Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
115° (1997), Vol. XXI, fasc. 1, pagg. 235-252

MARCO BIROLI (*)

Nonlinear Kato Measures and Nonlinear Subelliptic Schrödinger Problems (**)

ABSTRACT. — We give a definition of Kato Measures relative to a subelliptic p-laplacian and we prove the Harnack inequality and the local Hölder continuity for the local harmonics relative to a Schrödinger problem for a subelliptic p-laplacian with a Kato measure as potential.

Key words and phrases: Schrödinger type problems, Subelliptic operators, Harnack inequality.
A.M.S. Classification: 31C05, 35B65, 35J60.

Misure di Kato in caso nonlineare e problemi nonlineari sottoellittici tipo Schrödinger

SUNTO. — Si definisce la classe delle misure di Kato relative a un p-laplaciano subellittico e si provano la diseguaglianza di Harnack e la hölderianità locale per le armoniche locali relative a un problema tipo Schrödinger per un p-laplaciano subellittico e con una misura di Kato come potenziale.

1. - INTRODUCTION

Let $X_i = \sum_{j=1}^{N} a_{ij} D_{x_j}$, $i = 1, \ldots, m$, $a_{ij} \in C^\infty(R^N)$, be vector fields on R^N satisfying the Hörmander condition (i.e. the vector fields X_i and their commutators up to the order k span all the diractions in R^N). We denotes $X_i^* = -\sum_{j=1}^{N} D_{x_j} a_{ij}$. In the following Xu denotes the vector $(X_1 u, \ldots, X_m u)$.

A distance relative to the vector fields X_i can be defined as

$$d(x, y) = \sup\{\phi(x) - \phi(y); \phi \in C_0^\infty(R^N), |X\phi| \leq 1 \text{ a.e.}\} \tag{1.1}$$

see [3], [4], [10], [13], [14], [16], [18].

(*) Indirizzo dell'Autore: Dipartimento di Matematica «F. Brioschi», Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy; e-mail: marbir@mate.polimi.it

(**) Memoria presentata il 4 dicembre 1997 da Marco Biroli, socio dell'Accademia.

We recall that, for x, y in a compact set K and $d(x, y) \leqslant 1$, there exists a constant $c \geqslant 1$ (that may depend on K) such that

$$\frac{1}{c}\,|x-y| \leqslant d(x, y) \leqslant c\,|x-y|^{1/(k+1)}$$

so the topology defined by d on R^N is locally equivalent to the euclidean one ([10], [13], [14], [16], [18]).

In the following we denote by $B(x, r)$ (B_r, B) the ball of center x and radius r relative to the distance d and we recall that given a compact set K there exists constants c_0, $R_0 > 0$ (that may depend on K) such that if $x \in K$, $r < R \leqslant R_0$

$$m(B(x, r)) \geqslant c_0 \left(\frac{r}{R}\right)^n m(B(x, R))\,; \tag{1.2}$$

where m denotes (here ans in the following) the Lebesgue measure on R^N the exponent n (possibly different from N) depends on k and we say that n is the *intrinsic dimension* relative to our problem ([10], [13], [14], [16], [18]).

We denote by $Q(x, r)$ the standard cubes associated with the fields X_i, $i = 1, \ldots, m$ defined in [14]. We recall that given a compact set K there exists a constant $c_1 \geqslant 1$, that may depend on K, such that $B(x, r/c_1) \subseteq Q(x, r) \subseteq B(x, c_1 r)$; moreover from Theorem 3.1 in [14] we may assume, without loss of generality, $m(\partial Q(x, r)) = 0$ for $r \leqslant R_0$.

We now define the Sobolev spaces relative to the vector fields X_i. Let Ω be a *bounded* connected open set in R^N; the space $W^{1,p}(\Omega, X)$, $p \in (1, +\infty)$, is defined as the completion of the space $C^\infty(\overline{\Omega})$ for the norm

$$\|u\|_{1,p,\Omega} = (\|u\|^p_{L^p(\Omega)} + \|Xu\|^p_{(L^p(\Omega))^m})^{1/p}\,.$$

We observe that the functions $u \in W^{1,p}(\Omega, X)$ are in $L^p(\Omega)$ and $Xu \in (L^p(\Omega))^m$.

By $W^{1,p}_{\mathrm{loc}}(\Omega, X)$ we denote the space of the functions in $W^{1,p}(A, X)$ for every open set A such that $\overline{A} \subseteq \Omega$.

The space $W^{1,p}_0(\Omega, X)$, $p \in (1, +\infty)$, is defined as the closure of the space $C_0^\infty(\overline{\Omega})$ in $W^{1,p}(\Omega, X)$.

