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On the Lower Semicontinuity and Relaxation Properties of Certain Classes of Variational Integrals (**)

SUMMARY. — We prove some lower semicontinuity and relaxation results, with respect to the strong topology of $L^1(\Omega)$, for integral functionals of the type $u \in BV(\Omega) \mapsto \int_\Omega f(x, \nabla u)\, dx$.

Sulle proprietà di semicontinuità inferiore e rilassamento di certe classi di integrali variazionali

RIASSUNTO. — Vengono provati alcuni risultati di semicontinuità inferiore e rilassamento, nella topologia forte di $L^1(\Omega)$, per funzionali integrali del tipo $u \in BV(\Omega) \mapsto \int_\Omega f(x, \nabla u)\, dx$.

0. - INTRODUCTION

In this paper we are concerned with lower semicontinuity and relaxation, in the $L^1_{loc}(\Omega)$ topology, of integral of the type

$$I(\Omega, u) = \int_\Omega f(x, \nabla u)\, dx \tag{0.1}$$

where Ω is an open subset of R^n.

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In 1961 Serrin proved, see [S], that, if the function $f: (x, s, z) \in \Omega \times R \times \times R^n \mapsto [0, +\infty[$ is convex with respect to the z variable and satisfies suitable uniform continuity assumptions with respect to the remaining ones, then the integral functional

$$u \in W^{1,1}_{loc}(\Omega) \mapsto \int_\Omega f(x, u, \nabla u)\, dx$$

is lower semicontinuous with respect to the $L^1_{loc}(\Omega)$ topology. He also introduced a "relaxed" functional defined as follows

$$(0.2) \quad \mathfrak{F}(\Omega, \cdot): u \in BV_{loc}(\Omega) \mapsto \inf \left\{ \liminf_h \int_{\Omega_h} f(x, u_h, \nabla u_h)\, dx \,|\, (\Omega_h)_h \text{ increasing}, \right.$$
$$\left. \Omega = \bigcup_h \Omega_h,\, u_h \in C^1(\Omega_h)\, \forall h \in N,\, u_h \to u \text{ in } L^1_{loc}(\Omega) \right\}$$

and in 1964 with Goffman, see [GS], gave an integral representation theorem on $BV(\Omega)$ for it when Ω is a bounded open set, the integrand f depends only on the z variable and it is convex. More precisely they proved that

$$(0.3) \quad \mathfrak{F}(\Omega, u) = \int_\Omega f(\nabla u)\, dx + \int_\Omega f^\infty \left(\frac{dD^s u}{d|Du|} \right) d|Du|$$

where f^∞ is the recession function of f (see section 1 also for the definition of BV-spaces).

A similar result, but in a different framework, has been proved in [CEDA1]. Lower semicontinuity and relaxation problems in L^1_{loc} topologies have been considered by Dal Maso in [DM2], where the Goffman-Serrin representation theorem has been extended to the case in which the integrand f depends also on (x, s) and satisfies suitable growth and continuity assumptions. In [BoDM] the case in which f depends only on (x, z), verifies the following linear growth condition $0 \leq f(x, z) \leq a(x) + \gamma|z|$, where $a \in L^1(\Omega)$ and $\gamma \in R$, but is not necessarely continuous with respect to the x variable, has been considered and some relaxation results, in $L^1(\Omega)$ topology, have been proved, see Theorem 4.1 in [BoDM]. When the integrand f depends only on (s, z), a L^1-lower semicontinuity result, covering several cases of discontinuous behaviours of f with respect to the variable s, has been proved by De Giorgi, Buttazzo, Dal Maso, see [DBD]. Finally, in the same order of ideas, some cases of dependence on the x variable have been treated in [A].

In this paper we consider an open subset Ω of R^n and we prove some lower semicontinuity and relaxation results, always in the $L^1_{loc}(\Omega)$ topology, for the functional in

(0.1). More precisely we prove that, see Theorem 2.3, if f satisfies

$$(0.4)\qquad \begin{cases} f:(x,z)\in\Omega\times R^n\mapsto[0,+\infty[, \\ f(x,\cdot) \text{ is convex for almost every } x\in\Omega, \\ f(\cdot,z) \text{ is lower semicontinuous for every } z\in R^n. \end{cases}$$

and

$$(0.5)\qquad \begin{cases} \forall A \subset\subset \Omega \exists \lambda_A:[0,+\infty[\mapsto[0,+\infty[\\ \text{increasing, continuous in zero with } \lambda_A(0)=0 \\ \text{such that for all compact } K \text{ of } A \text{ exists } x_K \text{ in } K \text{ verifying} \\ f(x_K,z)\le f(x,z)+\lambda_A(\operatorname{diam} K)\{1+f(x,z)\}, \quad \forall x\in K, \forall z, \end{cases}$$

then the functional

$$(0.6)\qquad G(\Omega,\cdot): u\in BV_{loc}(\Omega)\mapsto \int_\Omega f(x,\nabla u)\,dx + \int_\Omega f^\infty\left(x,\frac{dD^s u}{d|Du|}\right)d|Du|$$

is lower semicontinuous in the $L^1_{loc}(\Omega)$ topology (for a.e. $x\in\Omega$ $f^\infty(x,\cdot)$ being the recession function of $f(x,\cdot)$). We observe that such results allow us to treat also cases when f is not uniformly continuous or even discontinuous with respect to the x variable.

Furthermore, if f satisfies (0.4), (0.5),

$$(0.7)\qquad f(x,z)\le f(x_K,z)+\lambda_A(\operatorname{diam} K)\{1+f(x_K,z)\}, \quad \forall x\in K, \forall z,$$

and

$$(0.8)\qquad f(\cdot,0)\in L^1(\Omega)$$

we prove that, see Theorem 3.8,

$$(0.9)\qquad \inf\left\{\liminf_h \int_{\Omega_h} f(x,\nabla u_h)\,dx, (u_h)_h\subseteq C^1(\Omega), u_h\to u \text{ in } L^1_{loc}(\Omega)\right\} =$$

$$= \int_\Omega f(x,\nabla u)\,dx + \int_\Omega f^\infty\left(x,\frac{dD^s u}{d|Du|}\right)d|Du|, \qquad \forall u\in BV_{loc}(\Omega).$$

By such result we also deduce that, see Corollary 3.11,

$$\mathfrak{F}(\Omega,u)=G(\Omega,u) \quad \text{for every } u\in BV_{loc}(\Omega).$$

One of the main tools utilized in order to prove our relaxation theorem is an inner regularity result on the whole $L^1_{loc}(\Omega)$ for the functional in the left-hand side of (0.9), see Theorem 3.6.

Finally we observe that our assumptions seem to be different from those existing in literature, indeed we assume no growth conditions and no uniform continuity assumptions on f with respect to the x variable.

1. - Notations and preliminary results

We first recall some definitions concernig increasing set functions (see [DGL]).

Let Ω and A be open subsets of $\boldsymbol{R}^n$; we say that $A \subset\subset \Omega$ if $\overline{A}$ is a compact subset of Ω and that a family $\mathcal{F}$ of open subsets of $\boldsymbol{R}^n$ is dense if whenever A_1, A_2 are open sets with $A_1 \subset\subset A_2$ there exists $B \in \mathcal{F}$ such that $A_1 \subset\subset B \subset\subset A_2$.

Let F be a real function defined on the set of all open subset of $\boldsymbol{R}^n$; we say that F is increasing if

$$A_1 \subseteq A_2 \Rightarrow F(A_1) \leq F(A_2).$$

For an increasing function F we define the inner regular envelope F_- as the function defined for every open set Ω by

$$F_-(\Omega) = \sup_{A \subset\subset \Omega} F(A). \tag{1.1}$$

In this paper we consider functionals of the type $F(\Omega, u)$ where Ω is an open set of $\boldsymbol{R}^n$, u belongs to a suitable functional space and the set function $F(\cdot, u)$ is increasing. For every Ω we set $F_-(\Omega, u) = (F(\cdot, u))_-(\Omega)$.

Let Ω be an open subset of $\boldsymbol{R}^n$, we denote by $BV(\Omega)$ the set of the functions in $L^1(\Omega)$ whose distributional partial derivatives are Radon measures with bounded total variation on Ω. We recall that, see [G], [EG], [DGCP], [Z], if $u \in BV(\Omega)$, then the total variation of Du on Ω is given by

$$\int_\Omega |Du| = \sup\left\{ \int_\Omega u \operatorname{div} g \, dx \colon g \in C_0^1(\Omega; \boldsymbol{R}^n),\ |g(x)| \leq 1 \quad \text{for every } x \in \Omega \right\},$$

moreover, by the Radon-Nikodim's theorem, for all $u \in BV(\Omega)$ we have

$$(Du)(E) = \int_E \nabla u(x)\, dx + D^s u(E) \quad \text{for every Borel set } E,$$

where we denote by ∇u the Radon-Nikodim's derivative of Du and by $D^s u$ the singular part of Du (both taken with respect to Lebesgue measure). We denote by $BV_{loc}(\Omega)$ the set of functions on Ω which are in $BV(A)$ for every $A \subset\subset \Omega$.

Let $f\colon z \in \boldsymbol{R}^n \mapsto f(z) \in [0, +\infty[$ be a convex function; it is well known that for every $z \in \boldsymbol{R}^n$ the limit $\lim_{t \to 0^+} tf(z/t)$ exists so denote by f^∞ the recession function of f defined as $f^\infty\colon z \in \boldsymbol{R}^n \mapsto \lim_{t \to 0^+} tf(z/t) \in [0, +\infty[$. The recession function is important in the study of relaxation prolems for integral functionals on BV as it has been proved by Goffman and Serrin, (see [GS]).

