Bounded Solutions Relative to a Vibrating and not Tensioned String

Summary. — We examine the problem of the motion of a vibrating and not tensioned string, when some friction is present. In particular, using the model already studied in a recent paper, we investigate the asymptotic behaviour of the solutions of the Cauchy-Dirichlet problem. We prove both the existence of a bounded solution in \( \mathbb{R} \), and the boundedness of all solutions in \( \mathbb{R}^+ \).

Soluzioni limitate relative a una corda vibrante non completamente tesa

Riassunto. — Si considera il problema della corda non completamente tesa, studiando il comportamento asintotico delle soluzioni del problema di Cauchy-Dirichlet, in presenza di attrito. Utilizzando un modello introdotto in un precedente lavoro, si ottengono teoremi di limitatezza per \( t \to + \infty \) per tutte le soluzioni, nonché un teorema di esistenza di una soluzione limitata in tutto \( \mathbb{R} \).

1. - The problem and the model

In a recent paper [1] we studied a possible mathematical model of the motion of a vibrating and not tensioned string with fixed ends (for different models concerning other problems see, for instance, [2], [3], [4], [5], [6], [7], [8]). Following a classical procedure, we substituted the string with a system of \( n \) suitable elements, connected by hinges at the points \( P_j^{(n)} \) sliding traversally and without friction. We started studying a system of \( n \) ordinary differential equations, suggested by this problem. Taking then \( n \to \infty \), the solution of the discrete model approaches, in a suitable sense, a solution of a partial differential equation which we assumed


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as the model of the problem considered above. This procedure allowed us to state an existence theorem for the Cauchy-Dirichlet problem.

The purpose of the present paper is to investigate the asymptotic behaviour of the solutions, when some friction is present.

Taking \( t \to +\infty \), the first aim is to have appropriate assumptions on the external force, so that all solutions are bounded, and, moreover, to find a bound good for all solutions.

Later, considering \( t \in \mathbb{R} \), we look for a bounded solution.

Supposing that the string, with proper length \( A \), has fixed ends at the points \( A(0,0) \) and \( B(L,0) \), with \( L < A \), and that some elastic reaction arises when its length reaches \( A \), we introduce the following notations, where \( n \in \mathbb{N}, \ i = -2, -1, ..., n + 1, \ n + 2, \)

\[
b^{(n)}_i = \frac{L}{n}, \quad n \in \mathbb{N};
\]

\[
A = \{ x: 0 \leq x \leq L \};
\]

\[
A_i^{(n)} = \{ x: (i - 1/2)b^{(n)} \leq x \leq (i + 1/2)b^{(n)} \};
\]

\[
A_1^{(n)} = \{ x: b^{(n)} \leq x \leq (i + 1)b^{(n)} \};
\]

\[
T_n = \{ t: -n \leq t \leq n + 1 \};
\]

\[
R(t) = \{ t: t \leq t < +\infty \};
\]

\[
Q_{\tau} = \{(x, t, \eta): x \in A, \eta \in T_0 \}, \quad t \in \mathbb{R};
\]

\[
Q = \{(x, t): x \in A, t \in \mathbb{R} \};
\]

\[
\{f_i^{(n)}\} = \{f_n^{(n)}, f_{n-1}^{(n)}, ..., f_{n+1}^{(n)}, f_{n+2}^{(n)} \};
\]

\[
\delta f_i^{(n)} = f_{i+1}^{(n)} - f_i^{(n)};
\]

\[
S = \{f_i^{(n)}\}: f_n^{(n)} = -f_{n-1}^{(n)}, f_{n-1}^{(n)} = -f_{n-2}^{(n)}, f_{n-2}^{(n)} = -f_{n-3}^{(n)}, f_{n-3}^{(n)} = -f_{n-4}^{(n)}, f_{n-4}^{(n)} = f_n^{(n)} = 0 \};
\]

\[
L^p_{\text{loc}}(T) = \left\{ f(t): \sup_{t \in T} \left( \int_0^\infty |f(t + \eta)|^p \, d\eta \right)^{1/p} < +\infty \right\};
\]

obviously, for the sake of simplicity, we shall drop out the \((n)\) when no confusion will be possible; moreover, if necessary, we shall extend functions defined on \( A \) or \( Q \) as follows:

\[
f(-x,t) = -f(x,t), \quad f(2L-x,t) = -f(x,t), \quad (x,t) \in \bigcup_{i \in \mathbb{Z}} A_i^{(n)} \times \mathbb{R}.
\]

The function which defines the stress-strain relationship obtained when considering
the discrete model introduced above is

\[
\Phi(\alpha) = \begin{cases} 
L \left( \sqrt{1 + \alpha^2} - \frac{A}{L} \right) \frac{\alpha}{\sqrt{1 + \alpha^2}}, & |\alpha| \geq \sqrt{(A/L)^2 - 1} = \tilde{a}, \\
0, & |\alpha| < \sqrt{(A/L)^2 - 1} = \tilde{a}, \end{cases}
\]

in the classical case, when the proper length of the string is \( A < L \), setting \( \Phi(\alpha) = \alpha \), we have the Hooke's law.

Moreover the physical problem suggest to examine two families of discrete Cauchy–Dirichlet problems (depending on \( h \)): the first one for \( t \geq 0, 0 < \mu < 2 \),

\[
\left\{ \begin{array}{l}
z_i^\mu - \frac{\delta}{b} \varphi \left( \frac{\delta}{b} z_i \right) + h^\mu \frac{\delta^4}{b^4} z_{i-2} + z'_i(t) = f_i(t), \quad i = 1, 2, \ldots, n - 1, \\
z_0(t) = z_n(t) = 0, \quad z_{-1}(t) = -z_1(t), \quad z_{-2}(t) = -z_2(t), \\
z_{n-1}(t) = -z_{n+1}(t), \quad z_{n-2}(t) = -z_{n+2}(t), 
\end{array} \right.
\]

\[
z_i(0) = \tilde{z}_i, \quad z'_i(0) = \tilde{z}'_i;
\]

this problem allows us to study the behaviour of the solutions for \( t \to +\infty \).