We recall, [11], [13], [14], that, given a compact set K, the following Poincaré inequality in a ball B_r with center x in a compact set K and radius $r \leqslant \overline{R}_0$ holds

$$\int_{B_r} |u - u_r|^p\,dx \leqslant c_2 r^p \int_{B_r} |Xu|^p\,dx\,, \tag{1.3}$$

for a function $u \in W^{1,p}(B_{2r}, X)$, where the constant c_2 may depend on K (in the following we assume $R_0 = \overline{R}_0$), [13], [14].

As consequence of (1.3), the Sobolev inequalities relative to B_r and to the intrinsic dimension n and to every exponent $s \geqslant p$ also hold (see [5], [11]). Moreover if $u \in$

$\in W_0^{1,p}(B_r, X)$ the following modification of (1.3) also holds

$$\int_{B_r} |u|^p \, dx \le c_2' r^p \int_{B_r} |Xu|^p \, dx\,; \tag{1.3'}$$

so

$$\|u\|_{0,1,p,\Omega} = \|Xu\|_{L^p(\Omega)^m}$$

is an equivalent norm on $W_0^{1,p}(\Omega, X)$ (the proof is well known and founded on a contradiction argument).

The notion of *p-capacity* of a set $E \subseteq \overline{E} \subseteq \Omega$ is defined by the relation

$\operatorname{cap}_p(E, \Omega) =$

$$= \inf\left\{\int_\Omega |Xu|^p \, dx;\ u \in W_0^{1,p}(\Omega, X) \text{ with } u \ge 1 \text{ a.e. on aneighborhood of } E\right\}.$$

We can prove that for a set $E \subseteq \overline{E} \subseteq \Omega$ we have $\operatorname{cap}_p(E, \Omega) = 0$ iff $\operatorname{cap}_p(E, \Omega') = 0$ for every bouned open set Ω' such that $E \subseteq \overline{E} \subseteq \Omega$ (the proof is the same as for the usual Newtonian capacity). We say that a property is verified *p-quasi everywhere* (p-q.e.) in Ω, if the set where the property fails has zero capacity in Ω.

A function u is *p-quasicontinuous* in Ω if for every $\varepsilon > 0$ there exists a closed set E_ε with $\operatorname{cap}_p(E_\varepsilon) \le \varepsilon$ such that u is continuous in $\Omega - E_\varepsilon$. Let u be in $W^{1,p}(\Omega, X)$, there exists a locally p-quasi continuous (in Ω) function v (the p-quasicontinuous representative of u) such that $u = v$ p-q.e. (in Ω), see [12]. In the following we identify u with its p-quasi continuous representative.

We denote by $W^{-1,q}(\Omega, X)$, $1/p + 1/q = 1$, the dual space of $W_0^{1,p}(\Omega, X)$ considered as a subspace of the distributions on Ω and by $\|\cdot\|_{-1,q,\Omega}$. Let μ be a Radon measure (on Ω) in $W^{-1,q}(\Omega, X)$; then μ does not charge sets of zero p-capacity. Moreover if $u \in W_0^{1,p}(\Omega, X)$, then

$$\int_\Omega u \, d\mu = \langle \mu, u \rangle_{q,p}$$

where $\langle .,. \rangle_{q,p}$ denotes the duality between $W^{-1,q}(\Omega, X)$ and $W_0^{1,p}(\Omega, X)$ and we identify u with its p-quasi continuous representative (the proof easily follows from the density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega, X)$).

Definition 1.1: *A Radon measure μ on Ω is in the Kato space $K^q(\Omega)$ iff*

$$\lim_{r \to 0} \sup_{x \in \Omega} \int_0^r \left(|\mu|(B(x,\delta) \cap \Omega) \frac{\delta^p}{m(B(x,\delta) \cap \Omega)} \right)^{q-1} \frac{d\delta}{\delta} = 0$$

where $1/p + 1/q = 1$.

The Radon measure μ is in $K^q_{\text{loc}}(\Omega)$ if it is in $K^q(A)$ for every open set $A \subseteq \overline{A} \subseteq \Omega$.

As proved in section 2 every measure in $K^q(\Omega)$ is in $W^{-1,q}(\Omega, X)$.

We are now ready to give the results that are the object of the paper.

Let $\mu \in K^q(\Omega)$, $1/p + 1/q = 1$, $1 < p < n$; a function $u \in W^{1,p}_{loc}(\Omega, X)$ is a local solution (subsolution, supersolution) in Ω of the problem

$$\sum_{i=1}^{m} X_i^*(|Xu|^{p-2} X_i u) + \mu |u|^{p-2} u = 0 \tag{1.4}$$

if

$$\sum_{i=1}^{m} \int_{\Omega} |Xu|^{p-2} X_i u X_i v + \int_{\Omega} |u|^{p-2} uv \, d\mu = (\leq, \geq) 0$$

for every $v \in W^{1,p}(\Omega, X)$ (positive) with compact support in Ω.

