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R. GRANDE - P. SANTUCCI (*)

Anisotropic Sobolev-Besicovitch Spaces and a Semielliptic Differential Equation (**)

SUMMARY. — In this work we introduce the anisotropic Sobolev-Besicovitch spaces $W_p^{s,\alpha}$ and $H_p^{s,\alpha}$ of almost periodic functions and we present some of their properties. An embedding theorem of Sobolev type for the space $H_p^{s,\alpha}$ in spaces of continuous functions, is also given. Further, the results obtained are applied to the study of a semielliptic differential equation.

Spazi di Sobolev-Besicovitch anisotropi e una equazione differenziale semiellittica

SOMMARIO. — In questo lavoro si definiscono gli spazi di Sobolev-Besicovitch anisotropi $W_p^{s,\alpha}$ e $H_p^{s,\alpha}$ e si presentano alcune delle loro proprietà. Si dimostra un teorema di immersione continua per lo spazio $H_p^{s,\alpha}$ in spazi di funzioni continue. Inoltre i risultati ottenuti sono applicati allo studio di una equazione differenziale semiellittica.

1. - INTRODUCTION

This short note is concerned with some questions about spaces of almost periodic functions, which can be considered as the anisotropic counterpart of the Sobolev-Besicovitch $H_p^{s,\alpha}$ and $W_p^{s,\alpha}$ spaces.

The study of the isotropic Sobolev-Besicovitch spaces, introduced by Shubin [14], Pankov [13] and in a different way by Avantageggiati [2], has been developed recently by Iannacci et Al. [10], Dell'Acqua-Santucci [9] and Bruno-Grande [7]. These authors obtain continuous and compact embeddings in B_p^s spaces as well

(*) Indirizzo degli Autori: Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università di Roma «La Sapienza», Via A. Scarpa, 16, 00161 Roma.

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as in the C_0^0 space, if subspaces (that contain periodic functions) with fixed frequency spectrum are considered.

As we will show, also in the anisotropic case, it is possible to obtain some continuous embedding results, comparable with the classical embeddings, if the structure of the spectrum possesses certain summability properties.

Now we briefly describe the contents of the paper. In Section 2 we define the anisotropic Sobolev-Besicovitch spaces $W_{\varphi}^{\alpha, \beta}$ and $H_{\varphi}^{\alpha, \beta}$ and here we present some of their elementary properties. Further in this section, an equivalent norm in the space $W_{\varphi}^{\alpha, \beta}$ is given.

A simple continuous embedding result for the spaces $H_{\varphi}^{\alpha, \beta}$ in spaces of regular almost periodic functions is obtained in Section 3. As we have mentioned before, the structure of the spectrum plays a crucial role in our proof.

Finally, the last section is reserved to the study of the following semielliptic differential operator

$$\Delta_{\mu} f = -\frac{\partial^2 f}{\partial x^2} + \frac{\partial^4 f}{\partial y^4} + \mu f, \quad \mu > 0,$$

in the anisotropic Sobolev-Besicovitch spaces framework and its spectral analysis.

2. - ANISOTROPIC SOBOLEV-BESICOVITCH SPACES

We refer the reader to the monographs by Amerio-Prouse [1], Corduneanu [8], Levitan [11], Levitan-Zhikov [12] for the classical theory of the uniformly almost periodic functions and to the papers by Avantaggiati [1], Avantaggiati-Bruno-Iannacci [5], Bruno-Grande [6], Pankov [13] for the main properties of the B_{φ}^{α} spaces. Moreover, for more information about isotropic Sobolev-Besicovitch spaces the reader is referred to the papers quoted in the introduction.

Here we only recall that $\mathcal{P}(\mathbb{R}^N)$ is the vector spaces over \mathbb{C} of the generalized trigonometric polynomials

$$P(x) = \sum_{j=1}^w c_j e^{i\lambda_j \cdot x},$$

where $x, \lambda^j \in \mathbb{R}^N$ ($N \in \mathbb{N}$), $\lambda^j \cdot x = \lambda_1^j x_1 + \lambda_2^j x_2 + \dots + \lambda_N^j x_N \in \mathbb{C}$ for $j = 1, 2, \dots, w$.

If $c_j \neq 0$, $\forall j = 1, \dots, w$ the set

$$\sigma(P) = \{\lambda^1, \lambda^2, \dots, \lambda^w\} \subset \mathbb{R}^N,$$

is called the *spectrum* of P and the function $a(\cdot; P)$ defined in $\mathbb{R}^N \times \mathcal{P}(\mathbb{R}^N)$, by means of

the asymptotic mean

$$a(\lambda; P) = \lim_{T \rightarrow \infty} \frac{1}{|Q_T|} \int_{Q_T} P(x) e^{-i\lambda x} dx,$$

where $Q_T = [-T, T]^N$ and $|Q_T| = (2T)^N$, is called the Bohr transform of the polynomial P .

It is easy to show that

$$a(\lambda; P) = \begin{cases} c_j, & \text{if } \lambda = \lambda' \in \sigma(P), \\ 0, & \text{if } \lambda \notin \sigma(P). \end{cases}$$

Therefore, an arbitrary $P \in \mathcal{P}(\mathbb{R}^N)$ can be written in the following form

$$P(x) = \sum_{\lambda \in \sigma(P)} a(\lambda; P) e^{i\lambda x}.$$

Now, let $N = N_1 + N_2$, that is $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. According to Avantaggiati [4] we denote by A_1 the projection of a subset $A \subset \mathbb{R}^{N_1 + N_2}$ over the subspace \mathbb{R}^{N_1} . For every $\lambda' \in \mathbb{R}^{N_1}$ let $A_2(\lambda')$ denote the set of all $\lambda'' \in \mathbb{R}^{N_2}$ such that $(\lambda', \lambda'') \in A$.

