla_H w^\varepsilon|^{p-2} \nabla_H w^\varepsilon \cdot \nabla_H (u^\varepsilon |\phi|^{p-2}\phi)\, d\xi \, .$$

c) It remains to find the limit of

$$B_\varepsilon^\star := \int_\Omega |\nabla_H w^\varepsilon|^{p-2} \nabla_H w^\varepsilon \cdot \nabla_H (u^\varepsilon |\phi|^{p-2}\phi)\, d\xi \, .$$

For that, we will prove the following

LEMMA 3.2: *Let $v^\varepsilon \in W^{1,p}_{0,H}(\Omega)$ such that $v^\varepsilon = 0$ in $T^{a(\varepsilon)}$ and (v^ε) converges to v in $L^p(\Omega)$. Let $M_\varepsilon = \int_\Omega |\nabla_H w^\varepsilon|^{p-2} \nabla_H w^\varepsilon \cdot \nabla_H v^\varepsilon \, d\xi$.*

Then M_ε converges to $\alpha C \int_\Omega v \, d\xi$ and

(3.16) $$C = 2^N \left(\frac{\nu - p}{p-1} \right)^{p-1} \nu \frac{1}{|B|} \int_B |\nabla_H \varrho|^p d\xi \, .$$

Applying this lemma with $v^\varepsilon = u^\varepsilon |\phi|^{p-2}\phi$, it follows that

$$B_\varepsilon^\star \to \alpha C \int_\Omega |\phi|^{p-2} \phi u \, d\xi \, .$$

PROOF OF LEMMA 3.2: Using regularity results in [CDG], Green formula and since $v^\varepsilon = 0$ in $B_k^{a(\varepsilon)}$, we get

$$M_\varepsilon = \sum_{k \in K(\Omega)} \int_{B_k^\varepsilon \setminus B_k^{a(\varepsilon)}} |\nabla_H w^\varepsilon|^{p-2} \nabla_H w^\varepsilon \cdot \nabla_H v^\varepsilon \, d\xi =$$

$$= \sum_{k \in K(\Omega)} \int_{\partial B_k^\varepsilon} v^\varepsilon |\nabla_H w^\varepsilon|^{p-2} \nabla_H w^\varepsilon \cdot \sigma n \, dH^{N-1} .$$

But $\nabla_H w^\varepsilon = \dfrac{\partial w^\varepsilon}{\partial \varrho} \nabla_H \varrho$ and from (3.14)

$$\frac{\partial w^\varepsilon}{\partial \varrho}\Big|_{\varrho=\varepsilon} = \frac{(p-\nu)}{(p-1)} \frac{\varepsilon^{(p-\nu)/(p-1)-1}}{\varepsilon^{(p-\nu)/(p-1)} - a(\varepsilon)^{(p-\nu)/(p-1)}}$$

and hence

$$\left|\frac{\partial w^\varepsilon}{\partial \varrho}\right|^{p-2} \frac{\partial w^\varepsilon}{\partial \varrho}\Big|_{\varrho=\varepsilon} \approx \left(\frac{\nu-p}{p-1}\right)^{p-1} \frac{\varepsilon^{1-\nu}}{a(\varepsilon)^{p-\nu}} \approx \left(\frac{\nu-p}{p-1}\right)^{p-1} \alpha 2^N \varepsilon\,.$$

Consequently

$$M_\varepsilon \approx \varepsilon\alpha 2^N \left(\frac{\nu-p}{p-1}\right)^{p-1} \sum_{k\in K(\Omega)} \int_{\partial B_k^\varepsilon} v^\varepsilon |\nabla_H \varrho|^{p-2} \nabla_H \varrho \cdot \sigma n \, dH^{N-1}.$$

Since $\sigma n = \sigma(\nabla\varrho/|\nabla\varrho|) = \nabla_H\varrho/|\nabla\varrho|$, we obtain

$$M_\varepsilon \approx \varepsilon\alpha 2^N \left(\frac{\nu-p}{p-1}\right)^{p-1} \sum_{k\in K(\Omega)} \int_{\partial B_k^\varepsilon} v^\varepsilon \frac{|\nabla_H \varrho|^p}{|\nabla\varrho|} \, dH^{N-1}.$$