In fact, for suitable \( \{\tilde{z}^{(n)}\} \) and \( \{\tilde{z}'^{(n)}\} \), taking \( n \to +\infty \), the polygonal \( \tilde{\gamma}^{(n)}(x, t) \), connecting the points \( P_i^{(n)}(t) = \{\tilde{u}_i^{(n)}(t), \tilde{z}_i^{(n)}(t)\} \), approaches a solution of the following variational problem

\[
\int_0^t \left\{ \left( y'(\eta), u'(\eta) \right)_{L^2} - \left( \varphi(Dy(\eta)), Du(\eta) \right)_{L^2} - \left( y'(\eta), u(\eta) \right)_{L^2} + \\
+ \left( f(\eta), u(\eta) \right)_{L^2} \right\} \, d\eta + \left( \tilde{y}'(0), u(0) \right)_{L^2} - \left( y'(t), u(t) \right)_{L^2} = 0, \quad t \in \mathbb{R}^+ \text{ a.e.},
\]

\[
y(0) = \tilde{\gamma}.
\]

In this case, studying the asymptotic behaviour for \( t \to +\infty \), we want a family of solutions with similar properties, therefore we examine the family of solutions of (1.4) (1.5) which we can approach with a sequence of polygonals \( \tilde{\gamma}^{(n)}(x, t) \). Then, bearing in mind the boundary conditions, we shall say that \( y(x, t) \) is a solution of the problem above considered if

i) \( y(t) \in L^2 (\mathbb{R}^+; H^1_0) \cap H^1 (\mathbb{R}^+; L^2) \);

ii) (1.4) holds \( \forall u \in L^2 (\mathbb{R}^2; H^1_0) \cap H^1 (\mathbb{R}^+; L^2) \).

iii) \( \forall T \in \mathbb{R}^+ \) there exists a sequence of polygonals \( \tilde{\gamma}^{(n)}(x, t) \) approaching \( y(x, t) \) weakly in \( L^2 (0, T; H^1_0) \cap H^1 (0, T; L^2) \).
The second discrete problem, where we set $t \geq -n$ changing initial conditions (1.3) into

\begin{equation}
(1.6) \quad z_i(-n) = 0, \quad z'_i(-n) = 0,
\end{equation}

allows us to study the behaviour of solutions on $\mathbb{R}$. In fact, taking $n \to +\infty$, the functions $\beta^{(n)}(x,z)$ approach a solution of the variational problem

\begin{equation}
(1.7) \quad \int_{-n}^{n} \left\{ (y'(\eta), u'(\eta))_{L^2} - (\varphi(Dy(\eta)), Du(\eta))_{L^2} +
\right.
\left.
- (y'(\eta), u(\eta))_{L^2} + (f(\eta), u(\eta))_{L^2} \right\} d\eta +
\left\{ (y'(t'), u(t'))_{L^2} - (y'(t''), u(t''))_{L^2} = 0, \quad t', t'' \in \mathbb{R}^+ \text{ a.e.} \right. \right.
\end{equation}

Now, since we are looking for a bounded solution on $\mathbb{R}$, we say that $y(x,t)$ is a solution of the problem if

i) $y(t) \in L^2_{\text{loc}}(\mathbb{R}; H^1_0) \cap H^1_{\text{loc}}(\mathbb{R}; L^2)$;

ii) (1.7) holds $\forall u \in L^2_{\text{loc}}(\mathbb{R}; H^1_0) \cap H^1_{\text{loc}}(\mathbb{R}; L^2)$.

The equations (1.4) and (1.7), similar to the equation studied in [1], have a linear dissipative term which takes into account the friction.

Finally setting

\begin{equation}
(1.8) \quad \left\{ \begin{array}{ll}
\Phi(\alpha) = \int_0^{\alpha} \varphi(\beta) d\beta = \left\{ \frac{L}{2} \left[ \sqrt{1 + \alpha^2 - \frac{A}{L}} \right]^2, \quad |\alpha| \geq \bar{\alpha},
\right. \\
0, \quad |\alpha| < \bar{\alpha},
\end{array} \right.
\end{equation}

\begin{equation}
\begin{array}{l}
E^{(n)}(z_i(t)) = \sum_{i=1}^{n} \left\{ \frac{1}{2} z_i'^2(t) + \Phi \left( \frac{\delta}{b} z_{i-1}(t) \right) + h \frac{\delta^2}{b^2} z_i z_{i-2} \right\},
\end{array}
\end{equation}

\begin{equation}
\begin{array}{l}
E(z_i(t)) = \sum_{i=1}^{n} \left\{ \frac{1}{2} z_i'^2(t) + \Phi \left( \frac{\delta}{b} z_{i-1}(t) \right) \right\},
\end{array}
\end{equation}

\begin{equation}
\begin{array}{l}
E(y(t)) = \frac{1}{2} \|y'(t)\|_{L^2}^2 + \|\Phi(Dy(t))\|_{L^2},
\end{array}
\end{equation}

we observe that $E(z_i(t))$ and $E(y(t))$ have an obvious physical meaning.

In fact, in the classical case, where $\varphi(\alpha) = \alpha$, the quantities

\begin{equation}
(1.9) \quad \left\{ \begin{array}{l}
E(z_i(t)) = \frac{1}{2} \sum_{i=1}^{n} \left\{ z_i'^2(t) + \left( \frac{\delta}{b} z_i(t) \right)^2 \right\},
\end{array} \right.
\end{equation}

give the «energy» of the system or of the string respectively, the unique rest position...
corresponds to the null solution, and we can assume (1.9) as a square of the norm in the «energy» space.

By analogy with the classical case we shall say that the functions \( E^{(a)}(z_t) \) and \( E(y) \) give the energy of the solutions of (1.2), (1.7) respectively. Nevertheless neither \( E^{(a)}(z_t) \), nor \( E(y) \) are any norms, in fact we have infinite rest positions corresponding to solutions not equal to zero.

In \( \S 2 \) we give the statements on the continuous problem; in \( \S 3 \) we give the statements on the discrete problem useful for the study of the continuous problem; in \( \S 4 \) and \( \S 5 \) we give the proofs of the statements of \( \S 2 \) and \( \S 3 \) respectively.

Finally we recall in \( \S 6 \) some classical results and useful formulas.

2. - THE CONTINUOUS PROBLEM

Supposing \( f(t) \in L^2_{\text{loc}}(R^+, L^2) \), we set

\[
\|f\|_{L^2_{\text{loc}}(R^+, L^2)} = \sup_{t \in R^+} \left( \int_0^1 \int_{x + \eta} f^2(x, t + \eta) \, dx \, d\eta \right)^{1/2} < \infty.
\]

Now we can state the following theorems.

**Theorem 2.1 (Boundedness of solutions):** Let

(2.1) \( \|f\|_{L^2_{\text{loc}}(R^+, L^2)} = K < \infty \),

(2.2) \( \bar{y} \in H^1_0, \quad \bar{y}' \in L^2 \).

Then there exists at least a solution \( y = y(t) \) in the sense indicated above (see \( \S 1 \)) of the Cauchy-Dirichlet problem (1.4), (1.5).

Moreover there exists a positive constant \( M \) (depending on \( K, \bar{y}, \bar{y}' \)) such that

(2.3) \( E(y(t)) \leq M, \quad \|Dy(t)\|_{L^2} \leq M, \quad \|y'(t)\|_{L^2} \leq M, \quad t \in R^+, \text{ a.e.} \).