With these notations, we can describe A in the following way

$$A = \{(\lambda', \lambda'') \in \mathbb{R}^{N_1 + N_2} \text{ such that } \lambda' \in A_1 \text{ and } \lambda'' \in A_2(\lambda')\}.$$

It is possible that $A_2(\lambda')$ does not depend on λ' , that is $A = A_1 \times A_2$, but in general $A_2(\lambda')$ varies when λ' does. These notations are extended to subsets of $\mathbb{R}^{N_1 + N_2}$ and in particular to the subset of A . Then if $I \subset A$ and $I \neq \emptyset$ we denote by I_1 the projection of I over \mathbb{R}^{N_1} and therefore I can be described as

$$I = \{(\lambda', \lambda'') \in \mathbb{R}^{N_1 + N_2} \text{ such that } \lambda' \in I_1 \text{ and } \lambda'' \in I_2(\lambda')\},$$

and we have

$$I \subset A \Rightarrow I_1 \subset A_1 \quad \text{and} \quad I_2(\lambda') \subset A_2(\lambda').$$

According to the notations given above, if $x, \lambda \in \mathbb{R}^{N_1 + N_2}$ we set $x = (x', x'')$, $\lambda = (\lambda', \lambda'')$ with $x' \in \mathbb{R}^{N_1}$, $x'' \in \mathbb{R}^{N_2}$, Therefore, a trigonometric polynomial $P \in \mathcal{P}(\mathbb{R}^{N_1 + N_2})$ can be represented in the following way: $J = \sigma(P)$

$$P(x) = P(x', x'') = \sum_{\lambda \in \sigma(P)} a(\lambda; P) e^{i\lambda x} = \sum_{\lambda' \in J_1} \sum_{\lambda'' \in J_2(\lambda')} a(\lambda', \lambda''; P) e^{i\lambda' x'} e^{i\lambda'' x''}.$$

From now on, we restrict ourselves to the case $N = 2$, that is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. In this situation we simply write $(x', x'') = (x, y) \in \mathbb{R}^2$.

For any $q \in [1, +\infty]$ and $\mathbf{s} = (s_1, s_2)$, $s_1, s_2 \in \mathbb{N}_0$, we set $\forall P \in \mathcal{P}(\mathbb{R}^2)$

$$(2.1) \quad \|P\|_{\mathbf{s}, q} := \|P\|_q + 6 \left\| \frac{\partial^{\mathbf{s}} P}{\partial x^{s_1}} \right\|_q + 6 \left\| \frac{\partial^{\mathbf{s}} P}{\partial y^{s_2}} \right\|_q.$$

Let us observe that, for any fixed q , (2.1) defines a norm on $\mathcal{D}(\mathbb{R}^2)$ and moreover $\|\cdot\|_{\mathbf{s}, q}$ is equivalent to $\|\cdot\|_q$.

DEFINITION 2.1: For any fixed $q \in [1, +\infty]$ we denote by $W_{\mathcal{D}}^{s, q}(\mathbb{R}^2)$ the completion of $\mathcal{D}(\mathbb{R}^2)$ with respect to the norm $\|\cdot\|_{\mathbf{s}, q}$ defined by (2.1).

These spaces are called anisotropic Sobolev-Besicovitch spaces of (vectorial) order \mathbf{s} and of type W .

Hence we can define the norm on the space $W_{\mathcal{D}}^{s, q}$ in the following way:

$$\|f\|_{\mathbf{s}, q} := \|f\|_q + \left\| \frac{\partial^\alpha f}{\partial x^{\alpha'}} \right\|_q + \left\| \frac{\partial^\alpha f}{\partial y^{\alpha''}} \right\|_q.$$

Of course, $W_{\mathcal{D}}^{s, q}(\mathbb{R}^2) \subset B_{\mathcal{D}}^s(\mathbb{R}^2)$, $\forall \mathbf{s} = (s_1, s_2)$, $s_1, s_2 \in \mathbb{N}_0$, $\forall q \geq 1$.

In a similar way, for any $q \in [1, +\infty]$, $\mathbf{s} = (s_1, s_2)$, $s_1, s_2 \in \mathbb{R}$, and $\forall P \in \mathcal{D}(\mathbb{R}^2)$, with $J = \sigma(P)$ let us consider the norm $\|\cdot\|_{\mathbf{s}, q}$ defined as follows

$$(2.2) \quad \|P\|_{\mathbf{s}, q}^{q'} = \sum_{\lambda' \in J_1} \sum_{\lambda'' \in J_2(\lambda')} [(1 + |\lambda'|^2)^{s_1/2} + (1 + |\lambda''|^2)^{s_2/2}]^{q'} |a(\lambda', \lambda''); P|^{q'},$$

where $q' = q/(q-1)$.

DEFINITION 2.2: We denote by $H_{\mathcal{D}}^{s, q}(\mathbb{R}^2)$ the completion of $\mathcal{D}(\mathbb{R}^2)$ with respect to the norm $\|\cdot\|_{\mathbf{s}, q}$ defined by (2.2).

These spaces are called anisotropic Sobolev-Besicovitch spaces of type H .

For any $f \in H_{\mathcal{D}}^{s, q}(\mathbb{R}^2)$, with $J = \sigma(f)$, we introduce the norm

$$\|f\|_{\mathbf{s}, q}^{q'} = \sum_{\lambda' \in J_1} \sum_{\lambda'' \in J_2(\lambda')} [(1 + |\lambda'|^2)^{s_1/2} + (1 + |\lambda''|^2)^{s_2/2}]^{q'} |a(\lambda', \lambda''); f|^{q'}.$$

Observe that, $\forall \mathbf{s} = (s_1, s_2)$, such that $s_1, s_2 \geq 0$, $\forall q \geq 2$, we have $H_{\mathcal{D}}^{s, q}(\mathbb{R}^2) \subset B_{\mathcal{D}}^s(\mathbb{R}^2)$, with continuous embedding.