THEOREM 1.1: *Let u be a local positive solution in Ω of* (1.4). *There exists a constant $\overline{R}$ such that for every ball $B(x, r) \subsetneq B(x, 8r) \subsetneq \Omega$ with $r \leq \overline{R}$ we have*

$$\sup_{B(x,r)} u \leq C \inf_{B(x,r)} u$$

where C is a constant depending only on p, c_0, c_1, c_2.

THEOREM 1.2: *Let u be a local solution in Ω of* (1.4). *Then $u \in C(\Omega)$.*

Moreover if $\mu(B(x, r)) \leq c(x) r^{n-p+\varepsilon}$, where $c(x)$ is a continuous function defined on Ω and $r \leq \overline{R}'$, then u is locally Hölder continuous.

We recall that the definition and the properties of a Kato measure relative to a Dirichlet form have been given in [6] and in [19] the corresponding Schrödinger problem is studied.

In section 2 we give some preliminary results, that are interesting in itself. We observe that the proof of Lemma 2.2 follows by a refinement of the method used in [1] and [20] to prove analogous results in the framework of euclidean spaces or of polynomial Lie groups. In section 3 we prove Theorems 1.1 and 1.2.

We finally remark that the results in Theorems 1.1, 1.2 have been announced at WCNA 96, Athens, July 1996.

REMARK 1.1: We observe that the results in Theorems 1.1 and 1.2 have a generalization to the case of operators

$$\sum_{i=1}^{m} X_i^* A_i(x, u, Xu)$$

where

$$|A_i(x, s, t)| \leq K_1 + K_2 |t|^p,$$

$$(A_i(x, s, t) - A_i(x, s, t')) \cdot (t - t') \geq K_3 |t - t'|^p$$

and K_1, K_2, K_3 are positive constants.

We will also remark that the result in Lemma 2.9 generalizes previous results [11], where the nonlinear elliptic operator are considered and the assumption $\mu(B(x,r)) \leq$ $\leq c_2(x) r^{n-p+\varepsilon}$, $r \leq \overline{R}'$, is taken into account.

2. - Preliminaries

We observe that the results in Theorems 1.1 and 1.2 are of local type; then we may assume, without loss of generality, that a ball B^* with center in the support of μ is given such that, if we denote by R_0 (c_0) the corresponding radius (constant) in (1.2), the support of μ is contained in the ball $(1/100)B$ where B is a ball with radius less than $400\,D$ with the same center as B^* (D denotes the diameter of the support of μ and we assume $c_1 10^4\ D \leq R_0$).

We denote

$$I(x,y) = \phi(d(x,y)) \frac{d(x,y)}{m(B(x, d(x,y)))} \quad \text{and} \quad I\mu(x) = \int I(x,y)\, d\mu(y),$$

where $\phi(d(x,y))$ is a function such that $\phi \in C^\infty$, $\phi(t) = \phi(-t)$, $\phi(t) = 1$ for $0 < t < 100\ D$, $\phi(t) = 0$ for $t > 200\,D$ and $\phi'(t) \leq 200/D$.

We observe that $I\mu(x) = 0$ if $x \in B^c$ (E^c denotes the complement of the set E).

Lemma 2.1: *Let μ be a positive Radon measure such that the preceding assumptions hold; then*

$$\int (I\mu(x))^q\, dx < +\infty \tag{2.1}$$

implies $\mu \in W^{-1,q}(R^N, X)$, $q > n/n-1$.

Proof: We recall that a positive Radon measure μ with compact support can be approximated by a sequence of positive Radon measures in $L^q(R^N)$ with support in a neighbourhood of $\operatorname{supp}(\mu)$. Then it is enough to prove that

$$\|\mu\|^q_{-1,q} \leq C \int (I\mu(x))^q\, dx\,, \tag{2.2}$$

where μ (with compact support K) has a density in $L^q(R^N, X)$ and C is an increasing function of K, i.e. $C(K_1) \leq C(K_2)$ if K_1 and K_2 are compact sets with $K_1 \subsetneq K_2$ ($\|\mu\|_{-1,q} = \|\mu\|_{-1,q,R^N}$).