THEOREM 1.1: *Let Ω be a bounded open set, $f: \mathbf{R}^n \mapsto [0, +\infty[$ be convex. Then for every $u \in BV_{loc}(\Omega)$ we have*

$$(1.2)\quad \int_\Omega f(\nabla u(x))\,dx + \int_\Omega f^\infty\left(\frac{dD^s u}{d|Du|}\right) d|Du| =$$

$$= \inf\left\{\liminf_h \int_{\Omega_h} f(\nabla u_h)\,dx : (\Omega_h)_h \text{ increasing},\ \Omega = \cup_h \Omega_h,\right.$$

$$\left. u_h \in C^1(\Omega_h),\ \forall h \in \mathbf{N},\ u_h \to u \text{ in } L^1_{loc}(\Omega)\right\}.$$

REMARK 1.2: If Ω is a bounded open set then Theorem 1.1 implies that the functional

$$u \in BV_{loc}(\Omega) \mapsto \int_\Omega f(\nabla u(x))\,dx + \int_\Omega f^\infty\left(\frac{dD^s u}{d|Du|}\right) d|Du|$$

is lower semicontinuous in the $L^1_{loc}(\Omega)$ topology.

In this paper we consider functions of the type (Ω being an open subset of $\mathbf{R}^n$) $f: (x, z) \in \Omega \times \mathbf{R}^n \mapsto f(x, z) \in [0, +\infty[$ such that for a.e. $x \in \Omega$ $f(x, \cdot)$ is convex and we denote by f^∞ the recession function of $f(x, \cdot)$, given by $f^\infty(x, \cdot): (x, z) \in \Omega \times \mathbf{R}^n \mapsto \lim_{t \to 0^+} tf(x, z/t)$.

Let Ω be a bounded open set of $\mathbf{R}^n$ and let $\varepsilon > 0$; we put $\Omega_\varepsilon^- = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.

If Ω is an open subset of $\mathbf{R}^n$ and f is a function as in (0.4) we set

$$(1.3)\quad I(\Omega, \cdot): u \in BV_{loc}(\Omega) \mapsto \int_\Omega f(x, \nabla u)\,dx;$$

$$(1.4)\quad \bar{I}(\Omega, \cdot): u \in BV_{loc}(\Omega) \mapsto \inf\left\{\liminf_h I(\Omega, u_h): (u_h)_h \subseteq C^1(\Omega), u_h \to u \text{ in } L^1_{loc}(\Omega)\right\};$$

$$(1.5)\quad G(\Omega, \cdot): u \in BV_{loc}(\Omega) \mapsto \int_\Omega f(x, \nabla u)\,dx + \int_\Omega f^\infty\left(x, \frac{dD^s u}{d|Du|}\right) d|Du|.$$

PROPOSITION 1.3: *Let Ω be a bounded open set, $f: \mathbf{R}^n \mapsto [0, +\infty[$ be convex and let $I(\Omega, \cdot)$ be the functional in (1.3). Then for every $u \in BV_{loc}(\Omega)$ we have*

$$(1.6)\quad \bar{I}_-(\Omega, u) = \int_\Omega f(\nabla u)\,dx + \int_\Omega f^\infty\left(\frac{dD^s u}{d|Du|}\right) d|Du|.$$

PROOF: Let $G(\Omega,\cdot)$ be given by (1.5) and let $u \in BV_{loc}(\Omega)$. We first show that $G(\Omega,u) \geq \bar{I}_-(\Omega,u)$. If $G(\Omega,u) = +\infty$ the above inequality is trivial so we can suppose $G(\Omega,u) < +\infty$. Theorem 1.1 implies that, for every $A \subset\subset \Omega$, $G(\Omega,u) \geq \bar{I}(A,u)$ so

$$G(\Omega,u) \geq \bar{I}_-(\Omega,u).$$

To show the opposite inequality let us observe that $\bar{I}_-(\Omega,u) = \lim_k \bar{I}(\Omega^-_{1/k},u)$. For every $k \in N$ let $(u^k_h)_h \subseteq C^1(\Omega^-_{1/k})$ be such that

$$\begin{cases} u^k_h \to u & \text{in } L^1_{loc}(\Omega^-_{1/k}), \\ \bar{I}(\Omega^-_{1/k},u) = \lim_h \int_{\Omega^-_{1/k}} f(\nabla u^k_h)\,dx, \end{cases}$$

and let $B_k = \overline{\Omega^-_{1/(k-1)}}$. Then there exist $\bar{h}(k)$ and $h^*(k) \in N$ such that

$$\bar{I}(\Omega^-_{1/k},u) \geq \int_{\Omega^-_{1/k}} f(\nabla u^k_h)\,dx - \frac{1}{k}, \qquad \forall h \geq \bar{h}(k) \tag{1.7}$$

and

$$\|u^k_h - u\|_{L^1(B_k)} < \frac{1}{k}, \qquad \forall h \geq h^*(k). \tag{1.8}$$

If we put $h_k = \max\{\bar{h}(k), h^*(k)\}$ and $u_k = u^k_{h_k}$, then by (1.7) and (1.8) we have that

$$\begin{cases} u_k \in C^1(\Omega^-_{1/k}), \\ \bar{I}(\Omega^-_{1/k},u) \geq \int_{\Omega^-_{1/k}} f(\nabla u_k)\,dx - \frac{1}{k}, \\ \|u_k - u\|_{L^1(B_k)} < \frac{1}{k}. \end{cases} \tag{1.9}$$

By (1.9) and Theorem 1.1 we obtain

$$\bar{I}_-(\Omega,u) = \lim_k \bar{I}(\Omega^-_{1/k},u) \geq \liminf_k \int_{\Omega^-_{1/k}} f(\nabla u_k)\,dx \geq G(\Omega,u). \tag{1.10}$$ ∎

We recall the following lemma (see Lemma 2.2 in [CEDA2]).

LEMMA 1.4: *Let A be a bounded open set, f be as in (0.4) with $\Omega = A$ and let us assume that $\forall z \in R^n$ $f(\cdot,z) \in L^1(A)$. Let $(m_k)_k \subseteq L^\infty(A;R^n)$ and $m \in L^\infty(A;R^n)$ be such that*

1) $m_k(x) \to m(x)$ almost everywhere in A;

2) $\sup_h \| |m_h| \|_{L^\infty(\Omega)} < +\infty$. Then

$$\lim_h \int_A f(x, m_h(x))\,dx = \int_A f(x, m(x))\,dx. \tag{1.11}$$

For every subset Ω of R^n and $i \in \{1, 2, \dots, n\}$ we denote by Ω_i the projection of Ω on the i-th axis and given $(x_1, x_2, \dots, x_n) \in R^n$ we put $S_x^i = \Omega \cap \{y \in R^n \,|\, y_i = x_i\}$. Obviously we have that:

$$\Omega = \bigcup_{x \in \Omega_i} S_x^i .$$

Lemma 1.5: *Let Ω be an open set and let μ be a positive Borel measure with $\mu(\Omega) < +\infty$. Then for every $i \in \{1, 2, \dots, n\}$ the set*

$$A_i = \{x \in \Omega_i \,|\, \mu(S_x^i) > 0\}$$

is at most countable.

Lemma 1.6: *Let Ω be a bounded open set and let μ be a positive Borel measure with $\mu(\Omega) < +\infty$. For every $\varepsilon > 0$ there exist $m_\varepsilon \in N$ and $Q_1^\varepsilon, Q_2^\varepsilon, \dots, Q_{m_\varepsilon}^\varepsilon$ open disjoint rectangles whose sides are parallel to the coordinates axes such that for every $j \in \{1, \dots, m_\varepsilon\}$ diam $Q_j^\varepsilon < \varepsilon$ and*

$$\mu(\partial(Q_j^\varepsilon \cap \Omega) \cap \Omega) = 0 .$$

Proof: Let $\varepsilon > 0$. For every $i \in \{1, \dots, n\}$ let $(\Pi_k^{\varepsilon,i})_{k \in Z}$ be a sequence of hyperplanes orthogonal to the i-th axis. For the sake of simplicity we denote by x_k the intersection of $\Pi_k^{\varepsilon,i}$ with the i-th axis and we suppose that $0 < (\varepsilon / 2\sqrt{n}) < (x_{k-1})_i - (x_k)_i < (\varepsilon / \sqrt{n})$, $\forall k \in Z$. Let $I_i = \{x \in \Omega_i : \mu(S_x^i) > 0\}$. If $\mu(S_{x_k}^i) = 0$ for every $k \in Z$ and for every $i \in \{1, 2, \dots, n\}$ the hyperplanes $(\Pi_k^{\varepsilon,i})_{k \in Z, i = \{1, \dots, n\}}$ determine a partition of R^n, up to a μ-null set, in open disjoint rectangles whose sides are parallel to the coordinates axes. Being Ω bounded there exists $m_\varepsilon \in N$ such that only m_ε of these rectangles do not intersect Ω. If we call $Q_1^\varepsilon, \dots, Q_{m_\varepsilon}^\varepsilon$ these rectangles the thesis follows.