Using the same technique as in the proof of Theorem 4.1 in [II], it is possible to prove that

- i) $W_{\mathcal{D}}^{s, q}(\mathbb{R}^2) \hookrightarrow H_{\mathcal{D}}^{s, q}(\mathbb{R}^2)$, $\forall q \in [1, 2]$, $\forall \mathbf{s} = (s_1, s_2)$, $s_1, s_2 \in \mathbb{N}$,
- ii) $H_{\mathcal{D}}^{s, q}(\mathbb{R}^2) \hookrightarrow W_{\mathcal{D}}^{s, q}(\mathbb{R}^2)$, $\forall q \geq 2$, $\forall \mathbf{s} = (s_1, s_2)$, $s_1, s_2 \in \mathbb{N}$,

from which $W_{\mathcal{D}}^{s, 2}(\mathbb{R}^2) = H_{\mathcal{D}}^{s, 2}(\mathbb{R}^2)$, $\forall \mathbf{s} = (s_1, s_2)$, $s_1, s_2 \in \mathbb{N}$ follows.

In the space $W_{\mathcal{D}}^{s, 2}(\mathbb{R}^2)$ we define a different norm

$$\|f\|_{\mathbf{s}, 2}^2 := \sum_{0 \leq m_1, m_2, m_1+m_2 \leq 1} \left\| \frac{\partial^{m_1+m_2} f}{\partial x^{m_1} \partial y^{m_2}} \right\|_2^2.$$

PROPOSITION 2.3: The norms $\|\cdot\|_{\mathbf{s}, 2}$ and $\|\cdot\|_{\mathbf{s}, 2}^M$ are equivalent.

PROOF: We show first that if $f \in W_{\phi}^{m_1, m_2}(\mathbb{R}^2)$, then it possesses the strong mixed derivatives $\partial^{m_1+m_2} f / \partial x^{m_1} \partial y^{m_2}$ in B_{ϕ}^2 , provided $0 \leq m_1/s_1 + m_2/s_2 \leq 1$.

To this end, let $(P_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}^2)$ be a Cauchy sequence in $W_{\phi}^{m_1, m_2}(\mathbb{R}^2)$ such that it converges to f . Moreover, we observe that for any pair of real numbers $1 < p < \infty$, $1 < q < \infty$ such that $1/p + 1/q = 1/r \leq 1$ the following inequality

$$(2.3) \quad \frac{(ab)^r}{r} \leq \frac{a^r}{p} + \frac{b^r}{q} \quad \forall a, b \geq 0,$$

holds. By (2.3) with $p = s_1/m_1$, $q = s_2/m_2$ and the Parseval equality, we get

$$\begin{aligned} \left\| \frac{\partial^{m_1+m_2} P_n}{\partial x^{m_1} \partial y^{m_2}} \right\|_2^2 &= \sum_{\lambda' \in J_1} \sum_{\lambda'' \in J_2(\lambda')} (|\lambda'|^{m_1} |\lambda''|^{m_2})^2 |a(\lambda', \lambda'') P_n|^2 \leq \\ &\leq c \sum_{\lambda' \in J_1} \sum_{\lambda'' \in J_2(\lambda')} (|\lambda'|^{1/s_1} + |\lambda''|^{1/s_2})^{2s} |a(\lambda', \lambda'') P_n|^2 \leq \\ &\leq c \sum_{\lambda' \in J_1} \sum_{\lambda'' \in J_2(\lambda')} [(1 + |\lambda'|^2)^{r/2} + (1 + |\lambda''|^2)^{r/2}]^{2s} |a(\lambda', \lambda'') P_n|^2 \leq \\ &\leq c \sum_{\lambda' \in J_1} \sum_{\lambda'' \in J_2(\lambda')} [(1 + |\lambda'|^2)^{r/2} + (1 + |\lambda''|^2)^{r/2}]^2 |a(\lambda', \lambda'') P_n|^2 = \\ &= c \|P_n\|_{s, 2}^2 \leq c' \|P_n\|_2^2, \end{aligned}$$

where c and c' are suitable positive constants. Thus, the sequence $((\partial^{m_1+m_2} P_n) / (\partial x^{m_1} \partial y^{m_2}))_{n \in \mathbb{N}}$ is a Cauchy sequence in $B_{\phi}^2(\mathbb{R}^2)$, hence it defines the strong mixed derivative $(\partial^{m_1+m_2} f) / (\partial x^{m_1} \partial y^{m_2})$ of f in $B_{\phi}^2(\mathbb{R}^2)$. Passing to the limit for $n \rightarrow \infty$ in the previous inequality, we obtain

$$\left\| \frac{\partial^{m_1+m_2} f}{\partial x^{m_1} \partial y^{m_2}} \right\|_2^2 \leq c' \|f\|_{s, 2}^2,$$

and therefore

$$\sum_{0 \leq m_1/s_1 + m_2/s_2 \leq 1} \left\| \frac{\partial^{m_1+m_2} f}{\partial x^{m_1} \partial y^{m_2}} \right\|_2^2 \leq c'' \|f\|_{s, 2}^2,$$

where c'' is also a suitable positive constant. This establishes one half of the desired inequality. On the other hand, also by the Parseval equality, it follows