Let $G(x,y)$ be the fundamental solution in a bounded open set A of the operator $\sum_{i=1}^{m} X_i^* X_i$ [5]; then if $2B \subsetneq\subsetneq A$ and $(x,y) \in B \times B$, if the radius of B is less than $50\,D$, we have

$$|XG(\cdot, y)| \leq C I(x,y).$$

Let u be the weak solution of

$$\sum_{i=1}^{m} X_i^* X_i u = \mu, \quad \text{in } A; \quad u \in W_0^{1,2}(A, X).$$

Then

$$|Xu(x)| \leq C I\mu(x)$$

for $x \in B$ (where C is the constant corresponding to B), so

$$\|\mu\|_{-1,q} \leq \|Xu\|_{L^p(B)} \leq C \int (I\mu(x))^p \, dx . \quad \blacksquare$$

Lemma 2.2: *Let the preceding assumptions on μ hold; then*

$$\int \left(\int_0^{800D} \left(|\mu|(B(x,\delta)) \frac{\delta^p}{m(B(x,\delta))} \right)^{q-1} \frac{d\delta}{\delta} \right) d|\mu|(x) < +\infty$$

implies $\mu \in W^{-1,q}(R^N, X)$, $q > n/(n-1)$, and

$$\|\mu\|_{-1,q}^{q} \leq C \int \left(\int_0^{800D} \left(|\mu|(B(x,\delta)) \frac{\delta^p}{m(B(x,\delta))} \right)^{q-1} \frac{d\delta}{\delta} \right) d|\mu|(x)$$

where C is a constant, that depends increasingly on the diameter D of the support of μm and $1/p + 1/q = 1$.

Proof: It is enough to prove our result in the case μ positive.
Let us denote

$$M\mu(x) = \sup_{0<r<400D} \mu(B(x,r)) \frac{r}{m(B(x,r))} .$$

At first we prove that the result follows from the estimate

$$\int (I\mu(x))^p \, dx = \int_B (I\mu(x))^p \, dx \leq C_1 \int_B M\mu(x)^p \, dx . \tag{2.3}$$

Define

$$J_q \mu(x) = \left[\int_0^{400D} \left(\frac{\mu(B(x,\varrho))}{m(B(x,\varrho))} \varrho \right)^q \frac{d\varrho}{\varrho} \right]^{1/q} .$$

Let x be in B; then we have

$$J_q\mu(x) \geq \left[\int_\delta^{2\delta} \left(\frac{\mu(B(x,\varrho))}{m(B(x,\varrho))}\,\varrho\right)^q \frac{d\varrho}{\varrho}\right]^{1/q} \geq \frac{2^n}{c_0}\,\frac{\mu(B(x,\delta))}{m(B(x,\delta))}\,\delta(\log 2)^{1/q}$$

for every $0 < \delta < 400\,D$. Then

$$J_q\mu(x) \geq CM\mu(x)\,. \tag{2.4}$$

From (2.3) we obtain

$$\int_B (I\mu(x))^q\,dx \leq \int_B (J_q\mu(x))^q\,dx\,. \tag{2.5}$$

We estimate now the term in the right hand side of (2.5).

Let $x \in B$ for every $\varrho \leq 400\,D$ we have

$$\int\left(\frac{\mu(B(x,\varrho))}{m(B(x,\varrho))}\right)^q dx = \int\left(\frac{\mu(B(x,\varrho))^{q-1}}{m(B(x,\varrho))^q}\int_{d(x,y)<\varrho} d\mu(y)\right)dx =$$

$$= \int\left(\int_{d(x,y)<\varrho} \frac{\mu(B(x,\varrho))^{q-1}}{m(B(x,\varrho))^q}\,dx\right)d\mu(y) \leq C\int\left(\int_{d(x,y)<\varrho}\frac{\mu(B(x,\varrho))^{q-1}}{m(B(x,4\varrho))^q}\,dx\right)d\mu(y) \leq$$

$$\leq C\int\left(\int_{d(x,y)<\varrho}\frac{\mu(B(x,\varrho))^{q-1}}{m(B(y,\varrho))^q}\,dx\right)d\mu(y) \leq C\int\frac{\mu(B(y,2\varrho))^{q-1}}{m(B(y,2\varrho))^{q-1}}\,d\mu(y)$$

where C denotes (here and in the following) possibly different constants depending on c_0, and n.

Then

$$\int_B (J_q\mu(x))^q\,dx = \int_B dx \int_0^{400\,D}\left(\frac{\mu(B(x,\varrho))}{m(B(x,\varrho))}\,\varrho\right)^q \frac{d\varrho}{\varrho} = \tag{2.6}$$

$$= \int_0^{400\,D}\left[\int_B\left(\frac{\mu(B(x,\varrho))}{m(B(x,\varrho))}\right)^q dx\right]\varrho^{q-1}\,d\varrho \leq C\int_0^{400\,D}\left[\int\left(\frac{\mu(B(x,2\varrho))}{m(B(x,2\varrho))}\right)^{q-1} d\mu(x)\right]\varrho^{q-1}\,d\varrho \leq$$

$$\leq C\int d\mu(x)\int_0^{400\,D}\left(\frac{\mu(B(x,\varrho))}{m(B(x,\varrho))}\,\varrho^{p}\right)^{q-1}\frac{d\varrho}{\varrho}\,.$$

From (2.4), (2.5) and (2.6) the result of Lemma 2.2.