**Theorem 2.2 (Asymptotic behaviour):** Let (2.2) hold and there exists \( \beta > 1/2 \) such that \( t^{\beta}f(t) \in L^2(R^+, L^2) \), with

(2.4) \( \|t^{\beta}f(t)\|_{L^2(R^+, L^2)} = K \).

Then

(2.5) \( E(y(t)) \in L^1(R^+, L^2) \).

(2.5) implies that \( \forall \varepsilon > 0 \) there exists a positive constant \( t_0 \) depending on \( \varepsilon, \bar{y}, \bar{y}' \), such that

(2.6) \( \|E(y(t))\|_{L^2_{\text{loc}}(R^+)} \leq \varepsilon, \quad \|y'(t)\|_{L^2_{\text{loc}}(R^+)} \leq \varepsilon, \)

(2.7) \( \|Dy(t)\|_{L^2_{\text{loc}}(R^+)} \leq \varepsilon + \varepsilon^2. \)
**Theorem 2.3** (Boundeness in \( \mathbb{R} \)): Let

\[
\|f\|_{L^2_{\eta}((\mathbb{R}_t, L^2))} = \sup_{t \in \mathbb{R}} \left\{ \int_{\mathbb{R}_t} \int_{-\infty}^{\infty} f^2(x, t + \eta) \, dx \right\}^{1/2} = K < + \infty.
\]

Then there exists at least a solution of (1.7) bounded in \( \mathbb{R} \) (in the sense indicated in § 1).

3. - The discrete problem

The first result that we state is the following lemma concerning a Cauchy-Dirichlet problem for the system (1.2)

\[
\begin{align*}
(3.1) \quad & z_i^* - \frac{\delta}{b} \psi \left( \frac{\delta}{b} z_{i-1} \right) + b^n \frac{\delta^4}{b^2} z_{i-2} + z_i^* - f_i(t) = 0, \quad i = 1, 2, \ldots, n - 1, \quad t \leq r, \\
(3.2) \quad & z_i(t) = \tilde{z}_i, \quad z_i'(t) = \tilde{z}_i'.
\end{align*}
\]

**Lemma 3.1:** Assume that there exists a constant \( C > 0 \) such that

\[
\sum_{i=1}^{n} \|f_i(t)\|_{L^2_{\eta}((R^n_+, R^2))} = \sup_{t \in R^n_+} \left\{ \sum_{i=1}^{n} \int_{R^n_+} \left( f_i(t + \eta) \right)^2 \, d\eta \right\} \leq \frac{C}{b}.
\]

Then there exists a constant \( C_1 > 0 \) such that

\[
E^{(n)}(z^{(n)}(i + 1)) \leq \sup \left\{ E^{(n)}(z^{(n)}(i)), \frac{C_1}{b} \right\}.
\]

We can then state the following Theorems.

**Theorem 3.2:** Assume that (3.3) holds and moreover

\[
\sum_{i=1}^{n} |z_i|^2 \leq \frac{C_1}{b}, \quad \sum_{i=1}^{n} |z_i'|^2 \leq \frac{C_1}{b}, \quad \sum_{i=1}^{n} \left( \frac{\delta}{b} z_i \right)^2 \leq \frac{C_1}{b}, \quad \sum_{i=1}^{n} \left( \frac{\delta^2}{b^2} z_i \right)^2 \leq \frac{C_1}{b} - \mu^2.
\]

Then the solution \( \{z_i^{(n)}(t)\} \in S \) of (3.1), (3.2), which is unique, is bounded in \( R^n_+ \). Precisely there exists a constant \( C_1 \), depending on \( C \), such that

\[
\sup_{t \in R^n_+} E^{(n)}(z^{(n)}(t)) \leq \frac{C_1}{b}.
\]

Obviously we have

\[
\sum_{i=1}^{n} |z_i|^2 (t) \leq \frac{C_1}{b}, \quad \sum_{i=1}^{n} b^n \left( \frac{\delta^2}{b^2} z_{i-2}(t) \right)^2 \leq \frac{C_1}{b}.
\]
and, by (6.6), there exists a constant $C_2$ such that

$$
\sum_{i=1}^{n} z_i^2(t) \leq \frac{C_2}{b}, \quad \sum_{i=1}^{n} \left( \frac{\partial}{\partial t} z_{i-1}(t) \right)^2 \leq \frac{C_2}{b}.
$$

**THEOREM 3.3:** Assume that (3.5) holds and there exists $\beta > 1/2$ such that

$$
\sum_{i=1}^{n} \int \eta^{2\beta} f_i^2(\eta) \, d\eta \leq \frac{C_2}{b}.
$$

Then the «Energy» of the solution of (3.1), (3.2) $E^{(n)}(z_i^{(n)}(t))$ vanishes for $t \to +\infty$. Moreover there exists a constant $C_3$, depending on $C$, such that

$$
\int_0^\infty E^{(n)}(z_i^{(n)}(t)) \, dt \leq \frac{C_3}{b}.
$$

Obviously (3.10) implies that

$$
\int_0^\infty \left\{ \sum_{i=1}^{n} z_i^2(t) \right\} \, dt \leq \frac{C_3}{b};
$$

moreover, for $\epsilon > 0$ there exists a constant $t_\epsilon$ (depending only on $\epsilon$) such that

$$
\sup_{t > t_\epsilon} \int_t^{t+\epsilon} \left\{ \sum_{i=1}^{n} \left( \frac{\partial}{\partial t} z_{i-1}(\eta) \right)^2 \right\} \, d\eta \leq \frac{1}{b} (\alpha^2 + \epsilon).
$$

**COROLLARY 3.4:** Let (3.3) hold and

$$
\bar{z}_i = \bar{z}_i^{(n)} = 0; \quad f_i(t) = 0, \quad \text{for } t < \tau.
$$

Then the solution of (3.1), (3.2) is bounded in $R$. Precisely there exists a constant $C_4$, depending only on $C$, such that

$$
\sup_{t \in R} E^{(n)}(z_i^{(n)}(t)) \leq \frac{C_4}{b}.
$$

4. - PROOF OF THE STATEMENTS OF § 2

We can prove the theorems of § 2 using the results concerning the discrete problem considered in § 1. In order to do it, we fix the following connections between the continuous and the discrete problems.