$$\begin{aligned} \|f\|_{s, 2}^2 &= \left(\|f\|_2^2 + \left\| \frac{\partial^{\nu} f}{\partial x^{\nu}} \right\|_2^2 + \left\| \frac{\partial^{\nu} f}{\partial y^{\nu}} \right\|_2^2 \right) \leq c \left(\|f\|_2^2 + \left\| \frac{\partial^{\nu} f}{\partial x^{\nu}} \right\|_2^2 + \left\| \frac{\partial^{\nu} f}{\partial y^{\nu}} \right\|_2^2 \right) = \\ &= c \left(\sum_{\lambda' \in J_1} \sum_{\lambda'' \in J_2(\lambda')} |a(\lambda', \lambda'') f|^2 + \sum_{\lambda' \in J_1} \sum_{\lambda'' \in J_2(\lambda')} |\lambda'|^{2\nu} |a(\lambda', \lambda'') f|^2 + \right. \end{aligned}$$

$$\begin{aligned}
& + \sum_{\lambda' \in J_1} \sum_{\lambda'' \in J_2(\lambda')} |\lambda''|^{2n_2} |a(\lambda', \lambda''; f)|^2 = \\
& = c \sum_{\lambda' \in J_1} \sum_{\lambda'' \in J_2(\lambda')} (1 + |\lambda'|^{2n_1} + |\lambda''|^{2n_2}) |a(\lambda', \lambda''; f)|^2 \leq \\
& \leq c \sum_{\lambda' \in J_1} \sum_{\lambda'' \in J_2(\lambda')} \left[\sum_{0 \leq m_1/n_1 + m_2/n_2 \leq 1} |\lambda'|^{2m_1} |\lambda''|^{2m_2} |a(\lambda', \lambda''; f)|^2 \right] = \\
& = c \sum_{0 \leq m_1/n_1 + m_2/n_2 \leq 1} \sum_{\lambda' \in J_1} \sum_{\lambda'' \in J_2(\lambda')} \left| a \left(\lambda', \lambda''; \frac{\partial^{m_1 + m_2} f}{\partial x^{m_1} \partial y^{m_2}} \right) \right|^2 = \\
& = c \sum_{0 \leq m_1/n_1 + m_2/n_2 \leq 1} \left\| \frac{\partial^{m_1 + m_2} f}{\partial x^{m_1} \partial y^{m_2}} \right\|_2^2 \leq c (\|f\|_{4,2}^M)^2.
\end{aligned}$$

This establishes the other half of the desired inequality and completes the proof of the proposition. ■

3. - AN EMBEDDING RESULT

Let A be a nonempty subset \mathbb{R}^N . According to Avantaggiati [4], we said that A satisfies the α) condition if

α) A is a countable semigroup in $\mathbb{R}^N(+)$ with a finite number of generators, which is contained in a convex cone in $\mathbb{R}^N(+)$, that is

$$A = \{n_1 \lambda_1^* + \dots + n_d \lambda_d^* \text{ such that } (n_1, \dots, n_d) \in \mathbb{N}_0^d \setminus \{0\}\}$$

with the hypothesis

- i) $\lambda_1^*, \dots, \lambda_d^*$ are \mathbb{Z} -linearly independent;
- ii) $\overline{\text{conv}}\{\lambda_1^*, \dots, \lambda_d^*\} \cap \{0\} = \emptyset$.

Moreover, we said that A satisfies the β) condition if there exists a positive number β such that

$$\sum_{\lambda \in A} \frac{1}{|\lambda|^\gamma} < +\infty, \quad \forall \gamma > \beta.$$

It is known that if A satisfies the α) condition, then it satisfies the β) condition and the number β is equal to the number of generators of the semigroup. See Avantaggiati [4] and Iannacci et al. [10] Lemma 6.1.

Now, let us fix the sets $A_1, A_2 \subset \mathbb{R}$ satisfying the condition α) (with β_1 and β_2 generators respectively) and consider almost periodic functions f such that $\sigma(f) \subset A_1 \times A_2$.

Let us set

$$B_{\varphi}^g(A_1 \times A_2) = \{f \in B_{\varphi}^g(\mathbb{R}^2) : \sigma(f) \subset A_1 \times A_2\},$$

and, analogously, $W_{\varphi}^{g, \sigma}(A_1 \times A_2)$, $H_{\varphi}^{g, \sigma}(A_1 \times A_2)$ and $C_{\varphi}^g(A_1 \times A_2)$. Observe that these spaces are separable.

Then the following result holds.

THEOREM 3.1: *Let $1 < q < \infty$ and $m_1, m_2 \in \mathbb{N}$ be such that*

$$\frac{1}{s_1} \left(m_1 + \frac{\beta_1}{q} \right) + \frac{1}{s_2} \left(m_2 + \frac{\beta_2}{q} \right) < 1,$$

then

$$\left\| \frac{\partial^{m_1 + m_2} f}{\partial x^{m_1} \partial y^{m_2}} \right\|_q \leq c \|f\|_{W_{\varphi}^{g, \sigma}},$$

for all $f \in H_{\varphi}^{g, \sigma}(A_1 \times A_2)$, where $s = (s_1, s_2)$ and c is a suitable positive constant.

PROOF: Choose γ_1, γ_2 with $\gamma_1 > \beta_1$, $\gamma_2 > \beta_2$, such that the following inequality

$$\frac{1}{s_1} \left(m_1 + \frac{\gamma_1}{q} \right) + \frac{1}{s_2} \left(m_2 + \frac{\gamma_2}{q} \right) < 1,$$

holds. Clearly

$$(3.1) \quad \frac{\gamma_1}{s_1} + \frac{\gamma_2}{s_2} < \left[1 - \left(\frac{m_1}{s_1} + \frac{m_2}{s_2} \right) \right] q.$$

On the other hand, the inequality

$$(3.2) \quad |\lambda'|^{m_1} |\lambda''|^{m_2} \leq (|\lambda'|^{r_1} + |\lambda''|^{r_2})^{m_1/r_1 + m_2/r_2},$$

also holds, since $0 \leq m_1/s_1 + m_2/s_2 < 1$.