To end the proof we have to prove (2.3). At first we prove that (2.3) is consequence of the following property:

(H) there exists $a > 1$ and b such that for every $\lambda > 0$ and $0 < \varepsilon \leq 1$ we have

$$m(\{I\mu(x) > a\lambda\}) \leq b\varepsilon^{\alpha/(n-1)} m(\{I\mu(x) > \lambda\}) + m(\{x \in B;\, M\mu(x) > \varepsilon\lambda\}).$$

We multiply (H) by λ^{q-1} and we integrate on $(0, R)$

$$\int_0^R m(\{I\mu(x) > a\lambda\})\lambda^{q-1}\, d\lambda \leq$$

$$\leq b\varepsilon^{\alpha/(n-1)} \int_0^R m(\{I\mu(x) > \lambda\})\lambda^{q-1}\, d\lambda + \int_0^R m(\{x \in B;\, M\mu(x) > \varepsilon\lambda\})\lambda^{q-1}\, d\lambda.$$

By a change of variables we obtain

$$a^{-q} \int_0^R m(\{I\mu(x) > a\lambda\})\lambda^{q-1}\, d\lambda \leq$$

$$\leq b\varepsilon^{\alpha/(n-1)} \int_0^R m(\{I\mu(x) > \lambda\})\lambda^{q-1} d\lambda + \varepsilon^{-q} \int_0^R m(\{x \in B;\, M\mu(x) > \lambda\})\lambda^{q-1}\, d\lambda.$$

We choose ε such that $b\varepsilon^{\alpha/(n-1)} \leq (1/2)a^{-q}$ so we have

$$a^{-q} \int_0^R m(\{I\mu(x) > a\lambda\})\lambda^{q-1} d\lambda \leq 2\varepsilon^{-q} \int_0^R m(\{x \in B;\, M\mu(x) > \lambda\})\lambda^{q-1} d\lambda.$$

We go to the limit as $r \to +\infty$ and we obtain

$$a^{-q} \int (I\mu(x))^q\, dx \leq 2\varepsilon^{-q} \int_B (M\mu(x))^q.$$

We end the proof by proving that (H) holds.

We denote

$$I'\mu(x) = \int I'(x, y)\, d\mu(y)$$

where $I'(x, y) = \phi(d(x, y)) \dfrac{d(x, y)}{m(Q(x, d(x, y)))}$; then there exists constants C_2 and C_3 such that

$$C_2 I'\nu(x) \leq I\nu(x) \leq C_3 I'\nu(x)$$

where ν satisfies the same assumptions as μ and the constants C_2 and C_3 does not depend on $N\nu$; then it is enough to prove (H) replacing I by I'.

The set $\{I'\mu > \lambda\}$, $\lambda > 0$, is open (due to lower semicontinuity of $I'\mu(x)$) and with closure contained in $(1/2)B$; then there exists a Whitney covering of the set by balls $B(x_i, r_i) = B_i$), such that every point x in the set is covered by at most Q balls and such that for every B_i there exists a point x with $d(x, B_i) \leq 8r_i$ and $I'\mu(x) \leq \lambda$.

Consider a ball $B' \in \{B_i\}$ with radius r and the set $\{x \in B'; I'\mu > a'\lambda\}$, $a' > 1$; suppose that its intersection with the set $\{M\mu \leq \varepsilon\lambda\}$ is not empty.

Denote $P = 24B'$ and by μ_1 the restriction of μ to P and by μ_2 the measure $\mu - \mu_1$.

From Theorem 2.1 pg. 71 [9] using the same methods as in Theorem 1.1.1 and Propositions 3.1.2, 3.1.4 in [1] we get

$$m\left(\left\{x \in B'; I'\mu_1 > \frac{a'\lambda}{2}\right\}\right) \leq m\left(\left\{x \in B'; I\mu_1 > \frac{a'\lambda}{2C_3}\right\}\right) \leq A\left(\frac{1}{a'\lambda}\int d\mu_1\right)^{n/(n-1)}$$