If there exist i and k such that $x_k \in I_i$ then, being by Lemma 1.6 I_i at most countable, there exists $y_k \in \Omega_i - I_i$ such that

$$\begin{cases} (x_{k-1})_i < (y_k)_i < (x_{k+1})_i, \\ 0 < (y_k)_i - (x_{k-1})_i < \dfrac{\varepsilon}{\sqrt{n}}, \\ 0 < (x_{k+1})_i - (y_k)_i < \dfrac{\varepsilon}{\sqrt{n}}. \end{cases}$$

We set $P_k^i = \{y \in R^n : y_i = (y_k)_i\}$ and we replace the hyperplane $\Pi_k^{\varepsilon,i}$ by P_k^i. Being Ω

bounded the number of the integers $k \in Z$ such that $x_k \in I_i$ is finite, say $k_1, k_2, \dots, k_l$. Then by considering the hyperplanes $(\Pi_k^{i,t})_{k \in Z - \{k_1, \dots, k_l\}} \cup P_{k_1}^i \cup \dots \cup P_{k_l}^i$ we fall in the previous case and the thesis follows. ■

2. - Semicontinuity

Let Ω be an open subset of R^n, let f be as in (0.4) verifying (0.5). In this section we prove a lower semicontinuity result, in the $L^1_{loc}(\Omega)$ topology, for the functional $G(\Omega, \cdot)$ given by (1.5).

Proposition 2.1: *Let f be as in (0.4) verifying (0.5). Then for every $z \in R^n$ $f^\infty(\cdot, z)$ is lower semicontinuous and for every $A \subset\subset \Omega$ and every compact set K of A it results*

$$f^\infty(x_K, z) \leq f^\infty(x, z) + \lambda_A (\operatorname{diam} K) f^\infty(x, z), \qquad \forall x \in K,\ \forall z \in R^n \tag{2.1}$$

λ_A, x_K being given by (0.5).

Proof: For every $z \in R^n$ the function $f^\infty(\cdot, z)$ is lower semicontinuous since it is the supremum of the family of lower semicontinuous functions $x \in \Omega \mapsto tf(x, z/t)$.

Let $A \subset\subset \Omega$, $K \subseteq A$ be a compact set, λ_A and x_k given by (0.5), then

$$tf\left(x_K, \frac{z}{t}\right) \leq tf\left(x, \frac{z}{t}\right) + \lambda_A (\operatorname{diam} K)\left\{t + tf\left(x, \frac{z}{t}\right)\right\} \qquad \forall t \in]0, +\infty[.$$

If $t \to 0^+$ we have (2.1). ■

Lemma 2.2: *Let Ω be an open set, f be as in (0.4) verifying (0.5) and let $A \subset\subset \Omega$. Then the functional $G(A, \cdot)$ is lower semicontinuous on $BV(A)$ in the $L^1_{loc}(A)$ topology.*

Proof: Let $u \in BV(A)$. By Lemma 1.6 for every $k \in N$ there exist m disjoint rectangles of R^n, $Q_1^k, Q_2^k, \dots, Q_m^k$, whose sides are parallel to the coordinates axes such that, if $A_j^k = A \cap Q_j^k$, we have $|Du|(\partial A_j^k \cap A) = 0$ and $\operatorname{diam} A_j^k < 1/k$ for every $k \in N$ and for every $j = 1, 2, \dots, m$. Let $(u_h)_h \subseteq BV(A)$ such that $u_h \to u$ in $L^1_{loc}(A)$. We can suppose that the limit below exists and

$$\lim_h \int_A f(x, \nabla u_h)\, dx + \int_A f^\infty\left(x, \frac{dD^s u_h}{d|Du_h|}\right) d|Du_h| < +\infty. \tag{2.2}$$

It results

$$G(A, u_k) = \int_A f(x, \nabla u_k)\,dx + \int_A f^\infty\left(x, \frac{dD^s u_k}{d|Du_k|}\right) d|Du_k| \geq \tag{2.3}$$

$$\geq \sum_{j=1}^{m} \int_{A_j^k} f(x, \nabla u_k)\,dx + \sum_{j=1}^{m} \int_{A_j^k} f^\infty\left(x, \frac{dD^s u_k}{d|Du_k|}\right) d|Du_k| .$$

Let B such that $A \subset\subset B \subset\subset \Omega$ and $\overline{A_j^k} \subseteq B$; by (0.5) and (2.1) there exist λ_B and $x_j^k \in \overline{A_j^k}$ such that

$$\begin{cases} f(x_j^k, z) \leq f(x, z) + \lambda_B\left(\frac{1}{k}\right)\{1 + f(x, z)\}, \\ f^\infty(x_j^k, z) \leq f^\infty(x, z) + \lambda_B\left(\frac{1}{k}\right)\{f^\infty(x, z)\}, \end{cases} \tag{2.4}$$

for every $x \in \overline{A_j^k}$ and for every z. Moreover by (2.2) we definitively have that

$$\int_A f(x, \nabla u_k)\,dx + \int_A f^\infty\left(x, \frac{dD^s u_k}{d|Du_k|}\right) d|Du_k| < +\infty$$

from which, together with (2.4), we infer that for every $k \in N$ and for every $j = 1, 2, \dots, m$

$$\int_{A_j^k} f(x_j^k, \nabla u_k)\,dx + \int_{A_j^k} f^\infty\left(x_j^k, \frac{dD^s u_k}{d|Du_k|}\right) d|Du_k| < +\infty .$$

By virtue of this, (2.3) and (2.4) we get

$$G(A, u_k) \geq \sum_{j=1}^{m}\left[\int_{A_j^k} f(x_j^k, \nabla u_k)\,dx + \int_{A_j^k} f(x, \nabla u_k) - \int_{A_j^k} f(x_j^k, \nabla u_k)\right] + \tag{2.5}$$

$$+ \sum_{j=1}^{m}\left[\int_{A_j^k} f^\infty\left(x_j^k, \frac{dD^s u_k}{d|Du_k|}\right) d|Du_k| + \int_{A_j^k} f^\infty\left(x, \frac{dD^s u_k}{d|Du_k|}\right) d|Du_k| -\right.$$

$$\left. - \int_{A_j^k} f^\infty\left(x_j^k, \frac{dD^s u_k}{d|Du_k|}\right) d|Du_k|\right] \geq \sum_{j=1}^{m} \int_{A_j^k} f(x_j^k, \nabla u_k)\,dx + \sum_{j=1}^{m} \int_{A_j^k} f^\infty\left(x_j^k, \frac{dD^s u_k}{d|Du_k|}\right) d|Du_k| -$$

$$- \sum_{j=1}^{m} \int_{A_j^k} \lambda_B\left(\frac{1}{k}\right)\{1 + f(x, \nabla u_k)\}dx - \sum_{j=1}^{m} \int_{A_j^k} \lambda_B\left(\frac{1}{k}\right) f^\infty\left(x, \frac{dD^s u_k}{d|Du_k|}\right) d|Du_k| .$$

By Remark 1.2 it results

$$\text{(2.6)}\quad \liminf_h \left(\sum_{j=1}^{m} \int_{A_j^k} f(x_j^k, \nabla u_h)\, dx + \sum_{j=1}^{m} \int_{A_j^k} f^\infty\left(x_j^k, \frac{dD^s u_h}{d|Du_h|}\right) d|Du_h| \right) \geq$$

$$\geq \sum_{j=1}^{m} \int_{A_j^k} f(x_j^k, \nabla u)\, dx + \sum_{j=1}^{m} \int_{A_j^k} f^\infty\left(x_j^k, \frac{dD^s u}{d|Du|}\right) d|Du| .$$

By (2.5) and (2.6) we get

$$\text{(2.7)}\quad \liminf_h G(A, u_h) \geq \sum_{j=1}^{m} \int_{A_j^k} f(x_j^k, \nabla u)\, dx + \sum_{j=1}^{m} \int_{A_j^k} f^\infty\left(x_j^k, \frac{dD^s u}{d|Du|}\right) d|Du| -$$

$$- \lambda_B\left(\frac{1}{k}\right) \limsup_h \left[\sum_{j=1}^{m} \int_{A_j^k} \{1 + f(x, \nabla u_h)\}\, dx + \sum_{j=1}^{m} \int_{A_j^k} f^\infty\left(x, \frac{dD^s u_h}{d|Du_h|}\right) d|Du_h| \right].$$

Once we observe that by (2.2)

$$\limsup_h \left[\sum_{j=1}^{m} \int_{A_j^k} \{1 + f(x, \nabla u_h)\}\, dx + \sum_{j=1}^{m} \int_{A_j^k} f^\infty\left(x, \frac{dD^s u_h}{d|Du_h|}\right) d|Du_h| \right] < +\infty ,$$

letting k go to $+\infty$ in (2.7) we obtain

$$\text{(2.8)}\quad \liminf_h G(A, u_h) \geq \liminf_k \left[\sum_{j=1}^{m} \int_{A_j^k} f(x_j^k, \nabla u)\, dx + \sum_{j=1}^{m} \int_{A_j^k} f^\infty\left(x_j^k, \frac{dD^s u}{d|Du|}\right) d|Du| \right].$$

Let $X = \bigcap_k \bigcup_j A_j^k$. Fixed $x \in X$ then we have that for every $k \in N$ there exists a unique $j_k \in N$ such that $x \in A_{j_k}^k$ and

$$\begin{cases} \sum_{j=1}^{m} \chi_{A_j^k}(x) f(x_j^k, \nabla u) = f(x_{j_k}^k, \nabla u), \\ \sum_{j=1}^{m} \chi_{A_j^k}(x) f^\infty\left(x_j^k, \frac{dD^s u}{d|Du|}\right) = f^\infty\left(x_{j_k}^k, \frac{dD^s u}{d|Du|}\right). \end{cases}$$