When we consider the problem (1.2), (1.3) firstly we choose a sequence $\{ \alpha^{(p)}(x) \}$ with $\alpha^{(p)} \in H^2 \cap H_b^1$, such that

$$
\lim \| \alpha^{(p)} - \bar{y} \|_{H_b^1} = 0.
$$
Then, if
\[ \| \alpha^{(p)} \|_{H^2 \cap H_0^1} = a^{(p)}, \]
and \( \{ n_p \} \subseteq \mathbb{N} \) is a sequence such that \( a^{(p)} \leq n_p \), we construct a new sequence \( \tilde{z}^{(n)}(x) \) setting successively, for example
\[ \tilde{z}^{(1)}(x) = \tilde{z}^{(2)}(x) = \ldots = \tilde{z}^{(n_1 - 1)}(x) = \frac{1}{a^{(1)}} \alpha^{(1)}(x), \]
\[ \tilde{z}^{(n_1)}(x) = \tilde{z}^{(n_1 + 1)}(x) = \ldots = \tilde{z}^{(n_1 - 1)}(x) = \alpha^{(1)}(x), \]
\[ \tilde{z}^{(n_p)}(x) = \tilde{z}^{(n_p + 1)}(x) = \ldots = \tilde{z}^{(n_p - 1)}(x) = \alpha^{(p)}(x), \quad p \in \mathbb{N}. \]
In such a way we have, \( \forall n_p + 1 \leq n \leq n_p + 1 \)
\[ \| \tilde{z}^{(n)} \|_{H^2 \cap H_0^1} \leq \| \alpha^{(p)} \|_{H^2 \cap H_0^1} \leq (n_p)^{\mu_4} \leq (n)^{\mu_4} = L^\mu_4 b^{n_3}. \]

Finally we set in (1.2)
\[ f_i^{(n)}(t) = \frac{1}{b^{(n)}} \int f(x, t) \, dx, \quad t \in \mathbb{R}^+, \]
and in (1.3)
\[ \tilde{z}_i^{(n)} = \frac{1}{b^{(n)}} \int \tilde{z}_i^{(n)}(x) \, dx, \quad \tilde{y}_i^{(n)} = \tilde{y}_i^{(n)} \int \tilde{y}_i^{(n)}(x) \, dx. \]

Considering the second problem we set
\[ f_i(t) = \begin{cases} \frac{1}{b^{(n)}} \int f(x, t) \, dx, & t \geq -n, \\ 0, & t < -n. \end{cases} \]

**Proof of Theorem 2.1:** As in [1] we divide the proof in the following steps:

a) Setting
\[ \delta^{(n)}(x, t) = z_i(t) + \frac{\delta}{b} z_i(t)(x - i\delta), \quad x \in \mathcal{D}_i^{(n)}, \]
it is possible to select a subsequence \( \{ \delta^{n,1}(t) \} \) such that
\[ \lim_{n \to \infty} \delta^{n,1}(t) = \gamma_1(t). \]
strongly in $L^2(\mathbb{T}_1; L^2)$, weakly* in $L^\infty(\mathbb{T}_1; H^1_0)$,

\begin{equation}
\lim_{a \to \infty} \delta^{(a,1)}(t) = y'_1(t)
\end{equation}

weakly* in $L^\infty(\mathbb{T}_1; L^2)$.

We have (4.7) and (4.8) directly from (4.6) and Theorem 3.2.

In fact (2.1) implies (3.3); (2.2), (4.2) and (4.4) imply (3.5), then by Theorem 3.2 (that is (3.7) and (3.8)) there exists a constant $K_1$, depending only on $\mathcal{C}$ (that is on $K, \bar{y}, \tilde{y}'$), such that

\begin{equation}
\|\delta^{(a)}(t)\|_{L^\infty(\mathbb{R}^+, H^1_0)} \leq K_1, \quad \|\delta^{(a,1)}(t)\|_{L^\infty(\mathbb{R}^+, L^2)} \leq K_1,
\end{equation}

b) We can extend $y_1(t)$ to $\mathbb{R}^+$.

In fact we can consider the intervals $(0, k), \forall k \in \mathbb{N}$, and choose the successive subsequences $\{\delta^{(a,1)}(t)\} \subseteq \{\delta^{(a, k-1)}(t)\}$ such that

\begin{equation}
\lim_{a \to \infty} \delta^{(a, k)}(t) = y_k(t)
\end{equation}

strongly in $L^2(0, k; L^2)$, weakly* in $L^\infty(0, k; H^1_0)$,

\begin{equation}
\lim_{a \to \infty} \delta^{(a, k)}(t) = y'_k(t)
\end{equation}

weakly* in $L^\infty(0, k; L^2)$. Moreover $y_k(t) = y_{k-1}(t)$ for $0 \leq t \leq k - 1$, and we can set

\[ y(t) = y_k(t), \quad t \in (0, k). \]

c) The function $y(t)$ satisfies (2.3).

In fact, $\forall t \in \mathbb{R}^+$, we can choose $k > t$ and considering the subsequence $\{\delta^{(a, k)}(t)\}$ we have (2.3) by (4.9), (4.10) and (4.11).

d) $y(t)$ is a solution of (1.4), (1.5) in the sense indicated in §1.

Dropping out the $k$ for the sake of simplicity, (1.5) holds by (4.1) and (4.4), since

\[ \delta^{(n)}(0) - \bar{y} = \delta^{(n)}(0) - \tilde{z}^{(n)} + \tilde{z}^{(n)} - \bar{y}. \]

Now we recall how to verify (1.4), for the reader’s convenience.
Assuming at first that \( u(t) \in H^{1, \infty}(R^+; L^2) \cap L^{\infty}(R^+; H^2 \cap H_0^1) \), we calculate

\[
\int_0^t \{(\dot{\xi}(\eta), u'(\eta))_{L^2} - (\varphi(D\dot{\xi}(\eta)), Du(\eta))_{L^2} - (\dot{\xi}(\eta), u(\eta))_{L^2} +
+(f(\eta), u(\eta))_{L^2}\} d\eta - (\dot{\xi}(t), u(t))_{L^2} + (\dot{\xi}(0), u(0))_{L^2}
\]

and pass to the limit for \( t \to +\infty \).

Setting

\[
g_i = \frac{\lambda}{b} \left\{ \frac{1}{b} \int_0^\xi \left( \int_0^x u_0(\xi, t) d\xi \right) \right\} \in H^1(0, k),
\]

and

\[
\mathcal{C}(t) = \frac{\lambda}{b} g_i(t), \quad \mathcal{G}(t) = g_i(t), \quad x \in \mathcal{A},
\]

we have very easily

\[
\lim_{n \to \infty} \|\mathcal{C}(t) - Du(t)\|_{L^{\infty}(0, k; L^2)} = 0,
\]

\[
\lim_{n \to \infty} \|\mathcal{G}(t) - u(t)\|_{L^{\infty}(0, k; L^2)} = 0,
\]

and there exists a constant \( G > 0 \) (depending only on \( u \)) such that

(4.12)

\[
\sum_{i=1}^k \left( \frac{\lambda^2}{b^2} g_i \right)^2 \leq \frac{G}{b}.
\]