Let x_0 be a point in B' such that $M\mu(x_0) \leq \varepsilon\lambda$ and denote by $B(x_0)$ the ball with center x_0 and radius $30r$. We have $P \subsetneq B(x_0)$; then

$$\int d\mu_1 \leq \int_P d\mu \leq \int_{B(x_0)} d\mu \leq C_4 M\mu(x_0)\, m(B(x_0))^{(n-1)/n} \leq C_4\varepsilon\lambda m(B(x_0))^{(n-1)/n}.$$

Hence

$$\left(\frac{1}{a'\lambda}\int d\mu_1\right)^{n/(n-1)} \leq C_4\left(\frac{\varepsilon}{a'}\right)^{n/(n-1)} m(B(x_0)) \leq C_5\left(\frac{\varepsilon}{a'}\right)^{n/(n-1)} m(B').$$

Then there exists b' such that

$$m\left\{x \in B'; I'\mu_1 > \frac{a'\lambda}{2}\right\} \leq b'\varepsilon^{n/(n-1)} m(B').$$

If $r < D$ and x_1 is such that $d(x_1, B') \leq 8r$; there exists a constant L such that $(1/10)d(x, y) \leq d(x_1, y) \leq 10d(x, y) \leq 20D$ for every $y \in P^c$ and $x \in B'$. If in addition, we have $I'\mu(x_1) \leq \lambda$ then

$$I'\mu_2(x) \leq \frac{1}{C_2} I\mu_2(x) \leq C_6 I\mu_2(x_1) \leq C_7 I'\mu(x_1) \leq C_7\lambda.$$

Choose $a' > 2C_7$; then $I'\mu_2(x) \leq a'\lambda/2$. If $r \geq D$ we obtain easily that $I'\mu_2(x) \leq M\mu(x_0)$ then for a suitable constant C_7 the preceding estimate hold again.

If $I'\mu(x) > a'\lambda$ we obtain $I'\mu_1(x) > a'\lambda/2$; then either

$$B' \subsetneq \{x \in B'; M\mu > \varepsilon\lambda\}$$

or

$$\{x \in B' ; I'\mu > a'\lambda\} \subsetneq \left\{x \in B' ; I'\mu_1 > \frac{a'\lambda}{2}\right\}.$$

In the second case we obtain

$$m(\{x \in B' ; I'\mu > a'\lambda\}) \leq b' \varepsilon^{n/(n-1)} m(B).$$

Summing up over $B' \in \{B_i\}$ we obtain

$$m(\{I'\mu(x) > a'\lambda\}) \leq b' \varepsilon^{n/(n-1)} m(\{I'\mu(x) > \lambda\}) + m(\{M\mu(x) > \varepsilon\lambda\})$$

and the result follows for $a = a' C_3 / C_2$, $b = b' C_2^{-n/(n-1)}$. ■

From Lemma 2.2 by a covering argument we obtain:

Lemma 2.2': *Let Let μ be a Radon measure on Ω and $D = \operatorname{diam}(\Omega)$; then*

$$\int \left(\int^{D} \left(|\mu|(B(x,\delta)) \frac{\delta^p}{m(B(x,\delta))} \right)^{q-1} \frac{d\delta}{\delta} \right) d|\mu|(x) < +\infty$$

implies $\mu \in W^{-1,q}(R^N, X)$, $q > n/(n-1)$, and

$$\|\mu\|^q_{-1,q} \leq C \int \left(\int_0^{D} \left(|\mu|(B(x,\delta)) \frac{\delta^p}{m(B(x,\delta))} \right)^{q-1} \frac{d\delta}{\delta} \right) d|\mu|(x)$$

where C is a constant, that depends increasingly on D and $1/p + 1/q = 1$.

Fom Lemma 2.2' we have:

Corollary 2.3: *Let μ be a Radon measure on Ω and assume $\mu \in K^q(\Omega)$; then $\mu \in W^{-1,q}(\Omega, X)$ and*

$$\|\mu\|^q_{-1,q,\Omega} \leq C|\mu|(\Omega) \left(\int_0^{\infty} \left(|\mu|(B(x,\delta) \cap \Omega) \frac{\delta^p}{m(B(x,\delta) \cap \Omega)} \right)^{q-1} \frac{d\delta}{\delta} \right) d\mu(x),$$

where C does not depend on Ω, but depends on D.