For every $k \in N$ it results $|x_{j_k}^k - x| \leq \operatorname{diam} A_{j_k}^k < 1/k$ so that $\lim_k x_{j_k}^k = x$, therefore by the lower semicontinuity of f we deduce that for every $x \in X$

$$(2.9)\quad \begin{cases} \liminf_k \sum_{j=1}^{m} \chi_{A_j^k}(x) f(x_j^k, \nabla u) \geq f(x, \nabla u), \\ \liminf_k \sum_{j=1}^{m} \chi_{A_j^k}(x) f^\infty\left(x_j^k, \frac{dD^s u}{d|Du|}\right) \geq f^\infty\left(x, \frac{dD^s u}{d|Du|}\right). \end{cases}$$

Since

$$A - X = A - \bigcap_k \bigcup_j A_j^k = \bigcup_k \left(A - \bigcup_j A_j^k\right) = \bigcup_{k,j} (A \cap \partial(A_j^k))$$

we have, by Lemma 1.6,

$$(2.10)\qquad L^n(A - X) = |Du|(A - X) = 0 .$$

By Fatou's Lemma, (2.9) and (2.10) it results

$$(2.11)\qquad \liminf_k \int_A \sum_{j=1}^{m} \chi_{A_j^k}(x) f(x_j^k, \nabla u)\, dx \geq \int_A f(x, \nabla u)\, dx$$

and analogously

$$(2.12)\qquad \liminf_k \int_A \sum_{j=1}^{m} \chi_{A_j^k}(x) f^\infty\left(x_j^k, \frac{dD^s u}{d|Du|}\right) d|Du| \geq \int_A f^\infty\left(x, \frac{dD^s u}{d|Du|}\right) d|Du| .$$

Finally by (2.8), (2.11), (2.12)

$$\liminf_h G(A, u_h) \geq \int_A f(x, \nabla u)\, dx + \int_A f^\infty\left(x, \frac{dD^s u}{d|Du|}\right) d|Du| = G(A, u) . \qquad \blacksquare$$

Let us show the semicontinuity theorem

THEOREM 2.3: *Let Ω be an open set and f be as in* (0.4) *verifying* (0.5). *Then the functional $G(\Omega, \cdot)$ is lower semicontinuous on $BV_{loc}(\Omega)$ in the strong topology of $L^1_{loc}(\Omega)$.*

PROOF: Let $A \subset\subset \Omega$, let $(u_h)_h \subseteq BV_{loc}(\Omega)$ and let $u \in BV_{loc}(\Omega)$ be such that $u_h \to u$ in $L^1_{loc}(\Omega)$. Then $G(\Omega, u_h) \geq G(A, u_h)$, moreover by Lemma 2.2 we have

$$\liminf_h G(\Omega, u_h) \geq \liminf_h G(A, u_h) \geq G(A, u) .$$

Finally

$$G(\Omega, u) = \sup_{A \subset\subset \Omega} G(A, u) \leq \liminf_h G(\Omega, u_h) . \qquad \blacksquare$$

By Theorem 2.3 we deduce the following corollary:

COROLLARY 2.4: *Let Ω be an open set, $a: x \in \Omega \mapsto a(x) \in [0, +\infty[$ be lower semicontinuous and $g: z \in \mathbf{R}^n \mapsto g(z) \in [0, +\infty[$ be convex. Then the functional*

$$u \in BV_{loc}(\Omega) \mapsto \int_\Omega a(x) g(\nabla u)\, dx + \int_\Omega a(x) g^\infty\left(\frac{dD^s u}{d|Du|}\right) d|Du|$$

is lower semicontinuous in the $L^1_{loc}(\Omega)$ topology.

PROOF: We observe that if we put $f(x, z) = a(x)g(z)$ then f verifies (0.4) and (0.5). Indeed, being a lower semicontinuous, for every compact K there exists $x_K \in K$ such that

$$a(x_K) \leq a(x), \quad \forall x \in K$$

so

$$a(x_K) g(z) \leq a(x) g(z), \quad \forall x \in K, \forall z.$$

By virtue of this the thesis follows from Theorem 2.3. ■

In particular if a satisfies the assumptions of Corollary 2.4 and $g(z) = |z|$ the functional

$$u \in BV_{loc}(\Omega) \mapsto \int_\Omega a(x) |\nabla u|\, dx + \int_\Omega a(x) \left| \frac{dD^s u}{d|Du|} \right| d|Du|$$

is lower semicontinuous in the strong topology of $L^1_{loc}(\Omega)$.

REMARK 2.5: We remark that Theorem 2.3 holds also for functions f which do not verify Serrin's conditions, see [S], as it can be proved with easy examples.

On the other side we observe that

REMARK 2.6: The functional $G(\Omega, \cdot)$ in (1.5) is not necessarely $L^1_{loc}(\Omega)$-lower semicontinuous if the integrand f is not semicontinuous, see the example in section 4 of [CS].

3. - RELAXATION

In this section we intend to prove a relaxation result for the functional in (1.4).

LEMMA 3.1: *Let Ω be an open set and let f verify* (0.5), (0.7). *Then, for every $z \in \mathbf{R}^n$, $f(\cdot, z) \in L^\infty_{loc}(\Omega)$.*

PROOF: The claim follows immediatly by (0.5) and (0.7). ■

Proposition 3.2: *Let Ω be an open set and let f verify (0.5) and (0.7). Then for every $z \in R^n$, $f(\cdot, z)$ is continuous on Ω.*

Proof: Let $z \in R^n$, $\varepsilon > 0$ and let K be a compact subset of Ω such that diam $K < \varepsilon$, moreover let A be an open set such that $K \subsetneq A \subset\subset \Omega$, then, by (0.5), there exist $x_\varepsilon \in K$ and λ_A such that

$$f(x, z) \geq f(x_\varepsilon, z) - \lambda_A(\varepsilon)\{1 + f(x, z)\}. \tag{3.1}$$

By Lemma 3.1 $f(\cdot, z)$ is bounded on A so that there exists a costant $C = C_A$ such that $f(x, z) \leq C$ for every $x \in \bar{A}$. Then, for every $y \in K$, by (3.1) and by (0.7) we have

$$f(x, z) \geq f(x_\varepsilon, z) - \lambda_A(\varepsilon)(1 + C) \geq f(y, z) - \lambda_A(\varepsilon)\{1 + f(x_\varepsilon, z)\} - \lambda_A(\varepsilon)(1 + C) \geq$$
$$\geq f(y, z) - 2\lambda_A(\varepsilon)(1 + C).$$

By interchanging the roles of x and y it results

$$|f(x, z) - f(y, z)| \leq M\lambda_A(\varepsilon), \quad \forall x, y \in K$$

where $M = 2(1 + C)$. ∎

For every $k \in \mathbf{N}$ let χ_k be a function in $C^1(R)$ verifying:

$$\chi_k(t) = \begin{cases} -k-1 & \text{if } t \leq -k-2, \\ t & \text{if } -k \leq t \leq k, \\ k+1 & \text{if } t \geq k+2, \end{cases} \tag{3.2}$$

and

$$0 \leq \frac{d\chi_k}{dt} \leq 1. \tag{3.3}$$

Lemma 3.3: *Let Ω be an open set, f be as in (0.4) verifying (0.8). For every $k \in \mathbf{N}$ let χ_k be a function as in (3.2) and (3.3). Then*

$$\bar{I}(\Omega, u) = \lim_k \bar{I}(\Omega, \chi_k(u)), \quad \forall u \in L^1_{loc}(\Omega), \tag{3.4}$$

$$\bar{I}_-(\Omega, u) = \lim_k \bar{I}_-(\Omega, \chi_k(u)), \quad \forall u \in L^1_{loc}(\Omega). \tag{3.5}$$

Proof: If $u \in L^1_{loc}(\Omega)$, then for every $k \in \mathbf{N}$ $\chi_k(u) \in L^1_{loc}(\Omega)$ and $\chi_k(u) \to u$ in $L^1_{loc}(\Omega)$. Being $\bar{I}(\Omega, \cdot)$ and $I_-(\Omega, \cdot)$ lower semicontinuous in $L^1_{loc}(\Omega)$ we have

$$\begin{cases} \bar{I}(\Omega, u) \leq \liminf_k \bar{I}(\Omega, \chi_k(u)), \\ \bar{I}_-(\Omega, u) \leq \liminf_k \bar{I}_-(\Omega, \chi_k(u)). \end{cases} \tag{3.6}$$

Let $(u_h)_h \subseteq C^1(\Omega)$ such that $u_h \to u$ in $L^1_{loc}(\Omega)$, $u_h(x) \to u(x)$ a.e. in Ω and

$$\bar{I}(\Omega, u) \geq \lim_h \int_\Omega f(x, \nabla u_h(x))\, dx \,. \tag{3.7}$$

For every $k \in N$ it results

$$\chi_k(u_h) \in C^1(\Omega), \quad \forall h \in N \quad \text{and} \quad \chi_k(u_h) \to \chi_k(u) \quad \text{in } L^1_{loc}(\Omega) \text{ if } h \to +\infty \,.$$