Moreover

\[
\int_0^t (\dot{\xi}(\eta), u'(|\eta))_{L^2} d\eta = b \int_0^t \sum_{i=1}^k \xi_i(\eta) g_i(\eta) d\eta,
\]

and then, by (6.3),

\[
\int_0^t (\dot{\xi}(\eta), u'(|\eta))_{L^2} d\eta = -b \int_0^t \sum_{i=1}^k \varphi \left( \frac{\lambda}{b} z_{i-1}(\eta) \right) \frac{\lambda}{b} g_{i-1}(\eta) +
+b^2 \frac{\lambda^2}{b^2} z_{i-2}(\eta) \frac{\lambda^2}{b^2} g_{i-2}(\eta) + z_i(\eta) g_i(\eta) + -f_i(\eta) g_i(\eta)
\]

\[
+ b \sum_{i=0}^k z_i' (t) g_i (t) - b \sum_{i=0}^k z_i' (0) g_i (0),
\]
where, by (3.7) and (4.12)

\begin{equation}
(4.13) \quad b^{1+\mu} \int_0^t \left( \sum_{i=1}^n \frac{\delta^2}{b^2} z_{i-2}(\eta) \frac{\delta^2}{b^2} \xi_{i-2}(\eta) \right) d\eta \leq \leq b^{1+\mu} \left( \int_0^t \sum_{i=1}^n b^n \left( \frac{\delta^2}{b^2} z_{i-2}(\eta) \right)^2 d\eta \right)^{1/2} \left( \int_0^t \sum_{i=1}^n \left( \frac{\delta^2}{b^2} \xi_{i-2}(\eta) \right)^2 d\eta \right)^{1/2} \leq \varepsilon b^{n-2} \varepsilon C
\end{equation}

and moreover

\begin{equation}
(4.14) \quad b \int_0^t \sum_{i=1}^n f_i(\eta) g_i(\eta) d\eta \rightarrow \int_0^t (f(\eta), u(\eta))_{L^2} d\eta,
\end{equation}

\begin{equation}
(4.15) \quad b \sum_{i=1}^n z_i(t) g_i(t) - b \sum_{i=1}^n z_i(0) g_i(0) \rightarrow (y(t), u(t))_{L^2} - (y(0), u(0))_{L^2}.
\end{equation}

Finally, in order to calculate

\begin{equation}
\lim_{b \to 0} b \int_0^t \left\{ \sum_{i=1}^n \beta \left( \frac{\delta}{b} z_{i-1}(\eta) \right) \frac{\delta}{b} \xi_{i-1}(\eta) \right\} d\eta,
\end{equation}

we consider a new sequence \( \{ \eta^{(n)} \} \) defined as follows

\begin{equation}
\eta^{(n)}(x) = \frac{\delta}{b} z_{i-1}^{(n)} + \frac{\delta^2}{b^2} z_{i-1}^{(n)} \left[ x - \left( i - \frac{1}{2} \right) b \right], \quad x \in \Delta_i^{(n)},
\end{equation}

\begin{equation}
\zeta^{(n)}(x) = \int_{-L/2}^x \eta^{(n)}(\xi) d\xi - \frac{L - x}{L} \int_{-L/2}^0 \eta(\xi) d\xi - \frac{x}{L} \int_{-L/2}^L \eta(\xi) d\xi = \frac{1}{2} (z_i + z_{i-1}) + \frac{\delta}{b} z_{i-1} \left[ x - \left( i - \frac{1}{2} \right) b \right] + \frac{\delta^2}{b^2} z_{i-1} \left[ x - \left( i - \frac{1}{2} \right) b \right]^2, \quad x \in \Delta_i^{(n)},
\end{equation}

which is more regular than \( \{ \eta^{(n)} \} \).

We can verify that

\begin{equation}
\| \zeta^{(n)}(t) \|_{L^2} \leq C_k, \quad \| \zeta^{(n)}(t) \|_{H^1} \leq C_k, \quad t \in [0, k],
\end{equation}

\begin{equation}
\| \zeta^{(n)} \|_{H^1([0,k] \cap H^2)} \leq C_k, \quad \| \zeta^{(n)}(t) \|_{H^2 \cap H^1} \leq b^{n-2} C_k, \quad t \in [0, k],
\end{equation}

and moreover

\begin{equation}
(4.16) \quad \| D \zeta^{(n)}(t) - D \delta^{(n)}(t) \|_{L^2} \leq b^{2-n} C.
\end{equation}
Finally proving as in [1] that
\[ \lim_{n \to \infty} \varphi(D\mathcal{D}^{(n)}) = \varphi(Dy) \]
weakly in \( L^2(0, k; L^2) \), by (4.16) we have
\[ \lim_{n \to \infty} \varphi(D\mathcal{D}^{(n)}) = \varphi(Dy), \]
weakly in \( L^2(0, k; L^2) \). Now bearing in mind (4.13), (4.14), (4.15), we complete the proof, by (4.17).

**Proof of Theorem 2.2.** Let \( k \in \mathbb{N}, \sigma > 0 \) and \( y \in H_{\mathcal{D}}^k(0, k; L^2) \cap L^2(0, k; H_{\mathcal{D}}^k) \); we set
\[ \|y(t)\|_{H^k}^2 = E(y(t)) + \sigma \|Dy(t)\|_{L^2}^2. \]

By (4.10) and (4.11) (dropping out the \( k \)), we have, \( \forall n \in \mathbb{N} \),
\[ \int_0^k \|y(t)\|_{H^k}^2 dt = \int_0^k \{E(y(t)) + \sigma \|Dy(t)\|_{L^2}^2 \} dt \leq \]
\[ \leq \int_0^k \min \lim_{n \to \infty} \|z^{(n)}(t)\|_{H^k}^2 dt \leq \int_0^k \min \lim_{n \to \infty} \{E(z^{(n)}(t)) + \sigma \|Dz^{(n)}(t)\|_{L^2}^2 \} dt \leq \]
\[ \leq C_{10} b \int_0^k \min \lim_{n \to \infty} E^{(n)}(z^{(n)}(t)) dt + \sigma \min \lim_{n \to \infty} \|Dz^{(n)}(t)\|_{L^2}^2 \]
Now (2.4) implies (3.5), then (3.10) and Theorem 3.3 give
\[ \int_0^k \|y(t)\|_{H^k}^2 dt \leq C_{11} + \sigma C_{12} \]
and therefore
\[ (4.18) \quad \int_0^k E(y(t)) dt = \int_0^k \|y(t)\|_{H^k}^2 dt - \int_0^k \sigma \|y(t)\|_{H^D}^2 dt \leq C_{11} + \sigma C_{12}, \quad \forall \sigma > 0. \]

Since (4.18) holds \( \forall \sigma > 0 \) we have
\[ \int_0^k E(y(t)) dt \leq C_{11}, \]
where \( C_{11} \) does not depend on \( k \), and \( E(y(t)) \geq 0 \), that is (2.5), and obviously (2.6).
order to have (2.7), we can choose, by (2.6), \( t_\epsilon \) so that

\[
\|E(y(t))\|_{L_2^2(R_\epsilon^2)} < \frac{L^2 c_2}{16(A^2 + L^2 \epsilon)} \epsilon
\]

and then by (6.6)

\[
\|Dy(t)\|_{L_2^2(R_\epsilon^2)} < \bar{\alpha}^2 + \frac{\epsilon}{2} + \frac{A^2 + L^2}{2L^2 \epsilon} \|\Phi(Dy(t))\|_{L_2^2(R_\epsilon^2)} \leq \bar{\alpha}^2 + \epsilon.
\]

**Proof of Theorem 2.3:** Consider the sequence (4.6), where now \( \{z_\epsilon(t)\} \) is the solution of (1.2), (1.6).