Therefore, by Hölder inequality and (3.2) we get

$$\begin{aligned} & \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} |\lambda'|^{m_1} |\lambda''|^{m_2} |a(\lambda', \lambda''); f| \leq \\ & \leq \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} (1 + |\lambda'|^{r_1} + |\lambda''|^{r_2})^{m_1/r_1 + m_2/r_2} |a(\lambda', \lambda''); f| \\ & = \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} (1 + |\lambda'|^{r_1} + |\lambda''|^{r_2}) \frac{1}{(1 + |\lambda'|^{r_1} + |\lambda''|^{r_2})^{1 - (m_1/r_1 + m_2/r_2)}} |a(\lambda', \lambda''); f| \leq \\ & \leq \left(\sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} [(1 + |\lambda'|^{r_1})^{1/2} + (1 + |\lambda''|^{r_2})^{1/2}]^{2r'} |a(\lambda', \lambda''); f|^{r'} \right)^{1/r'} \times \\ & \quad \times \left(\sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{1}{(1 + |\lambda'|^{r_1} + |\lambda''|^{r_2})^{1 - (m_1/r_1 + m_2/r_2)q}} \right)^{1/q}. \end{aligned}$$

Since (3.1) holds, we see that

$$\begin{aligned} \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{1}{(1 + |\lambda'|^{\alpha_1} + |\lambda''|^{\alpha_2})^{(1 - (m_1/\alpha_1 + m_2/\alpha_2))\beta}} &\leq \\ &\leq \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{1}{(1 + |\lambda'|^{\alpha_1} + |\lambda''|^{\alpha_2})^{\beta_1/\alpha_1 + \beta_2/\alpha_2}} = \\ &= \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{1}{(1 + |\lambda'|^{\alpha_1} + |\lambda''|^{\alpha_2})^{\beta_1/\alpha_1}} \frac{1}{(1 + |\lambda'|^{\alpha_1} + |\lambda''|^{\alpha_2})^{\beta_2/\alpha_2}} \leq \\ &\leq \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{1}{|\lambda'|^{\beta_1}} \frac{1}{|\lambda''|^{\beta_2}} = \sum_{\lambda' \in A_1} \frac{1}{|\lambda'|^{\beta_1}} \sum_{\lambda'' \in A_2} \frac{1}{|\lambda''|^{\beta_2}} < \infty. \end{aligned}$$

Therefore,

$$\sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} |\lambda'|^{m_1} |\lambda''|^{m_2} |a(\lambda', \lambda''; f)| < \infty$$

for any $f \in H_{\phi}^{\alpha}(A_1 \times A_2)$, and the double series

$$\sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} (i\lambda')^{m_1} (i\lambda'')^{m_2} a(\lambda', \lambda''; f) e^{i\lambda'x} e^{i\lambda''y},$$

converges absolutely and uniformly in \mathbb{R}^2 to some $g^* \in C_{\phi}^{\alpha}(A_1 \times A_2)$.

On the other hand, the double series

$$\sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} a(\lambda', \lambda''; f) e^{i\lambda'x} e^{i\lambda''y},$$

which represent f in $B_{\phi}^{\alpha}(A_1 \times A_2)$ converges absolutely in \mathbb{R}^2 .

Indeed, since $|\lambda'| > 1$ and $|\lambda''| > 1$ definitively (by virtue of the β) condition), we have

$$\begin{aligned} \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} |a(\lambda', \lambda''; f)| &= \sum_{|\lambda'| \leq 1} \sum_{|\lambda''| \leq 1} |a(\lambda', \lambda''; f)| + \\ &+ \sum_{|\lambda'| > 1} \sum_{|\lambda''| > 1} |a(\lambda', \lambda''; f)| \leq c + \sum_{|\lambda'| > 1} \sum_{|\lambda''| > 1} |\lambda'|^{m_1} |\lambda''|^{m_2} |a(\lambda', \lambda''; f)| < \infty, \end{aligned}$$

where c is a suitable positive constant.

Then $f \in C_{\phi}^{\alpha}(A_1 \times A_2)$ and $f(x, y) = \lim_{\alpha \rightarrow \infty} S_{\alpha}(x, y)$ where

$$S_{\alpha}(x, y) = \sum_{k=1}^{\alpha} \sum_{l=1}^{\alpha} a(\lambda_k', \lambda_l''; f) e^{i\lambda_k'x} e^{i\lambda_l''y}.$$

Observe that

$$g^*(x, y) = \lim_{\alpha \rightarrow \infty} \frac{\partial^{m_1 + m_2} S_{\alpha}(x, y)}{\partial x^{m_1} \partial y^{m_2}}$$

in the sense of $C_{\phi}^{\alpha}(A_1 \times A_2)$.

Thus,

$$g^s(x, y) = \frac{\partial^{m_1+m_2} f(x, y)}{\partial x^{m_1} \partial y^{m_2}},$$

and the proof of the theorem is now complete. ■

REMARK 3.2: The result given above can be considered as the almost periodic counterpart of Theorem 4.2.3 in Schmeisser-Triebel [15]. Assertions of such type are well known for the classical anisotropic Sobolev spaces. See, for example, Troisi [16], [17], [18] and the reference therein. However, we wish to point out that the role played by β_1 and β_2 in Theorem 3.1, is similar to the ones played by space dimensions in the classical case.

4. - APPLICATIONS TO A SEMIELLIPTIC OPERATOR

Let $s = (s_1, s_2) = (s_1, 2s_1)$ with $s_1 \in \mathbb{N}$. Then, $H_\phi^{s,2}(A_1 \times A_2)$ and $W_\phi^{s,2}(A_1 \times A_2)$ are the spaces considered in Section 3.