Being $f(x, \cdot)$ convex and $0 \leq d\chi_k / dt \leq 1$ we have

$$\begin{aligned} \int_\Omega f(x, \nabla \chi_k(u_h))\, dx &\leq \int_{|u_h| \leq k} f(x, \nabla u_h)\, dx + \int_{|u_h| > k+2} f(x, 0)\, dx + \\ &+ \int_{k < |u_h| < k+2} f(x, (\chi_k)' \nabla u_h)\, dx \leq \int_{|u_h| \leq k} f(x, \nabla u_h)\, dx + \int_{|u_h| \geq k+2} f(x, 0)\, dx + \\ &+ \int_{k < |u_h| < k+2} (\chi_k)' f(x, \nabla u_h)\, dx + \int_{k < |u_h| < k+2} (1 - \chi_k') f(x, 0)\, dx \leq \int_{|u_h| \leq k} f(x, \nabla u_h)\, dx + \\ &+ \int_{|u_h| \geq k+2} f(x, 0)\, dx + \int_{k < |u_h| < k+2} f(x, \nabla u_h)\, dx + \int_{k < |u_h| < k+2} f(x, 0)\, dx \leq \\ &\leq \int_{|u_h| < k+2} f(x, \nabla u_h)\, dx + \int_{|u_h| > k} f(x, 0)\, dx \leq \int_\Omega f(x, \nabla u_h)\, dx + \int_{|u_h| > k} f(x, 0)\, dx \,. \end{aligned} \tag{3.8}$$

Let us observe that if $h \to +\infty$ then:

$$\int_{|u_h| > k} f(x, 0)\, dx \to \int_{|u| \geq k} f(x, 0)\, dx \,. \tag{3.9}$$

By (3.8), (3.7) and (3.9) we conclude that

$$\bar{I}(\Omega, \chi_k(u)) \leq \liminf_h \int_\Omega f(x, \nabla \chi_k(u_h))\, dx \leq \bar{I}(\Omega, u) + \int_{|u| \geq k} f(x, 0)\, dx \,. \tag{3.10}$$

If $k \to +\infty$, by (0.8), we get

$$\limsup_k \bar{I}(\Omega, \chi_k(u)) \leq \bar{I}(\Omega, u) \,. \tag{3.11}$$

By (3.6) and (3.11) we have (3.4).

Let us show (3.5). If $A \subset\subset \Omega$, then by (3.10) we get

$$\bar{I}(A, \chi_k(u)) \leq \bar{I}(A, u) + \int_{|u| \geq k} f(x, 0)\, dx \leq \bar{I}_-(\Omega, u) + \int_{|u| \geq k} f(x, 0)\, dx \,, \tag{3.12}$$

from which we deduce that

$$\bar{I}_-(\Omega, \chi_k(u)) \leq \bar{I}_-(\Omega, u) + \int_{|u| \geq k} f(x, 0)\, dx \tag{3.13}$$

and finally that

$$\limsup_k \bar{I}_-(\Omega, \chi_k(u)) \leq \bar{I}_-(\Omega, u). \tag{3.14}$$

By (3.6) and (3.14) equality (3.5) follows.

Lemma 3.4: *Let Ω be an open set, f be as in (0.4) verifying (0.8). Let $u \in L^\infty(\Omega)$, $(u_h)_h \subseteq C^1(\Omega)$ be such that $u_h \to u$ in $L^1_{loc}(\Omega)$. Then there exists $(\tilde{u}_h)_h \subseteq C^1(\Omega)$ such that*

1) $\tilde{u}_h \to u$ in $L^1_{loc}(\Omega)$,
2) $\|\tilde{u}_h\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)} + 1$,
3) $\liminf_h \int_\Omega f(x, \nabla \tilde{u}_h)\, dx \leq \liminf_h \int_\Omega f(x, \nabla u_h)\, dx$.

Proof: Let $\tilde{u}_h = \chi_{\|u\|_{L^\infty}}(u_h)$ where $\chi_{\|u\|_{L^\infty}}$ verifies (3.2) and (3.3) with $k = \|u\|_{L^\infty}$. It results that $\tilde{u}_h \in C^1(\Omega)$ for every $h \in N$; $\tilde{u}_h \to \chi_{\|u\|_{L^\infty}}(u) = u$ in $L^1_{loc}(\Omega)$ and $\|\tilde{u}_h\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)} + 1$. By (3.8) we have

$$\int_\Omega f(x, \nabla \tilde{u}_h)\, dx \leq \int_\Omega f(x, \nabla u_h)\, dx + \int_{\{x \in \Omega:\, |u_h| > \|u\|_{L^\infty(\Omega)}\}} f(x, 0)\, dx,$$

moreover

$$\lim_h L^n(\{x \in \Omega: |u_h| > \|u\|_{L^\infty(\Omega)}\}) \to 0,$$

thus, by (0.8), the thesis follows. ■

Proposition 3.5: *Let Ω be an open set, f be as in (0.4) and let us assume that $\forall z \in R^n f(\cdot, z) \in L^1_{loc}(\Omega)$. Let $A, A_1, A_2 \subset\subset \Omega$, $u \in L^1_{loc}(A_1 \cup A_2)$. If $A \subset\subset A_1 \cup A_2$, then*

$$\bar{I}(A, u) \leq \bar{I}(A_1, u) + \bar{I}(A_2, u) \tag{3.15}$$

and, if $A \subseteq A_1 \cup A_2$, then

$$\bar{I}_-(A, u) \leq \bar{I}_-(A_1, u) + \bar{I}_-(A_2, u). \tag{3.16}$$

Proof: We first prove (3.15) when $u \in L^\infty(A_1 \cup A_2)$. Let $(u_h^1)_h \subseteq C^1(A_1)$ and

$(u_b^2)_b \subseteq C^1(A_2)$ be such that

$$\begin{cases} u_b^i \to u \text{ in } L^1_{loc}(A_i) \quad \text{and a.e. } i = 1, 2, \\ \bar{I}(A_i, u) \geq \limsup_b \int_{A_i} f(x, \nabla u_b^i)\, dx, \quad i = 1, 2. \end{cases} \tag{3.17}$$

By Lemma 3.4 we can suppose that

$$\|u_b^i\|_{L^\infty(A_i)} \leq \|u\|_{L^\infty(A_1 \cup A_2)} + 1, \quad i = 1, 2. \tag{3.18}$$

Let $B \subset\subset A_1$ be such that $A \subset\subset B \cup A_2$ and let

$$\varphi \in C_0^1(A_1):\ 0 \leq \varphi \leq 1, \quad \varphi = 1 \text{ on } B.$$

Setting

$$w_b = \varphi u_b^1 + (1 - \varphi) u_b^2 \tag{3.19}$$

we have that $w_b \to u$ in $L^1_{loc}(A)$. For every $t \in]0, 1[$, by the convexity of f, we have

$$\int_A f(x, t\nabla w_b)\, dx \leq t \int_A \varphi f(x, \nabla u_b^1)\, dx + \tag{3.20}$$

$$+ t \int_A (1 - \varphi) f(x, \nabla u_b^2)\, dx + (1 - t) \int_A f\left(x, \frac{t}{1-t}(u_b^1 - u_b^2)\nabla\varphi\right) dx \leq$$

$$\leq t \int_A f(x, \nabla u_b^1)\, dx + t \int_A f(x, \nabla u_b^2)\, dx + (1 - t) \int_A f\left(x, \frac{t}{1-t}(u_b^1 - u_b^2)\nabla\varphi\right).$$

Putting for every $b \in \mathbf{N}$ $m_b = (t/(1-t))(u_b^1 - u_b^2)\nabla\varphi$, then by (3.17), (3.18) and Lemma (1.4) we have

$$\limsup_b \int_A f\left(x, \frac{t}{1-t}(u_b^1 - u_b^2)\nabla\varphi\right) = \int_A f(x, 0)\, dx. \tag{3.21}$$

By (3.17),(3.20) and (3.21) it results

$$\bar{I}(A, tu) \leq t\bar{I}(A_1, u) + t\bar{I}(A_2, u) + (1 - t) \int_A f(x, 0)\, dx. \tag{3.22}$$

Finally, being $\bar{I}(A, \cdot)$ lower semicontinuous, we obtain

$$\bar{I}(A, u) \leq \liminf_{t \to 1^-} \bar{I}(A, tu) \leq \bar{I}(A_1, u) + \bar{I}(A_2, u). \tag{3.23}$$

So (3.15) is proved if $u \in L^\infty(A_1 \cup A_2)$.

Now we prove (3.16) if $u \in L^\infty(A_1 \cup A_2)$. Let $A' \subset\subset A$, B_1, B_2 such that $B_i \subset\subset A_i$

and $A' \subset\subset B_1 \cup B_2$ then by (3.15) it follows that

(3.24) $$\bar{I}(A', u) \leq \bar{I}(B_1, u) + \bar{I}(B_2, u) \leq \bar{I}_-(A_1, u) + \bar{I}_-(A_2, u)$$

and by (3.24) that

(3.25) $$\bar{I}_-(A, u) \leq \bar{I}_-(A_1, u) + \bar{I}_-(A_2, u).$$

We now show (3.15) and (3.16) when $u \in L^1_{loc}(A_1 \cup A_2)$. Let $k \in \mathbf{N}$ and let us consider $\chi_k(u)$ where χ_k verifies (3.2) and (3.3). The function $\chi_k(u)$ belongs to $L^\infty(A_1 \cup A_2)$ then by Lemma 3.3 and by (3.15) we infer

(3.26) $$\bar{I}(A, u) \leq \liminf_k \bar{I}(A, \chi_k(u)) \leq \limsup_k \bar{I}(A_1, \chi_k(u)) + \limsup_k \bar{I}(A_2, \chi_k(u)) =$$
$$= \bar{I}(A_1, u) + \bar{I}(A_2, u).$$

The proof of (3.16) is analogous. ■

Proposition 3.6: *Let Ω be an open set, f be as in* (0.4) *verifying* (0.8) *and let us assume that* $\forall z \in \mathbf{R}^n f(\cdot, z) \in L^1_{loc}(\Omega)$. *Then*