Consider now a function u such that

$$\sum_{i=1}^{m} \int_{\Omega} |Xu|^{p-2} X_i u X_i v \, dx = 0 \tag{2.4}$$

where $u \in W^{1,p}_{\mathrm{loc}}(\Omega, X)$ $v \in W^{1,p}(\Omega, X)$ with $\operatorname{supp}(v) \subsetneq \Omega$. We recall that the cut-off function ϕ between balls $B(x,t)$ and $B(x,s)$, $s < t < R_0/2$, in Ω satisfying the estimate

$$|X\phi| \leq \frac{C}{t-s} \quad \text{a.e. in } B(x,t)$$

has been constructed in [3], [4]. Moreover Sobolev and John-Nirenberg inequalities (relative to the dimension n) have been proved respectively in [5], [8], [11] and in [7]; so using the same methods as in [21] we can prove the following Harnack inequality:

Lemma 2.4: *There exists a constant* $\overline{R}$ *such that if* u *is a positive function satisfying* (2.4); *then for every ball* $B(x,r) \subseteq \overline{B(x, 20r)} \subseteq \Omega$, $2r < \overline{R}$, *we have*

$$\sup_{B(x,r)} u \leq C \inf_{B(x,r)} u \,.$$

See also [8] for the proof.

From the Lemma 2.4 we obtain, by standard methods:

Lemma 2.5: *There exists a constant* $\overline{R}$ *such that if* u *is a function satisfying* (2.4) *and* $\omega(x,r)$ *be the oscillation of* u *in* $B(x,r)$; *then, for* $B(x,\varrho) \subseteq B(x, 2\varrho) \subseteq B(x,R) \subseteq \subseteq \overline{B(x, 4R)} \subseteq \Omega$, $4R < \overline{R}$, *we have*

$$\omega(x,\varrho) \leq C\left(\frac{\varrho}{R}\right)^{\alpha} \omega(x,R)$$

where α *is a suitable constant in* $(0,1)$.

See also [8] for the proof.

Using a Caccioppoli type estimate and the Poincaré inequality we obtain the following energy estimate:

Lemma 2.6: *There exists a constant* $\overline{R}$ *such that if* u *is a function satisfying* (2.4); *then, for* $B(x,r) \subseteq B(x,2r) \subseteq B(x,R) \subseteq \overline{B(x, 4R)} \subseteq \Omega$, $4R < \overline{R}$, *we have*

$$\int_{B(x,r)} |Xu|^p \, dx \leq C\left(\frac{r}{R}\right)^{\beta - p} \frac{m(B(x,r))}{m(B(x,R))} \int_{B(x,R)} |Xu|^p \, dx \,.$$

Let $\mu \in K^q(\Omega)$, $1/p + 1/q = 1$; the function $u \in W^{1,p}_{\mathrm{loc}}(\Omega, X)$ is a solution (subsolution) in Ω of the problem

$$\sum_{i=1}^{m} X_i^*(|Xu|^{p-2} X_i u) = \mu \tag{2.5}$$

if

$$\sum_{i=1}^{m} \int_{\Omega} |Xu|^{p-2} X_i u X_i v \, dx = (\leq) \int_{\Omega} v \, d\mu \tag{2.6}$$

for every $v \in W^{1,p}(\Omega, X)$ (positive) $\operatorname{supp}(v) \subseteq \Omega$.

Following the same techniques as in [15] we can prove:

LEMMA 2.7: *There exists a constant $\overline{R}$ such that if u is a positive subsolution of (2.5); then*

$$\sup_{B(x,r)} u \leq C \left[\left(\frac{1}{m(B(x,2r))} \int_{B(x,2r)} u^p \, dx \right)^{1/p} + \right.$$
$$\left. + \sup_{y \in B(x,r)} \int_0^{2r} \left[\frac{\varrho^p}{m(B(y,\varrho))} |\mu| (B(y,\varrho)) \right]^{q-1} \frac{d\varrho}{\varrho} \right]$$

where $B(x,r) \subseteq \overline{B(x,2r)} \subseteq \Omega$, $2r < \overline{R}$.

An immediate consequence of Lemma 2.7 is the following result:

LEMMA 2.8: *There exists a constant $\overline{R}$ such that if u is a solution of (2.5); then*

$$\sup_{B(x,r)} |u| \leq C \left[\left(\frac{1}{m(B(x,2r))} \int_{B(x,2r)} |u|^p \, dx \right)^{1/p} + \right.$$
$$\left. + \sup_{y \in B(x,r)} \int_0^{2r} \left[\frac{\varrho^p}{m(B(y,\varrho))} |\mu| (B(y,\varrho)) \right]^{q-1} \frac{d\varrho}{\varrho} \right]$$

where $B(x,r) \subseteq \overline{B(x,2r)} \subseteq \Omega$, $2r < \overline{R}$.