Now, we set $s+2 = (s_1+2, 2s_1+4)$ and consider the following semielliptic operator

$$(4.1) \quad \alpha_\mu f = -\frac{\partial^2 f}{\partial x^2} + \frac{\partial^4 f}{\partial y^4} + \mu f, \quad \mu > 0,$$

where μ is a positive constant, in the framework of the anisotropic Sobolev-Besicovitch spaces.

THEOREM 4.1: Let α_μ as in (4.1) and $s_1 \in \mathbb{N}$. Then, there exists a constant $\mu_0 > 0$ such that α_μ is an isomorphic mapping from $W_\phi^{s+2,2}(A_1 \times A_2)$ onto $W_\phi^{s,2}(A_1 \times A_2)$, with $\mu > \mu_0$.

PROOF: Let $(P_n)_{n \in \mathbb{N}} \subset \mathcal{D}(A_1 \times A_2)$ such that $P_n \rightarrow f$ in $W_\phi^{s+2,2}(A_1 \times A_2)$. Hence $(P_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $W_\phi^{s+2,2}(A_1 \times A_2)$. Therefore, also $(\alpha_\mu P_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $W_\phi^{s,2}(A_1 \times A_2)$.

Indeed

$$\begin{aligned} \|\alpha_\mu P_n - \alpha_\mu P_m\|_{s,2} &= \|\alpha_\mu P_n - \alpha_\mu P_m\|_2 + \left\| \frac{\partial^i}{\partial x^i} [\alpha_\mu P_n - \alpha_\mu P_m] \right\|_2 + \\ &+ \left\| 1 \frac{\partial^{2i}}{\partial x^{2i}} [\alpha_\mu P_n - \alpha_\mu P_m] \right\|_2. \end{aligned}$$

On the other hand, by Proposition 2.3, we get

$$\begin{aligned} \|\alpha_n P_n - \alpha_n P_m\|_2 &\leq \left\| \frac{\partial^2}{\partial x^2} [P_n - P_m] \right\|_2 + \left\| \frac{\partial^4}{\partial x^4} [P_n - P_m] \right\|_2 + \|\mu [P_n - P_m]\|_2 \leq \\ &\leq c \|P_n - P_m\|_{k+2,2} \leq c \|P_n - P_m\|_{k+2,2}. \end{aligned}$$

In the same way, we obtain

$$\begin{aligned} \left\| \frac{\partial^{2l}}{\partial x^{2l}} [\alpha_n P_n - \alpha_n P_m] \right\|_2 &\leq \left\| \frac{\partial^{2l+2}}{\partial x^{2l+2}} [P_n - P_m] \right\|_2 + \left\| \frac{\partial^{2l+4}}{\partial x^{2l} \partial y^4} [P_n - P_m] \right\|_2 + \\ &+ \left\| \mu \frac{\partial^{2l}}{\partial x^{2l}} [P_n - P_m] \right\|_2 \leq c \|P_n - P_m\|_{k+2,2}, \end{aligned}$$

and

$$\left\| \frac{\partial^{2l}}{\partial y^{2l}} [\alpha_n P_n - \alpha_n P_m] \right\|_2 \leq c \|P_n - P_m\|_{k+2,2}.$$

Then, the following inequality

$$(4.2) \quad \|\alpha_n P_n - \alpha_n P_m\|_{k,2} \leq c \|P_n - P_m\|_{k+2,2}$$

holds. Hence, $(\alpha_n P_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $W_{\varphi}^{k,2}(A_1 \times A_2)$. Therefore, it defines the element $\alpha_\mu f$ in $W_{\varphi}^{k,2}(A_1 \times A_2)$.

Moreover, passing to the limit for $n \rightarrow \infty$, (4.2) also shows that α_μ is a continuous operator from $W_{\varphi}^{k+2,2}(A_1 \times A_2)$ in $W_{\varphi}^{k,2}(A_1 \times A_2)$.

Now, we observe that

$$(4.3) \quad a(\lambda', \lambda''; \alpha_\mu f) = (|\lambda'|^2 + |\lambda''|^2 + \mu) a(\lambda', \lambda''; f),$$

and so we get

$$\begin{aligned} (4.4) \quad \|f\|_{k+2,2} &\leq c \|f\|_{k+2,2} = \\ &= c \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} [(1 + |\lambda'|^2)^{k/2+1} + (1 + |\lambda''|^2)^{k/2+1}] |a(\lambda', \lambda''; f)|^2 \leq \\ &\leq c \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} [(1 + |\lambda'|^2)^{k/2} (1 + |\lambda''|^2) + \\ &+ (1 + |\lambda''|^2)^{k/2} (1 + |\lambda'|^2)^2] |a(\lambda', \lambda''; f)|^2 \leq \\ &\leq c \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} [(1 + |\lambda'|^2)^{k/2} + (1 + |\lambda''|^2)^k] \times \\ &\times [(1 + |\lambda'|^2) + (1 + |\lambda''|^2)^2] |a(\lambda', \lambda''; f)|^2 \leq \end{aligned}$$

$$\leq c \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} [(1 + |\lambda'|^2)^{\mu/2} + (1 + |\lambda''|^2)^{\mu}]^2 [(1 + |\lambda'|^2 + 1 + |\lambda''|^4 + 2|\lambda''|^2) |\alpha(\lambda', \lambda'')|]^2.$$