(3.27) $$\bar{I}(\Omega, u) = \bar{I}_-(\Omega, u) \quad \text{for every } u \in L^1_{loc}(\Omega).$$

Proof: It is trivial that $\bar{I}_-(\Omega, u) \leq \bar{I}(\Omega, u)$ so it is enough to prove only the opposite inequality. To this aim we can assume that $\bar{I}_-(\Omega, u) < +\infty$. We first suppose that $u \in L^\infty(\Omega)$. For every $\varepsilon > 0$ and $j \in \mathbf{N} \cup \{0\}$ let $A_j \subset\subset \Omega$ be such that

(3.28) $$\begin{cases} A_0 \subset\subset A_j \subset\subset A_{j+1} \subset\subset \Omega, \\ L^n(\partial A_j) = 0, \\ \bar{I}_-(\Omega, u) - \dfrac{\varepsilon}{2^j} \leq \bar{I}(A_j, u) \leq \bar{I}_-(\Omega, u). \end{cases}$$

then for every $j \in \mathbf{N} \cup \{0\}$ there exists $(u_h^j)_h \subseteq C^1(A_j)$ such that

(3.29) $$\begin{cases} u_h^j \to u \quad \text{in } L^1_{loc}(A_j); \quad \text{and a.e. in } A_j, \\ \bar{I}(A_j, u) = \lim_h \displaystyle\int_{A_j} f(x, \nabla u_h^j)\, dx. \end{cases}$$

By (3.29) we can assume that $\int_{A_j} f(x, \nabla u_h^j) < +\infty$ for every h and by Lemma 3.4 we can suppose that for every j and for every h

(3.30) $$\|u_h^j\|_{L^\infty(A_j)} \leq \|u\|_{L^\infty(\Omega)} + 1.$$

If we denote by $A_{-1} = \emptyset$, then the family $(A_{j+1} - \bar{A}_{j-1})_j : j \in \mathbf{N} \cup \{0\}$ is a locally finite open covering of Ω. Let $(\varphi_j)_j$ be a partition of unity relative to such covering, i.e.,

$$\text{(3.31)}\qquad \begin{cases} \forall j \in \mathbf{N}\cup\{0\}\ \varphi_j \in C_0^1(A_{j+1} - \overline{A}_{j-1}), \\ 0 \leq \varphi_j \leq 1, \\ \sum_{j=0}^{+\infty} \varphi_j = 1 \quad \text{in } \Omega. \end{cases}$$

For every $j \in \mathbf{N}$ we denote by $\bar{h}(j)$ an integer to be chosen later, and we put

$$\text{(3.32)}\qquad v_j = u^j_{\bar{h}(j)},$$

$$\text{(3.33)}\qquad w_\varepsilon = \sum_{j=1}^{+\infty} \varphi_{j-1} v_j.$$

It is clear that for every $x \in \Omega$ the sum on the right hand side of (3.33) has only a finite number of non zero terms, hence for every $\varepsilon > 0$ it results that $w_\varepsilon \in C^1(\Omega)$. Moreover for every $A \subset\subset \Omega$, $t \in]0, 1[$, by (3.33) we have

$$\text{(3.34)}\qquad \|tw_\varepsilon - u\|_{L^1(A)} \leq t\|w_\varepsilon - u\|_{L^1(A)} + (1-t)\|u\|_{L^1(A)} =$$
$$= t\left\| \sum_{j=1}^{+\infty} \varphi_{j-1}(v_j - u)\right\|_{L^1(A)} + (1-t)\|u\|_{L^1(A)} \leq$$
$$\leq t \sum_{j=1}^{+\infty} \int_{A \cap \operatorname{supp}\varphi_{j-1}} |u^j_{\bar{h}(j)} - u| + (1-t)\|u\|_{L^1(A)}.$$

By the convexity of f and the finiteness of $\int_{A_j} f(x, \nabla u^j_h)$ we obtain

$$\text{(3.35)}\qquad \int_\Omega f(x, t\nabla w_\varepsilon)\,dx = \int_\Omega f\left(x, t\left(\sum_{j=1}^{+\infty} \varphi_{j-1}\nabla v_j + v_j \nabla\varphi_{j-1}\right)\right)dx \leq$$
$$\leq t\int_\Omega f\left(x, \sum_{j=1}^{+\infty} \varphi_{j-1}\nabla v_j\right)dx + (1-t)\int_\Omega f\left(x, \frac{t}{1-t}\sum_{j=1}^{+\infty} v_j \nabla\varphi_{j-1}\right)dx \leq$$
$$\leq \int_\Omega \sum_{j=1}^{+\infty} \varphi_{j-1} f(x, \nabla v_j)\,dx + (1-t)\int_\Omega f\left(x, \frac{t}{1-t}\sum_{j=1}^{+\infty} v_j \nabla\varphi_{j-1}\right)dx \leq$$
$$\leq \int_{A_1} f(x, \nabla v_1)\,dx + \sum_{j=2}^{+\infty} \int_{A_j - \bar{A}_{j-2}} f(x, \nabla v_j)\,dx +$$

$$+(1-t)\int_\Omega f\left(x, \frac{t}{1-t}\sum_{j=1}^{+\infty} v_j \nabla\varphi_{j-1}\right)dx \le$$

$$\le \int_{A_1} f(x, \nabla v_1)\,dx + \sum_{j=2}^{+\infty}\left(\int_{A_j} f(x, \nabla v_j) - \int_{\overline{A}_{j-1}} f(x, \nabla v_j)\right)dx +$$

$$+(1-t)\int_\Omega f\left(x, \frac{t}{1-t}\sum_{j=1}^{+\infty} v_j \nabla\varphi_{j-1}\right)dx.$$

Let us fix $j \in N$, then we have

$$\begin{cases} \nabla\varphi_{j-1} = 0 & \text{in } A_0, \\ \sum_{i=1}^{+\infty} v_i \nabla\varphi_{i-1} = v_j \nabla\varphi_{j-1} + v_{j+1}\nabla\varphi_j & \text{in } A_j - \overline{A}_{j-1}, \end{cases}$$

so that

$$\int_\Omega f\left(x, \frac{t}{1-t}\sum_{j=1}^{+\infty} v_j \nabla\varphi_{j-1}\right)dx = \tag{3.36}$$

$$= \int_{A_0} f(x, 0)\,dx + \sum_{j=1}^{+\infty}\int_{A_j-\overline{A}_{j-1}} f\left(x, \frac{t}{1-t}(v_j\nabla\varphi_{j-1} + v_{j+1}\nabla\varphi_j)\right)dx.$$

We also remark that if $x \in A_j - \overline{A}_{j-1}$ it results $\sum_{i=1}^{+\infty}\varphi_i = \varphi_{j-1} + \varphi_j = 1$ and $\nabla\varphi_{j-1} + \nabla\varphi_j = 0$ so that by (3.30), Lemma 1.4 and an identification argument, for every $j \in N \cup \{0\}$ it results

$$\lim_{(h,k)\to(+\infty,+\infty)} \int_{A_j-\overline{A}_{j-1}} f\left(x, \frac{t}{1-t}(u_h^j\nabla\varphi_{j-1} + u_k^{j+1}\nabla\varphi_j)\right)dx = \tag{3.37}$$

$$= \lim_h \lim_k \int_{A_j-\overline{A}_{j-1}} f\left(x, \frac{t}{1-t}(u_h^j\nabla\varphi_{j-1} + u_k^{j+1}\nabla\varphi_j)\right)dx = \int_{A_j-\overline{A}_{j-1}} f(x, 0)\,dx$$

by (3.37) there exist $h'(j) \in N$ and $k(j) \in N$ verifying

$$\int_{A_j-\overline{A}_{j-1}} f\left(x, \frac{t}{1-t}(u_h^j\nabla\varphi_{j-1} + u_k^{j+1}\nabla\varphi_j)\right)dx \le \int_{A_j-\overline{A}_{j-1}} f(x, 0)\,dx + \frac{\varepsilon}{2^j} \tag{3.38}$$

for all $h \ge h'(j)$ and $k \ge k(j)$. By (3.29) we obtain that for every $j \in N - \{1\}$ there exists $\bar{b}(j) \in \mathbf{N}$ such that

$$(3.39) \qquad \bar{I}_-(\Omega, u) > \bar{I}(A_j, u) \geq \int_{A_j} f(x, \nabla u_b^j)\,dx - \frac{\varepsilon}{2^j}, \quad \forall b \geq \bar{b}(j),$$

moreover by (3.28) and the convergence of u_b^j to u in $L^1_{loc}(A_j)$ we get

$$\bar{I}_-(\Omega, u) - \frac{\varepsilon}{2^{j-2}} \leq \bar{I}(A_{j-2}, u) \leq \liminf_b \int_{A_{j-2}} f(x, \nabla u_b^j)\,dx$$

so that there exists $b^*(j) \in \mathbf{N}$ with

$$(3.40) \qquad \int_{A_{j-2}} f(x, \nabla u_b^j)\,dx \geq \bar{I}_-(\Omega, u) - \frac{\varepsilon}{2^{j-2}} \geq \bar{I}_-(\Omega, u) - \frac{2\varepsilon}{2^{j-2}}, \quad \forall b \geq b^*(j).$$