In order to study the previous integral, we introduce a function q_0^ε defined in Q^ε by:

$$q_0^\varepsilon = \begin{cases} \dfrac{\varrho^{p'}}{p'} - \dfrac{\varepsilon^{p'}}{p'} & \text{in } B^\varepsilon, \\ 0 & \text{in } Q^\varepsilon \setminus B^\varepsilon \end{cases}.$$

We have

$$\frac{\partial q_0^\varepsilon}{\partial \varrho} = \begin{cases} \varrho^{1/(p-1)} & \text{in } B^\varepsilon, \\ 0 & \text{in } Q^\varepsilon \setminus B^\varepsilon, \end{cases}$$

and therefore

$$\left|\frac{\partial q_0^\varepsilon}{\partial \varrho}\right| \leqslant \varepsilon^{1/p-1}.$$

Moreover (see [CDG]), q_0^ε is the solution of

$$\begin{cases} \Delta_H q_0^\varepsilon = \nu |\nabla_H \varrho|^p & \text{in } B^\varepsilon, \\ \dfrac{\partial q_0^\varepsilon}{\partial \varrho} \doteq \varepsilon^{1/(p-1)} & \text{on } \partial B^\varepsilon. \end{cases}$$

Let q^ε be the periodic extension of q_0^ε, (see (2.1)).
We have

$$|\nabla_H q^\varepsilon| = \left| \frac{\partial q_0^\varepsilon}{\partial \varrho} \right| |\nabla_H \varrho| \leqslant c\varepsilon^{1/p-1}.$$

This implies that

$$\|q^\varepsilon\|_{L^\infty} + \|\nabla_H q^\varepsilon\|_{L^\infty} \to 0 .$$

From Green formula,

$$\int_\Omega |\nabla_H q^\varepsilon|^{p-2} \nabla_H q^\varepsilon \cdot \nabla_H v^\varepsilon \, d\xi = \sum_{k \in K(\Omega)} \int_\Omega |\nabla_H q^\varepsilon|^{p-2} \nabla_H q^\varepsilon \cdot \nabla_H v^\varepsilon \, d\xi =$$

$$= -\nu \sum_{k \in K(\Omega)} \int_{B_k^\varepsilon} v^\varepsilon |\nabla_H \varrho|^p \, d\xi + \sum_{k \in K(\Omega)} \int_{\partial B_k^\varepsilon} v^\varepsilon |\nabla_H q^\varepsilon|^{p-2} \nabla_H q^\varepsilon \cdot \sigma n \, dH^{N-1}.$$

But

$$\sum_{k \in K(\Omega)} \int_{\partial B_k^\varepsilon} v^\varepsilon |\nabla_H q^\varepsilon|^{p-2} \nabla_H q^\varepsilon \cdot \sigma n \, dH^{N-1} =$$

$$= \sum_{k \in K(\Omega)} \int_{\partial B_k^\varepsilon} v^\varepsilon \left| \frac{\partial q^\varepsilon}{\partial \varrho} \right|^{p-2} \frac{\partial q^\varepsilon}{\partial \varrho} \frac{|\nabla_H \varrho|^p}{|\nabla \varrho|} dH^{N-1} = \varepsilon \sum_{k \in K(\Omega)} \int_{\partial B_k^\varepsilon} v^\varepsilon \frac{|\nabla_H \varrho|^p}{|\nabla \varrho|} dH^{N-1}$$

and

$$\int_\Omega |\nabla_H q^\varepsilon|^{p-2} \nabla_H q^\varepsilon \cdot \nabla_H v^\varepsilon \, d\xi \to 0 .$$

Consequently,

$$M_\varepsilon = \alpha 2^N \left(\frac{\nu - p}{p-1} \right)^{p-1} \nu \sum_{k \in K(\Omega)} \int_{B_k^\varepsilon} v^\varepsilon |\nabla_H \varrho|^p \, d\xi + o(\varepsilon) .$$

Let us now introduce the function h_0^ε:

$$h_0^\varepsilon = \begin{cases} \nu|\nabla_H \varrho|^p & \text{in } B^\varepsilon, \\ 0 & \text{in } Q^\varepsilon \setminus B^\varepsilon \end{cases}$$

and call h^ε its periodic extension (see (2.1)).