Since, $|\lambda''| > 1$ definitively, there exists $\mu > 0$ such that

$$|\lambda''|^2 \leq \mu |\lambda''|^4.$$

Moreover, for any $\mu \geq 2/(2\alpha + 1) = \mu_0$ the inequality

$$2 + |\lambda'|^2 + (2\alpha + 1)|\lambda''|^4 \leq (2\alpha + 1) \left(\frac{2}{2\alpha + 1} + |\lambda'|^2 + |\lambda''|^4 \right) \leq (2\alpha + 1)(\mu + |\lambda'|^2 + |\lambda''|^4),$$

holds. Now, we set $c_1 = c \cdot (2\alpha + 1)$. Then, we have

$$\begin{aligned} (4.5) \quad \|f\|_{\mathbb{R}^{2,2}}^2 &\leq c_1 \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} [(1 + |\lambda'|^2)^{\mu/2} + (1 + |\lambda''|^2)^{\mu}]^2 (\mu + |\lambda'|^2 + |\lambda''|^4) |\alpha(\lambda', \lambda'')| f]^2 \leq \\ &\leq c_1 \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} [(1 + |\lambda'|^2)^{\mu/2} + (1 + |\lambda''|^2)^{\mu}]^2 |\alpha(\lambda', \lambda'')| \alpha_p f]^2 = \\ &= c_1 \|\alpha_p f\|_{\mathbb{R}^{2,2}}^2 \leq c \|\alpha_p f\|_{\mathbb{R}^{2,2}}^2. \end{aligned}$$

Therefore, (4.5) shows that the operator α_p has a continuous inverse, provided $\mu > \mu_0$.

It remains to prove that α_p is also onto. To this end, we fix $u \in W_{\alpha}^{s+2,2}(A_1 \times A_2)$ such that

$$u(x, y) = \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \alpha(\lambda', \lambda'') u e^{i\lambda' x} e^{i\lambda'' y}.$$

It is easy to verify, by means (4.3) and (4.5), that

$$f(x, y) = \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{\alpha(\lambda', \lambda'') u}{(|\lambda'|^2 + |\lambda''|^4 + \mu)} e^{i\lambda' x} e^{i\lambda'' y},$$

belongs to $W_{\alpha}^{s+2,2}(A_1 \times A_2)$ and $\alpha_p f = u$. The proof of the theorem is now complete. ■

Now, we deal with spectral analysis for the differential operator

$$(4.6) \quad \alpha(D) = -\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial y^4},$$

in the space $B_{\alpha}^s(A_1 \times A_2)$, already defined in Section 3.

It is known, see [4], that an almost periodic fundamental solution $E_{A_1 \times A_2}$ of the differential operator (4.6) can be constructed and

$$E_{A_1 \times A_2} \sim \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{1}{|\lambda'|^2 + |\lambda''|^4} e^{i\lambda'x} e^{i\lambda''y}.$$

We claim that the operator

$$T: f \rightarrow E_{A_1 \times A_2} * f \sim \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{a(\lambda', \lambda''; f)}{|\lambda'|^2 + |\lambda''|^4} e^{i\lambda'x} e^{i\lambda''y},$$

from $B_{\beta}^2(A_1 \times A_2)$ to $B_{\beta}^2(A_1 \times A_2)$ is compact and self-adjoint. First of all, observe that, by virtue of the β condition, there exists $\lambda' \in A_1$ and $\lambda'' \in A_2$ such that

$$|\lambda'| \leq |\lambda'|, \quad \forall \lambda' \in A_1 \quad \text{and} \quad |\lambda''| \leq |\lambda''|, \quad \forall \lambda'' \in A_2.$$

Let us set

$$\mu_0 = \min_{A_1} |\lambda'|^2 = |\lambda'|^2, \quad \nu_0 = \min_{A_2} |\lambda''|^4 = |\lambda''|^4,$$

then we have

$$\|T\| = \frac{1}{\mu_0 + \nu_0}.$$

Indeed, the Parseval equality gives

$$(4.7) \quad \|E_{A_1 \times A_2} * f\|_2^2 = \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{|a(\lambda', \lambda''; f)|^2}{(|\lambda'|^2 + |\lambda''|^4)^2} \leq \frac{1}{(\mu_0 + \nu_0)^2} \|f\|_2^2,$$

$$\forall f \in B_{\beta}^2(A_1 \times A_2).$$

Choose $f(x) = e^{i\lambda'x} e^{i\lambda''y}$ in (4.7), we obtain the equality.

Now, let $\beta = \{f \in B_{\beta}^2(A_1 \times A_2), \|f\|_2 < 1\}$, the open unit ball of $B_{\beta}^2(A_1 \times A_2)$. We shall show that $\overline{T(\beta)}$ is a compact subset of $B_{\beta}^2(A_1 \times A_2)$. Observe first that $\overline{T(\beta)}$ is closed and bounded. Thus, by the compactness criterion [6], we need to show that $T(\beta)$ is B_{β}^2 -equicontinuous and B_{β}^2 -equi-almost periodic.

If $b = (b', b'')$, we have,

$$\begin{aligned} \|T_b T f - T f\|_2^2 &= \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{|a(\lambda', \lambda''; f)|^2 |e^{i(\lambda' b' + \lambda'' b'')} - 1|^2}{(|\lambda'|^2 + |\lambda''|^4)^2} \leq \\ &\leq \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{|a(\lambda', \lambda''; f)|^2 |\lambda' b' + \lambda'' b''|^2}{(|\lambda'|^2 + |\lambda''|^4)^2} \leq \\ &\leq c \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{|a(\lambda', \lambda''; f)|^2 (|\lambda'|^2 + |\lambda''|^4) (|b'|^2 + |b''|^2)}{(|\lambda'|^2 + |\lambda''|^4)^2} \leq \\ &\leq \frac{c}{\mu_0 + \nu_0} \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} |a(\lambda', \lambda''; f)|^2 (|b'|^2 + |b''|^2) \leq \frac{1}{\mu_0 + \nu_0 (|b'|^2 + |b''|^2)}, \end{aligned}$$

where c is a suitable positive constant.

Therefore

$$\|r_\varepsilon T f - T f\|_2 < \varepsilon, \quad \forall f \in \mathcal{B},$$

provided $|b'|^2 + |b''|^2$ sufficiently small.