Being $A \cap \operatorname{supp} \varphi_{j-1} \subset\subset A_j$ for every $j \in \mathbf{N}$ and being $\lim_b u_b^j = u$ in $L^1_{loc}(A_j)$ it is clear that for every $j \in \mathbf{N}$ there exists $b^{**}(j) \in \mathbf{N}$ such that

$$(3.41) \qquad \int_{A \cap \operatorname{supp} \varphi_{j-1}} |u_b^j - u|\,dx \leq \frac{\varepsilon}{2^j}, \quad \forall b \geq b^{**}(j).$$

By (3.38), (3.39), (3.40) and (3.41) we deduce the existence of

$$\bar{b}(j) > \max\{\bar{b}(j), b^*(j), b^{**}(j), b'(j)\} \quad \text{and} \quad k(j)$$

such that $\bar{b}(j+1) \geq k(j)$ and

$$(3.42) \qquad \int_{A \cap \operatorname{supp} \varphi_{j-1}} |u^j_{\bar{b}(j)} - u|\,dx \leq \frac{\varepsilon}{2^j}, \quad \forall j \in \mathbf{N},$$

$$(3.43) \qquad \int_{A_j} f(x, \nabla u^j_{\bar{b}(j)})\,dx \leq \bar{I}_-(\Omega, u) + \frac{\varepsilon}{2^j}, \quad \forall j \in \mathbf{N},$$

$$(3.44) \qquad \int_{A_{j-2}} f(x, \nabla u^j_{\bar{b}(j)})\,dx \geq \bar{I}_-(\Omega, u) - \frac{2\varepsilon}{2^{j-2}}, \quad \forall j = 2, 3, \ldots,$$

$$(3.45) \qquad \int_{A_j - \bar{A}_{j-1}} f\left(x, \frac{t}{1-t}(u^j_{\bar{b}(j)} \nabla\varphi_{j-1} + u_k^{j+1} \nabla\varphi_j)\right) dx \leq \int_{A_j - \bar{A}_{j-1}} f(x, 0)\,dx + \frac{\varepsilon}{2^j},$$

$$\forall j \in \mathbf{N}, \quad k \geq k(j).$$

Choosing $k = \bar{h}(j+1)$ in (3.45) we also obtain

$$\text{(3.46)} \qquad \int_{A_j - \bar{A}_{j-1}} f\left(x, \frac{t}{1-t}(u^j_{\bar{h}(j)} \nabla\varphi_{j-1} + u^{j+1}_{\bar{h}(j+1)} \nabla\varphi_j)\right) dx \leq$$

$$\leq \int_{A_j - \bar{A}_{j-1}} f(x, 0)\, dx + \frac{\varepsilon}{2^j}, \qquad \forall j \in \mathbf{N}.$$

By (3.35), (3.36), (3.43), (3.44) and (3.46) we deduce that

$$\text{(3.37)} \qquad \int_\Omega f(x, t\nabla w_\varepsilon)\, dx \leq$$

$$\leq \bar{I}_-(\Omega, u) + \frac{\varepsilon}{2} + \sum_{j=2}^{+\infty}\left[\bar{I}_-(\Omega, u) + \frac{\varepsilon}{2^j} - \bar{I}_-(\Omega, u) + \frac{2\varepsilon}{2^{j-2}}\right] +$$

$$+ (1-t)\int_{A_1} f(x, 0)\, dx + (1-t)\sum_{j=1}^{+\infty}\left(\int_{A_j - \bar{A}_{j-1}} f(x, 0)\, dx + \frac{\varepsilon}{2^j}\right) =$$

$$= \bar{I}_-(\Omega, u) + (1-t)\int_\Omega f(x, 0)\, dx + 5\varepsilon + (1-t)\varepsilon$$

and by (3.34) and (3.42) that

$$\text{(3.48).} \qquad \|tw_\varepsilon - u\|_{L^1(A)} \leq \varepsilon + (1-t)\|u\|_{L^1(A)}.$$

For every $m \in \mathbf{N}$, let us chose $\varepsilon = 1/m$, $t = 1 - (1/m)$ and $w_m = (1 - 1/m) w_{1/m}$. By (3.48) we have that

$$\text{(3.49)} \qquad w_m \to u \quad \text{in } L^1_{loc}(\Omega)$$

and by (3.47) and (3.49)

$$\text{(3.50)} \qquad \bar{I}(\Omega, u) \leq \liminf_m \int_\Omega f(x, \nabla w_m)\, dx \leq \bar{I}_-(\Omega, u).$$

By Lemma 3.3, if $u \in L^1_{loc}(\Omega)$,

$$\bar{I}(\Omega, u) = \lim_k \bar{I}(\Omega, \chi_k(u)) = \lim_k \bar{I}_-(\Omega, \chi_k(u)) = \bar{I}_-(\Omega, u). \qquad \blacksquare$$

LEMMA 3.7: *Let Ω be an open subset of $\mathbf{R}^n$, f be as in* (0.4) *verifying* (0.5), (0.7) *and* (0.8). *Let $A \subset\subset \Omega$ with $L^n(\partial A) = 0$, and $G(A, u)$ given by* (1.5), *then*

$$\text{(3.51)} \qquad \bar{I}_-(A, u) = G(A, u), \qquad \forall u \in BV(A).$$

PROOF: Let us first observe that (0.5) and (0.7), by Lemma 3.1, imply that $\forall z \in \in \mathbf{R}^n f(\cdot, z) \in L^1_{\mathrm{loc}}(\Omega)$. By Theorem 2.3 and Proposition 3.6 we have $G(A, u) \leq \leq \bar{I}(A, u) = \bar{I}_-(A, u)$ for every $u \in BV(A)$.

Let us show the opposite inequality and observe that to do this it is not restrictive to assume that $G(A, u) < +\infty$.

By Lemma 1.6 for every $k \in \mathbf{N}$ there exist ν_k open disjoint rectangles, $Q_1^k, Q_2^k, \ldots, Q_{\nu_k}^k$, whose sides are parallel to the coordinates axes such that, setting $A_j^k = A \cap Q_j^k$, it results $|Du|((\partial A_j^k) \cap A) = 0, \forall k, j = 1, 2, \ldots, \nu_k$, diam $Q_j^k < 1/k$ and

$$G(A, u) \geq \sum_{j=1}^{\nu_k} G(A_j^k, u) = \sum_{j=1}^{\nu_k} G(\overline{A_j^k} \cap A, u). \tag{3.52}$$

Let $\varepsilon > 0$, and fix $k \in \mathbf{N}$ and $j \in \{1, 2, \ldots, \nu_k\}$ then there exists an open set B_j^k such that

$$\begin{cases} A_j^k \subset\subset B_j^k \subset\subset \Omega, \\ L^n(B_j^k) \leq 2L^n(A_j^k), \\ G(\overline{A_j^k} \cap A, u) \geq G(B_j^k \cap A, u) - \dfrac{\varepsilon}{\nu_k}, \end{cases}$$

then

$$G(A, u) \geq \sum_{j=1}^{\nu_k} G(\overline{A_j^k} \cap A, u) \geq \sum_{j=1}^{\nu_k} G(B_j^k \cap A, u) - \varepsilon. \tag{3.53}$$

Let $\Omega_j^k = B_j^k \cap A$ and B with $A \subset\subset B \subset\subset \Omega$; by (0.5) and Proposition 2.1 there exist $x_j^k \in \overline{\Omega_j^k}$ and λ_B such that

$$\begin{cases} f(x_j^k, z) \leq f(x, z) + \lambda_B\left(\dfrac{1}{k}\right)\{1 + f(x, z)\}, \\ f^\infty(x_j^k, z) \leq f^\infty(x, z) + \lambda_B\left(\dfrac{1}{k}\right)\{f^\infty(x, z)\}, \end{cases} \tag{3.54}$$

for every $x \in \overline{\Omega_j^k}$ and for every $z \in \mathbf{R}^n$. Now we define

$$\overline{I^{k,j}}(\Omega_j^k, u) = \inf\left\{\liminf_h \int_{\Omega_j^k} f(x_j^k, \nabla u_h)\, dx (u_h)_h \subseteq C^1(\Omega_j^k), u_h \to u \text{ in } L^1_{\mathrm{loc}}(\Omega_j^k)\right\}.$$

By Proposition 1.3 and Proposition 3.6 we have

$$\int_{\Omega_j^k} f(x_j^k, \nabla u)\, dx + \int_{\Omega_j^k} f^\infty\left(x_j^k, \frac{dD^s u}{d|Du|}\right) d|Du| = \overline{I^{k,j}}_-(\Omega_j^k, u) = \overline{I^{k,j}}(\Omega_j^k, u).$$