1. **It is possible to select a subsequence \( \{\delta^{(n,1)}_\epsilon(t)\} \) such that**

\[
\lim_{n \to \infty} \delta^{(n,1)}_\epsilon(t) = y_1(t)
\]

strongly in \( L^2(T_1; L^2) \), weakly* in \( L^\infty(T_1, H_0^1) \),

\[
\lim_{n \to \infty} \delta^{(n,1)}_{\epsilon}(t) = y_1'(t)
\]

weakly* in \( L^\infty(T_1; L^2) \).

We have (4.19) and (4.20) directly from (4.6) and Theorem 3.2.

In fact by (3.7) and (3.8) there exists a constant \( K_1 \), depending only on \( C_1 \), such that

\[
\|\delta^{(n)}(t)\|_{L_2^2(R, H_0^1)} \leq K_1, \quad \|\delta^{(n)}_{\epsilon}(t)\|_{L_2^2(R, L^2)} \leq K_1.
\]

2. **We can extend \( y_1(t) \) to \( \mathbb{R} \), as in the step b) of Theorem 2.2, setting**

\[
\lim_{n \to \infty} \delta^{(n, k)}(t) = y_\kappa(t)
\]

strongly in \( L^2(T_k; L^2) \), weakly* in \( L^\infty(T_k, H_0^1) \),

\[
\lim_{n \to \infty} \delta^{(n, k)}_{\epsilon}(t) = y_\kappa'(t)
\]

weakly* in \( L^\infty(T_k; L^2) \), and finally

\[
y(t) = y_\kappa(t), \quad t \in T.
\]

3. **The function \( y(t) \) satisfies (2.3).**

In fact we observe that \( \forall \tilde{t} \in \mathbb{R} \) we can choose \( k \in \mathbb{N} \) such that \( \tilde{t} + 1 \in T_k \) and we substitute \( T_k \) to the interval \((0, k)\) in the step c) of the proof of Theorem 3.1.

4. **\( y(t) \) is a solution of (1.7) in the sense indicated in \S 1.**

We reach this purpose as in the proof of Theorem 3.1.
Precisely, we observe first that (1.7) holds $\forall u \in H^{1,\infty}(R; L^2) \cap L^\infty(R; H^2 \cap H^1_0)$ with compact support in $R$.

In fact for such test function, $k$ large enough, we have

$$\int_R \{ (\delta^{(\alpha, k)}(\eta), u'(\eta))_{L^2} - (\varphi(D\delta^{(\alpha, k)}(\eta)), Du(\eta))_{L^2} +$$

$$-(\delta^{(\alpha, k)}(\eta), u(\eta))_{L^2} + (f(\eta), u(\eta))_{L^2} \} \, d\eta =$$

$$= \int_{I_\delta} \{ (\delta^{(\alpha, k)}(\eta), u'(\eta))_{L^2} - (\varphi(D\delta^{(\alpha, k)}(\eta)), Du(\eta))_{L^2} +$$

$$-(\delta^{(\alpha, k)}(\eta), u(\eta))_{L^2} + (f(\eta), u(\eta))_{L^2} \} \, d\eta,$$

$$\int_R \{ (y'(\eta), u'(\eta))_{L^2} - (\varphi(Dy(\eta)), Du(\eta))_{L^2} - (y' \eta), u(\eta))_{L^2} + (f(\eta), u(\eta))_{L^2} \} \, d\eta =$$

$$= \int_{I_\delta} \{ (y'(\eta), u'(\eta))_{L^2} - (\varphi(Dy(\eta)), Du(\eta))_{L^2} - (y' \eta), u(\eta))_{L^2} + (f(\eta), u(\eta))_{L^2} \} \, d\eta,$$

therefore we can pass to the limit for $n \to \infty$, exactly as in the proof of Theorem 2.2.

5. PROOF OF THE STATEMENTS OF § 3

PROOF OF LEMMA 3.1: In what follows, we shall drop out the notation $(n)$ for the sake of simplicity.

If

$$\sum_{i=1}^{i+1} \int_{I_\delta} z_i^2(\eta) \, d\eta \geq \frac{c}{b},$$

we have (3.4) by Schwartz inequality, setting in (6.4) $\tau = \tilde{\tau} + 1$, $\tau' = \tilde{\tau}$; in fact,

$$0 \leq E(z_i(\tilde{\tau} + 1)) = E(z_i(\tilde{\tau})) + \sum_{i=1}^{i+1} \int_{I_\delta} [f_i(\eta) z_i'(\eta) - z_i^2(\eta)] \, d\eta.$$

Let's now assume that

(5.1) $$\sum_{i=1}^{i+1} \int_{I_\delta} z_i^2(\eta) \, d\eta < \frac{c}{b}.$$
Then there exist \( t' \in [\tilde{t}, \tilde{t} + 1/4], t'' \in [t + 3/4, t + 1] \) such that

\[
\sum_{i=1}^{n} z_i^{(2)}(t') < 4 \frac{c}{b}, \quad \sum_{i=1}^{n} z_i^{(2)}(t'') < 4 \frac{c}{b},
\]

moreover, \( \forall t \in [\tilde{t}, \tilde{t} + 1] \), we have successively

\[
\sum_{i=1}^{n} z_i^2(t) \leq 2 \sum_{i=1}^{n} z_i^2(\tilde{u}) + 2 \sum_{i=1}^{n} \left( \int_{\tilde{u}}^{t} z_i' (\eta) d\eta \right) \leq \nonumber
\]

\[
\leq 2 \left( \sum_{i=1}^{n} z_i^2(\tilde{u}) + \sum_{i=1}^{n} \int_{\tilde{u}}^{t} z_i^{(2)}(\eta) d\eta \right) \leq 2 \left( \sum_{i=1}^{n} z_i^2(\tilde{u}) \frac{c}{b} \right)^{1/2},
\]

\[5.2\]

\[
\left( \sum_{i=1}^{n} z_i^2(t) \right)^{1/2} \leq \sqrt{2} \left( \sum_{i=1}^{n} z_i^2(\tilde{u}) \frac{c}{b} \right)^{1/2}.
\]