Thus

$$\nu \sum_{k \in K(\Omega)} \int_{B_k^\varepsilon} v^\varepsilon |\nabla_H \varrho|^p \, d\xi = \int_\Omega v^\varepsilon h^\varepsilon \, d\xi .$$

We can prove (see Lemma 1 in [BMT]) that

$$h^\varepsilon \rightharpoonup \nu \frac{1}{|B|} \int_B |\nabla_H \varrho|^p \, d\xi , \qquad \text{*-weakly in } L^\infty(\Omega)$$

and then weakly in $L^{p'}(\Omega)$. Consequently, since (v_ε) converges to v strongly in $L^p(\Omega)$,

$$M_\varepsilon \to \alpha 2^N \left(\frac{\nu - p}{p-1} \right)^{p-1} \nu \frac{1}{|B|} \int_B |\nabla_H \varrho|^p \, d\xi \int_\Omega v \, d\xi .$$

Moreover, for w_ε instead of v_ε, we obtain, since $w_\varepsilon \to 1$ in $L^p(\Omega)$,

$$\int_\Omega |\nabla_H w^\varepsilon|^p \, d\xi \to |\Omega| \alpha 2^N \left(\frac{\nu - p}{p-1} \right)^{p-1} \nu \frac{1}{|B|} \int_B |\nabla_H \varrho|^p \, d\xi .$$

But, from lemma 3.1, $\int_\Omega |\nabla_H w^\varepsilon|^p \, d\xi \to \alpha C |\Omega|$. Hence, we get (3.16) and the proof of lemma 3.2 is completed. ■

d) Finally, we obtain

$$\liminf_{\varepsilon \to 0} \int_\Omega |\nabla_H u^\varepsilon|^p \, d\xi \geq (1-p) \left(\int_\Omega |\nabla_H \phi|^p \, d\xi + \alpha C \int_\Omega |\phi|^p \, d\xi \right) +$$

$$+ p \int_\Omega |\nabla_H \phi|^{p-2} \nabla_H \phi \cdot \nabla_H u \, d\xi + p \alpha C \int_\Omega |\phi|^{p-2} \phi u \, d\xi .$$

Since this inequality holds for all $\phi \in C_0^\infty(\Omega)$, it is also true for all $\phi \in W_{H,0}^{1,p}(\Omega)$, in particular for $\phi = u$; thus

$$\liminf_{\varepsilon \to 0} \int_\Omega |\nabla_H u^\varepsilon|^p \, d\xi \geq \int_\Omega |\nabla_H u|^p \, d\xi + \alpha C \int_\Omega |u|^p \, d\xi .$$

STEP 4: End of the proof of Theorem 3.1.

Proposition 3.4: *The function u is the unique solution of (P) and (3.5) holds.*

Proof: Let $v \in C_0^\infty(\Omega)$. By proposition 3.2, there exists $(v^\varepsilon) \in W_{H,0}^{1,p}$ such that $\lim_{\varepsilon \to 0} F_\varepsilon(v^\varepsilon) = F(v)$. From proposition 3.3 and since u^ε is solution of (P^ε) we deduce

$$F(u) \leqslant \liminf_{k \to 0} F_{\varepsilon_k}(u^{\varepsilon_k}) \leqslant \limsup_{k \to 0} F_{\varepsilon_k}(u^{\varepsilon_k}) \leqslant \limsup_{k \to 0} F_{\varepsilon_k}(v^{\varepsilon_k}) \leqslant F(v).$$

Since $C_0^\infty(\Omega)$ is dense in $W_{H,0}^{1,p}$, it follows that

$$F(u) \leqslant \liminf_{k \to 0} F_{\varepsilon_k}(u^{\varepsilon_k}) \leqslant \limsup_{k \to 0} F_{\varepsilon_k}(u^{\varepsilon_k}) \leqslant F(v), \qquad \forall v \in W_{H,0}^{1,p}.$$

Therefore

$$F(u) \leqslant \inf\{F(v);\ v \in W_{H,0}^{1,p}\}.$$

Hence u is the unique solution of (P) and then all the sequence (u^ε) converges to u and we have

$$F(u) \leqslant \liminf_{\varepsilon \to 0} F_\varepsilon(u^\varepsilon) \leqslant \limsup_{\varepsilon \to 0} F_\varepsilon(u^\varepsilon) \leqslant F(v), \qquad \forall v \in W_{H,0}^{1,p}.$$

In particular for $v = u$, we get

$$F(u) = \lim_{\varepsilon \to 0} F_\varepsilon(u^\varepsilon)$$

and (3.5) follows.

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