By (3.54) and the finiteness of $G(A, u)$ we obtain the finiteness of

$$\int_{\Omega_j^k} f(x_j^k, \nabla u)\,dx + \int_{\Omega_j^k} f^\infty\left(x_j^k, \frac{dD^s u}{d|Du|}\right) d|Du|$$

for every $k \in \mathbb{N}$ and $j = 1, 2, \ldots, \nu_k$ from which, together with (3.53), we conclude that

$$G(A, u) \geq \sum_{j=1}^{\nu_k} \left[\int_{\Omega_j^k} f(x, \nabla u)\,dx + \int_{\Omega_j^k} f(x_j^k, \nabla u)\,dx - \int_{\Omega_j^k} f(x_j^k, \nabla u)\,dx \right] + \tag{3.55}$$

$$+ \sum_{j=1}^{\nu_k} \left[\int_{\Omega_j^k} f^\infty\left(x_j^k, \frac{dD^s u}{d|Du|}\right) d|Du| - \int_{\Omega_j^k} f^\infty\left(x_j^k, \frac{dD^s u}{d|Du|}\right) d|Du| + \right.$$

$$\left. + \int_{\Omega_j^k} f^\infty\left(x, \frac{dD^s u}{d|Du|}\right) d|Du| \right] - \varepsilon \geq$$

$$\geq \sum_{j=1}^{\nu_k} \left(\int_{\Omega_j^k} f(x_j^k, \nabla u)\,dx + \int_{\Omega_j^k} f^\infty\left(x_j^k, \frac{dD^s u}{d|Du|}\right) d|Du| \right) -$$

$$- \sum_{j=1}^{\nu_k} \int_{\Omega_j^k} \lambda_B\left(\frac{1}{k}\right) \{1 + f(x, \nabla u)\}\,dx - \sum_{j=1}^{\nu_k} \int_{\Omega_j^k} \lambda_B\left(\frac{1}{k}\right) f^\infty\left(x, \frac{dD^s u}{d|Du|}\right) d|Du| - \varepsilon =$$

$$= \sum_{j=1}^{\nu_k} \overline{I^{k,j}}(\Omega_j^k, u) - \sum_{j=1}^{\nu_k} \int_{\Omega_j^k} \lambda_B\left(\frac{1}{k}\right) \{1 + f(x, \nabla u)\}\,dx -$$

$$- \sum_{j=1}^{\nu_k} \int_{\Omega_j^k} \lambda_B\left(\frac{1}{k}\right) f^\infty\left(x, \frac{dD^s u}{d|Du|}\right) d|Du| - \varepsilon .$$

We now observe that there exists $(u_h^{k,j})_h \subseteq C^1(\Omega_j^k)$ such that

$$\begin{cases} u_h^{k,j} \to u & \text{in } L^1_{loc}(\Omega_j^k), \\ \overline{I^{k,j}}(\Omega_j^k, u) = \lim_h \int_{\Omega_j^k} f(x_j^k, \nabla u_h^{k,j})\,dx . \end{cases} \tag{3.56}$$

By (3.56), (0.7) being $G(A, u) < +\infty$ we obtain

$$(3.57) \quad \sum_{j=1}^{\nu_k} \overline{I^{k,j}}(\Omega_j^k, u) \geq \sum_{j=1}^{\nu_k} \liminf_h \int_{\Omega_j^k} f(x, \nabla u_h^{k,j})\, dx -$$

$$- \sum_{j=1}^{\nu_k} \limsup_h \int_{\Omega_j^k} \lambda_B\left(\frac{1}{k}\right)\{1 + f(x_j^k, \nabla u_h^{k,j})\}\, dx \geq$$

$$\geq \sum_{j=1}^{\nu_k} \bar{I}_-(\Omega_j^k, u) - \lambda_B\left(\frac{1}{k}\right)\left[\sum_{j=1}^{\nu_k}\left(L^n(\Omega_j^k) + \overline{I^{k,j}}(\Omega_j^k, u)\right)\right].$$

By (3.55) and (3.53) we have

$$(3.58) \quad \sum_{j=1}^{\nu_k} \overline{I^{k,j}}(\Omega_j^k, u) = \sum_{j=1}^{\nu_k} \int_{\Omega_j^k} f(x_j^k, \nabla u)\, dx + \int_{\Omega_j^k} f^\infty\left(x_j^k, \frac{dD^s u}{d|Du|}\right) d|Du| \leq$$

$$\leq G(A, u) + \sum_{j=1}^{\nu_k} \int_{\Omega_j^k} \lambda_B\left(\frac{1}{k}\right)\{1 + f(x, \nabla u)\}\, dx +$$

$$+ \int_{\Omega_j^k} \lambda_B\left(\frac{1}{k}\right)\left\{f^\infty\left(x, \frac{dD^s u}{d|Du|}\right)\right\} d|Du| + \varepsilon \leq$$

$$\leq G(A, u) + \lambda_B\left(\frac{1}{k}\right)\left[\sum_{j=1}^{\nu_k} L^n(\Omega_j^k) + \sum_{j=1}^{\nu_k} \int_{\Omega_j^k} f(x, \nabla u)\, dx +\right.$$

$$\left. + \sum_{j=1}^{\nu_k} \int_{\Omega_j^k} f^\infty\left(x, \frac{dD^s u}{d|Du|}\right) d|Du|\right] + \varepsilon \leq$$

$$\leq G(A, u) + \lambda_B\left(\frac{1}{k}\right) G(A, u) + \lambda_B\left(\frac{1}{k}\right) \sum_{j=1}^{\nu_k} L^n(\Omega_j^k) + \lambda_B\left(\frac{1}{k}\right)\varepsilon + \varepsilon \leq$$

$$\leq G(A, u)\left[1 + \lambda_B\left(\frac{1}{k}\right)\right] + \lambda_B\left(\frac{1}{k}\right) L^n(A) + \lambda_B\left(\frac{1}{k}\right)\varepsilon + \varepsilon.$$

Inequalyties (3.57) and (3.58) imply

$$(3.59)\qquad \sum_{j=1}^{\nu_k} \overline{I^{k,j}}(\Omega_j^k, u) \geq \sum_{j=1}^{\nu_k} \bar{I}_-(\Omega_j^k, u) -$$

$$-\lambda_B\left(\frac{1}{k}\right)\left[L^n(A) + G(A,u)\left[1 + \lambda_B\left(\frac{1}{k}\right)\right] + \lambda_B\left(\frac{1}{k}\right)L^n(A) + \lambda_B\left(\frac{1}{k}\right)\varepsilon + \varepsilon\right],$$

therefore by (3.55) and (3.59) we get

$$(3.60)\qquad G(A,u) \geq \sum_{j=1}^{\nu_k} \bar{I}_-(\Omega_j^k, u) -$$

$$-\lambda_B\left(\frac{1}{k}\right)\left[L^n(A) + G(A,u)\left[1 + \lambda_B\left(\frac{1}{k}\right)\right] + \lambda_B\left(\frac{1}{k}\right)L^n(A) + \lambda_B\left(\frac{1}{k}\right)\varepsilon + \varepsilon\right] -$$

$$-\lambda_B\left(\frac{1}{k}\right)\sum_{j=1}^{\nu_k} L^n(\Omega_j^k) - \lambda_B\left(\frac{1}{k}\right)G(A,u) - \lambda_B\left(\frac{1}{k}\right)\varepsilon - \varepsilon.$$

If $k \to +\infty$ then $\lambda_B(1/k) \to 0$ therefore, being $G(A,u) < +\infty$, $\sum_{j=1}^{\nu_k} L^n(\Omega_j^k) \leq$ $\leq 2L^n(A) < +\infty$, we have

$$(3.61)\qquad G(A,u) \geq \liminf_k \sum_{j=1}^{\nu_k} \bar{I}_-(\Omega_j^k, u) - \varepsilon.$$

Finally Proposition 3.5 and (3.61) imply that

$$G(A,u) \geq \bar{I}_-(A,u) - \varepsilon$$

if $\varepsilon \to 0$ we have (3.51). ■

THEOREM 3.8: *Let Ω be an open set, f be as in* (0.4) *verifying* (0.5), (0.7) *and* (0.8). *Let A be an open set of Ω then*

$$\bar{I}(A,u) = G(A,u)$$

for every $u \in BV_{loc}(A)$.

PROOF: For every $u \in BV_{loc}(\Omega)$ the set functions $G(\cdot,u)$ and $\bar{I}(\cdot,u)$ are increasing and inner regular and by Lemma 3.7 they agree on the family of open sets A such that $A \subset\subset \Omega$, $L^n(\partial A) = 0$. Being this family dense theorem follows. ■

COROLLARY 3.9: *Let Ω be an open set. Let $a: x \in \Omega \mapsto a(x) \in]0, +\infty[$ be continuous and $g: z \in R^n \mapsto g(z) \in [0, +\infty[$ be convex with $g(0) = 0$. Then*

$$\int_\Omega a(x) g(\nabla u)\, dx + \int_\Omega a(x) g^\infty\left(\frac{dD^s u}{d|Du|}\right) d|Du| =$$

$$= \inf\left\{\liminf_h \int_\Omega a(x) g(\nabla u_h)\, dx; (u_h)_h \subseteq C^1(\Omega); u_h \to u \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\}.$$

PROOF: The function $f(x,z) = a(x)g(z)$ verifies (0.4) and (0.5), see Corollary 2.4. If now K is a compact set there exists $x_K \in K$ such that $a(x_K) \leq a(x)\, \forall x \in K$ and therefore that $a(x_K)g(z) \leq a(x)g(z)$ for every $x \in K$ and for every z. Choosing $\lambda = (\max_K a(x) - a(x_K))/a(x_K)$ it is easy to prove that f verifies (0.7) so, by Theorem 3.8 the Corollary follows. ∎

In particular if a is as in Corollary 3.9 and $g(z) = |z|$ then

$$\int_\Omega a(x)|\nabla u|\, dx + \int_\Omega a(x)\left|\frac{dD^s u}{d|Du|}\right| =$$

$$= \inf\left\{\liminf_h \int_\Omega a(x)|\nabla u_h|\, dx; (u_h)_h \subseteq C^1(\Omega); u_h \to u \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\}.$$

COROLLARY 3.10: *Let Ω be an open set, f be as in* (0.4) *verifying* (0.5), (0.7) *and* (0.8). *Then*

$$\Im(\Omega, u) = G(\Omega, u)$$

for every $u \in BV_{\mathrm{loc}}(\Omega)$.

PROOF: The Corollary follows by Proposition 3.6 and Theorem 3.8. ∎

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