Setting now

\[
g^{(n)}(z_i) = \frac{1}{2} \sum_{i=1}^{n} \left\{ \frac{1}{2} \left( \frac{\delta^2}{b^2} z_{i-1}(t) \right)^2 + \frac{c}{b} \left( \frac{\delta}{b} z_{i-1}(t) \right) \frac{\delta}{b} z_{i-1}(t) \right\}
\]

and \( u_i = z_i \) in (6.3), we have by (3.3), (5.1) and (5.2)

\[5.3\]

\[
\int_{t}^{t'} g^{(n)}(z_i(\eta)) d\eta = \sum_{i=1}^{n} \left( \int_{t}^{t'} z_i^{(2)}(\eta) + f_i(\eta) z_i(\eta) + \frac{c}{b} z_i(\eta) \right) d\eta - \nonumber
\]

\[
- \sum_{i=1}^{n} z_i'(t') z_i(t') \right) + \sum_{i=1}^{n} z_i'(t') z_i(t') \leq \frac{c}{b} + \left( \frac{c}{b} \right)^{1/2} \left( \sum_{i=1}^{n} z_i^2(\tilde{u}) \right)^{1/2} + \nonumber
\]

\[
+ 12 \left( \frac{c}{b} \right)^{1/2} \sqrt{2} \left( \sum_{i=1}^{n} z_i^2(\tilde{u}) \frac{c}{b} \right)^{1/2} \leq c_0 \left[ \frac{1}{b} + \left( \frac{1}{b} \right)^{1/2} \left( \sum_{i=1}^{n} z_i^2(\tilde{u}) \right)^{1/2} \right],
\]

and also

\[
\int_{t'}^{t} \{ g^{(n)}(z_i(\eta)) + z_i^{(2)}(\eta) \} d\eta \leq c_0 \left[ \frac{1}{b} + \left( \frac{1}{b} \right)^{1/2} \left( \sum_{i=1}^{n} z_i^2(\tilde{u}) \right)^{1/2} \right] \leq \nonumber
\]

\[
\leq c_0 \left[ \frac{1}{b} + \left( \frac{1}{b} \right)^{1/2} \left( E^{(n)}(z_i(\tilde{u})) + \frac{c_{10}}{b} \right) \right]^{1/2}.
\]

Then there exists \( t^* \in [t', t''] \) such that

\[
g^{(n)}(z_i(t^*)) + \frac{1}{2} \sum_{i=1}^{n} z_i^{(2)}(t^*) \leq c_0 \left[ \frac{1}{b} + \left( \frac{1}{b} \right)^{1/2} \left( E^{(n)}(z_i(\tilde{u})) + \frac{c_{10}}{b} \right) \right]^{1/2},
\]
and, by (6.5),

\[ E^{(n)}(z_i(t^*)) \leq C_{10} \left[ \frac{1}{b} + \left( \frac{1}{b} \right)^{1/2} \left\{ E^{(n)}(z_i(\tilde{t})) \right\}^{1/2} \right]. \]  

(5.4)

Setting now \( u_i = z_i, \ t' = t^*, \ t'' = t \in [\tilde{t}, \tilde{t} + 1], \) (6.4) gives

\[ E^{(n)}(z_i(t)) = E^{(n)}(z_i(t^*)) + \sum_{i=1}^{n} \int_{t}^{t^*} \left[ f_i(\eta) z_i'(\eta) - z_i''(\eta) \right] d\eta, \]

and, by (3.3), (5.1), (5.4),

\[ E^{(n)}(z_i(t)) \leq C_{11} \left[ \frac{1}{b} + \left( \frac{1}{b} \right)^{1/2} \left\{ E^{(n)}(z_i(\tilde{t})) \right\}^{1/2} \right]. \]  

(5.5)

Now, if

\[ E^{(n)}(z_i(\tilde{t} + 1)) \leq E^{(n)}(z_i(\tilde{t})), \]

we have (3.4); if

\[ E^{(n)}(z_i(\tilde{t} + 1)) \geq E^{(n)}(z_i(\tilde{t})), \]

setting in (5.5) \( t = \tilde{t} + 1, \) we have

\[ E^{(n)}(z_i(\tilde{t} + 1)) \leq \frac{C_{12}}{b}, \]

and then (3.4) again.

**Proof of Theorem 3.2:** Setting in (6.4) \( t' = \tau, \ t'' = \tau \in (\tau, \tau + 1), \) (3.3) gives

\[ E^{(n)}(z_i(t)) = E^{(n)}(z_i(\tau)) + \]

\[ + \sum_{i=1}^{n} \int_{\tau}^{t} \left[ \frac{1}{4} f_i(\eta) - \left( \frac{1}{2} f_i(\eta) - z_i'(\eta) \right)^2 \right] d\eta \leq E^{(n)}(z_i(\tau)) + \frac{C}{b}. \]

Then, by (3.5) there exists a constant \( C_{13} \) depending on \( C, \tilde{z}_i, \tilde{z}_i', \) such that

\[ E^{(n)}(z_i(t)) \leq \frac{C_{13}}{b}, \quad \tau \leq t \leq \tau + 1. \]

By (3.4)

\[ E^{(n)}(z_i(t)) \leq \text{Sup} \left\{ \frac{C_{11}}{b}, \frac{C_{14}}{b} \right\}, \]

for suitable \( C_{14} \) and then there exists a constant \( C_1 \) depending on \( C, \tilde{z}_i, \tilde{z}_i', \) such that

(3.6) holds.
Proof of Theorem 3.3: Setting in (6.4) \( t' = \tau \), \( t'' = t > \tau \), we have successively

\[
E^{(n)}(z_t(t)) + \sum_{i=1}^{n} \int_{t}^{\tau} z_i^{(2)}(\eta) \, d\eta = E^{(n)}(z_t(\tau)) + \sum_{i=1}^{n} \int_{t}^{\tau} f_i(\eta) z_i'(\eta) \, d\eta,
\]

\[
\sum_{i=1}^{n} \int_{t}^{\tau} z_i^{(2)}(\eta) \, d\eta \leq E^{(n)}(z_t(\tau)) + \sum_{i=1}^{n} \int_{t}^{\tau} f_i(\eta) z_i'(\eta) \, d\eta,
\]

and, by (3.5), (3.7), (3.9)

\[
\sum_{i=1}^{n} \int_{t}^{\tau} z_i^{(2)}(\eta) \, d\eta \leq \frac{c}{b} + \left( \sum_{i=1}^{n} \int_{t}^{\tau} f_i^{(2)}(\eta) \, d\eta \right)^{1/2} \sqrt{\sum_{i=1}^{n} \int_{t}^{\tau} z_i^{(2)}(\eta) \, d\eta} \leq \sqrt{\frac{c}{b}} + \sqrt{3 \frac{c}{b}},
\]