It is easy to verify that $T(\mathcal{B})$ is also a B_φ^2 -equi-almost periodic set. To this end, fix $\varepsilon > 0$. Then, there exists $\lambda'_\varepsilon \in \mathcal{A}_1$ and $\lambda''_\varepsilon \in \mathcal{A}_2$ such that

$$\sum_{|\lambda'| > |\lambda'_\varepsilon|} \sum_{|\lambda''| > |\lambda''_\varepsilon|} \frac{|a(\lambda', \lambda'')|}{(|\lambda'|^2 + |\lambda''|^4)^2} < \frac{\varepsilon}{8},$$

$$\forall f \in \mathcal{B}.$$

Let $\zeta = (\zeta', \zeta'') \in \mathbb{R}^2$ a solution of the following system of inequalities:

$$(4.8) \quad |e^{i(\lambda'\zeta' + \lambda''\zeta'')} - 1| < \delta_\varepsilon, \quad \forall |\lambda'| \leq |\lambda'_\varepsilon|, \quad \forall |\lambda''| \leq |\lambda''_\varepsilon|,$$

where $\delta_\varepsilon > 0$ only depends on ε . Therefore, one has

$$\begin{aligned} \|r_\varepsilon T f - T f\|_2^2 &= \sum_{|\lambda'| \leq |\lambda'_\varepsilon|} \sum_{|\lambda''| \leq |\lambda''_\varepsilon|} \frac{|a(\lambda', \lambda'')|}{(|\lambda'|^2 + |\lambda''|^4)^2} |e^{i(\lambda'\zeta' + \lambda''\zeta'')} - 1|^2 + \\ &+ \sum_{|\lambda'| > |\lambda'_\varepsilon|} \sum_{|\lambda''| > |\lambda''_\varepsilon|} \frac{|a(\lambda', \lambda'')|}{(|\lambda'|^2 + |\lambda''|^4)^2} |e^{i(\lambda'\zeta' + \lambda''\zeta'')} - 1|^2 \leq \\ &\leq \delta_\varepsilon \sum_{|\lambda'| \leq |\lambda'_\varepsilon|} \sum_{|\lambda''| \leq |\lambda''_\varepsilon|} \frac{|a(\lambda', \lambda'')|}{(|\lambda'|^2 + |\lambda''|^4)^2} + 4 \frac{\varepsilon}{8} \leq \delta_\varepsilon \frac{1}{(\mu_0 + \nu_0)^2} + \frac{\varepsilon}{2}. \end{aligned}$$

Choose

$$\delta_\varepsilon = \frac{1}{2} \varepsilon (\mu_0 + \nu_0)^2,$$

then we have

$$\|r_\varepsilon T f - T f\|_2 < \varepsilon, \quad \forall f \in \mathcal{B}.$$

According to the Kronecker's Theorem, the system (4.8) has a relatively dense set in \mathbb{R}^2 of solutions. This shows that $T(\mathcal{B})$ is also a B_φ^2 -equi-almost periodic set.

Finally, to show that T is self-adjoint we simply observe that

$$E_{\mathcal{A}_1 \times \mathcal{A}_2}(t - \tau) = \overline{E_{\mathcal{A}_1 \times \mathcal{A}_2}(\tau - t)}.$$

From the previous argument, it follows that for the equation

$$(d(D) - z)w = f, \quad z \in \mathbb{C},$$

the Fredholm Alternative Theorem holds. It is possible to construct the resolvent oper-

ator $R(z, \alpha(D)) = (\alpha(D) - z)^{-1}$, and by a simple calculation we find

$$R(z, \alpha(D))f = \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{a(\lambda', \lambda''; f)}{|\lambda'|^2 + |\lambda''|^4 - z} e^{i\lambda'x} e^{i\lambda''y}.$$

Thus, we are ready to prove

THEOREM 4.2. For the equation

$$(4.9) \quad (\alpha(D) - z)u = f, \quad z \in \mathbb{C}, \quad f \in B_{\alpha}^2(A_1 \times A_2),$$

the following statements hold:

- i) The nonzero eigenvalues are positive real numbers and form an infinite non decreasing sequence μ_1, μ_2, \dots with $\mu_j \rightarrow \infty$.
- ii) Every positive real number μ , is an eigenvalue for the operator $\alpha(D)$ if and only if there exist a $(\lambda', \lambda'') \in A_1 \times A_2$ such that $|\lambda'|^2 + |\lambda''|^4 = \mu$. Moreover the multiplicity of μ is at most one.
- iii) If $z \neq \mu_j$, $j \in \mathbb{N}$, then the equation has a unique solution for each $f \in B_{\alpha}^2(A_1 \times A_2)$.
- iv) If $z = \mu_j$ for some $j \in \mathbb{N}$ the equation has a solution if and only if

$$a(\lambda', \lambda''; f) = 0,$$

for each (λ', λ'') such that $|\lambda'|^2 + |\lambda''|^4 = \mu_j$.

PROOF: All statements follow easily. ■

Regularity results for the solution of (4.9) are possible. For instance, it should be clear that $u \in H_{\alpha}^2(A_1 \times A_2)$. Then by Theorem 3.1, if

$$\frac{1}{2} \left(1 + \frac{\beta_1}{2} \right) + \frac{1}{2} \left(1 + \frac{\beta_2}{2} \right) < 1,$$

the function

$$u(x, y; z) = \sum_{\lambda' \in A_1} \sum_{\lambda'' \in A_2} \frac{a(\lambda', \lambda''; f)}{|\lambda'|^2 + |\lambda''|^4 - z} e^{i\lambda'x} e^{i\lambda''y},$$

which is almost periodic for any $z \in \mathbb{C}$, $z \neq |\lambda'|^2 + |\lambda''|^4$, $(\lambda', \lambda'') \in A_1 \times A_2$, belongs to $C_{\alpha}^1(A_1 \times A_2)$.

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