(5.6)

\[
\sum_{i=1}^{n} \int_{t}^{\tau} z_i^{(2)}(\eta) \, d\eta \leq \frac{c_{15}}{b}.
\]

Now (5.3), for \( t' = \tau \) and \( t'' = t > \tau \), gives

\[
\int_{t}^{\tau} g^{(n)}(z_t(\eta)) \, d\eta = \sum_{i=1}^{n} \int_{t}^{\tau} (z_i^{(2)}(\eta) + f_i(\eta) z_i(\eta)) \, d\eta + \sum_{i=1}^{n} \left[ \frac{1}{2} z_i^{(2)}(t) + z_i'(t) z_i(t) - \frac{1}{2} z_i^{(2)}(\tau) - z_i'(\tau) z_i(\tau) \right],
\]

and by (3.7), (3.8), (3.9) (5.6) and Theorem 3.2,

\[
\int_{t}^{\tau} E^{(n)}(z_t(\eta)) \, d\eta \leq \frac{c_{16}}{b} + \sum_{i=1}^{n} \int_{t}^{\tau} \eta^{2\beta} f_i(\eta) \eta^{-\beta} z_i(\eta) \, d\eta \leq \frac{c_{16}}{b} + \frac{c}{b} \sqrt{\frac{1}{1 - 2\beta} \left( \frac{1}{t^{2\beta - 1}} - \frac{1}{\tau^{2\beta - 1}} \right)},
\]

\[
\int_{t}^{\tau} E^{(n)}(z_t(\eta)) \, d\eta \leq \frac{c_{3}}{b},
\]

for suitable \( c_{3} \).

Corollary 3.4: This corollary follows obviously by theorem 3.2, setting in the proof \( \tau = -n \).
6. SOME AUXILIARY RESULTS

We recall here some classical results on the discrete problem.

6.1 EXISTENCE AND UNIQUENESS: Let \( t \in \mathbb{R}^*_+(r,t) \), and \( \{f_i(t)\} \in S \) a.e. in \( \mathbb{R}^*_+(r,t) \), \( f_i(t) \in L^2_{\text{loc}}(\mathbb{R}^*_+(r,t)) \); then the system (1.2)-(1.3), that we write as

\[
\begin{align*}
z_i(t) = & \tilde{z}_i + \int_t^s \nu_i(\eta) \, d\eta, \quad t \geq r, \\
v_i(t) = & \tilde{v}_i' + \int_t^s \left[ \frac{\partial}{\partial \eta} \varphi \left( \frac{\partial}{\partial \eta} z_{i-1}(\eta) \right) - b^{\alpha} \frac{\partial^4}{\partial \eta^4} z_{i-2}(\eta) + \varphi(z_{i-1}(\eta)) + -v_i(\eta) + f_i(\eta) \right] \, d\eta, \\
z_0 = & -z_s = 0, \quad z_{-1} = -z_{s-1}, \quad z_{-2} = -z_{s-2}, \quad z_{s+1} = -z_{s-1}, \quad z_{s+2} = -z_{s-2}
\end{align*}
\]

has a unique solution \( \in C^0(\mathbb{R}^*_+(r,t)) \cap H^1_{\text{loc}}(\mathbb{R}^*_+(r,t)) \); moreover \( z_i(t), v_i(t) \in S, \forall t \in \mathbb{R}^*_+(r,t) \).

6.2 VARIATIONAL AND «ENERGY» EQUATIONS: The solution \( \{z_i(t), v_i(t)\} \) of (6.1) satisfies moreover the variational equations

\[
\begin{align*}
\int_t^s \sum_{i=1}^n \left[ \frac{z_i''(t)}{b^{\alpha}} u_i + \varphi \left( \frac{\partial}{\partial \eta} z_{i-1}(\eta) \right) \frac{\partial}{\partial \eta} u_{i-1} + b^{\alpha} \frac{\partial^2}{\partial \eta^2} z_{i-2}(\eta) \frac{\partial^2}{\partial \eta^2} u_{i-2} + z_i' u_i - f_i u_i \right] \, d\eta = 0, \\
\forall t' \in \mathbb{R}^*_+(r,t), \forall \{u_i(t)\} \in S \text{ a.e. in } \mathbb{R}^*_+(r,t), \text{ and } u_i \in L^2_{\text{loc}}(\mathbb{R}^*_+(r,t)).
\end{align*}
\]

\[
\begin{align*}
\int_t^s \sum_{i=1}^n \left[ -z_i''(t) u_i + \varphi \left( \frac{\partial}{\partial \eta} z_{i-1}(\eta) \right) \frac{\partial}{\partial \eta} u_{i-1} + b^{\alpha} \frac{\partial^2}{\partial \eta^2} z_{i-2}(\eta) \frac{\partial^2}{\partial \eta^2} u_{i-2} + z_i' u_i - f_i u_i \right] \, d\eta + \\
+ \sum_{i=1}^n z_i'(t) u_i(t') - \sum_{i=1}^n z_i'(t) u_i(t') = 0, \\
\forall t' \in \mathbb{R}^*_+(r,t), \forall \{u_i(t)\} \in S \text{ a.e. in } \mathbb{R}^*_+(r,t), \text{ and } u_i \in H^1_{\text{loc}}(\mathbb{R}^*_+(r,t)).
\end{align*}
\]

Setting \( u_i = z_i' \) in (6.2) and integrating we have the «energy» equation

\[
\begin{align*}
E^{(s)}(z_i(t')) = & E^{(n)}(z_i(t')) + \sum_{i=1}^n \int_t^s [f_i(\eta) z_i'(\eta) - z_i^{(2)}(\eta)] \, d\eta, \quad \forall t', t'' \in \mathbb{R}^*_+(r,t).
\end{align*}
\]

Finally (1.1) and (1.8) give \( \varphi(\alpha) \alpha - 2\Phi(\alpha) = 0 \) if \( |\alpha| \leq \bar{\alpha} \) and, if \( |\alpha| \geq \bar{\alpha} \),

\[
\varphi(\alpha) \alpha - 2\Phi(\alpha) = L \left( \frac{1}{2} - \frac{\alpha^2}{A} - \frac{\alpha^2}{L} \right) \leq 0.
\]
Consequently we have

\[(6.5)\]

\[E^{(n)}(z_i) \leq \frac{1}{2} \sum_{i=1}^{N} z_i^2 + s^{(n)}(z_i).\]

Finally (1.8) gives \(\forall \omega > 0\) and \(\forall \alpha\)

\[(6.6)\]

\[\alpha^2 \leq \bar{\alpha}^2 + \omega + 4 \frac{A^2 + L^2 \omega}{L^2} \phi(\alpha).\]

REFERENCES