We apply the Lemma 2.8 to $(u - u_{2r})$, where u_{2r} is the average of u on $B(x, 2r)$, and by the Poincaré inequality we obtain:

LEMMA 2.9: *There exists a constant $\overline{R}$ such that if u is a solution of (2.5); then*

$$\operatorname{osc}(u, B(x,r)) \leq C \left[\left(\frac{r^p}{m(B(x,2r))} \int_{B(x,2r)} |Xu|^p \, dx \right)^{1/p} + \right. \tag{2.7}$$
$$\left. + \sup_{y \in B(x,r)} \int_0^{2r} \left[\frac{\varrho^p}{m(B(y,\varrho))} |\mu| (B(y,\varrho)) \right]^{q-1} \frac{d\varrho}{\varrho} \right]$$

where $B(x,r) \subseteq \overline{B(x,2r)} \subseteq \Omega$, $2r < \overline{R}$.

We have now to estimate the first term in the right hand side of (2.7):

LEMMA 2.10: *There exists a constant* $\overline{R}$ *such that if* u *is a solution of* (2.5); *then*

$$\frac{r^p}{m(B(x,r))}\int_{B(x,r)}|Xu|^p\,dx) \leq C\left[\left(\frac{r}{R}\right)^{\beta}\frac{R^p}{m(B(x,R))}\int_{B(x,R)}|Xu|^p\,dx) + \right.$$

$$\left. +\left(\sup_{y\in B(x,4R)}\int_0^{10^4 R}\left[\frac{\varrho^p}{m(B(y,\varrho))}\,|\mu|(B(y,\varrho))^{q-1}\,\frac{d\varrho}{\varrho}\right]\right)^{p}\right]$$

where $B(x,R)\subsetneq\overline{B(x,8R)}\subsetneq\Omega$, $8R<\overline{R}$.

PROOF: Let $w\in W^{1,p}(B(x,4R),X)$ be the solution of the problem

$$\sum_{i=1}^{m}\int_{\Omega}|Xw|^{p-2}X_iwX_iv\,dx=0$$

where $v\in W_0^{1,p}(B(x,4R),X)$ and $w-u\in W_0^{1,p}(B(x,4R),X)$.

From Lemma 2.6 we have

$$\int_{B(x,r)}|Xw|^p\,dx\leq C\left(\frac{r}{R}\right)^{\beta-p}\frac{m(B(x,R))}{m(B(x,r))}\int_{B(x,R)}|Xw|^p\,dx\,. \tag{2.8}$$

Let $g=w-u$; we have

$$\left[\int_{B(x,4R)}|Xg|^p\,dx\right]^{1/p}\leq C_1\|\mu\|^{q-1}_{W^{-1,p'}(B(x,4R),X)}\leq$$

$$\leq C_2\,\mu(B(x,4R))^{1/p}\left(\sup_{y\in B(x,4R)}\int_0^{10^4 R}\left[\frac{\varrho^p}{m(B(y,\varrho))}\,\mu(B(y,\varrho))\right]^{q-1}\frac{d\varrho}{\varrho}\right)^{1/p}$$

where we have used the Lemma 2.2.

Then

$$\left[\frac{(4R)^p}{m(B(x,4R))}\int_{B(x,4R)}|Xg|^p\,dx\right]^{1/p}\leq C_3\left(\mu(B(x,4R))\,\frac{(4R)^p}{m(B(x,4R))}\right)^{q-1}+ \tag{2.9}$$

$$+C_3\sup_{y\in B(x,4R)}\int_0^{10^4 R}\left(\frac{\varrho^p}{m(B(y,\varrho))}\,\mu(B(y,\varrho))^{q-1}\,\frac{d\varrho}{\varrho}\right)\leq$$

$$\leq\sup_{y\in B(x,4R)}\int_0^{10^4 R}\left(\frac{\varrho^p}{m(B(y,\varrho))}\,\mu(B(y,\varrho))^{q-1}\,\frac{d\varrho}{\varrho}\right).$$

From (2.8), (2.9) we obtain easily the result. ■

From Lemma 2.9 and 2.10 we obtain:

LEMMA 2.11: *There exists a constant $\overline{R}$ such that if u is a solution of (2.5); then*

$$\operatorname{osc}(u, B(x,r)) \leqslant C\left[\left(\frac{r}{R}\right)^{\beta} \frac{(R)^{p}}{m(B(x,R))} \int\limits_{B(x,R)} |Xu|^{p}\,dx) + \right.$$

$$\left. + \left(\sup_{y \in B(x,4R)} \int\limits_{0}^{10^{p}R} \left[\frac{\varrho^{p}}{m(B(y,\varrho))}\, |\mu|(B(y,\varrho))\right]^{1/(p-1)} \frac{d\varrho}{\varrho}\right)^{p}\right]$$

where $B(x,R) \subsetneq \overline{B(x,16R)} \subsetneq \Omega$, $16R < \overline{R}$.

An easy consequence of the Lemma 2.11 is the following result:

COROLLARY 2.12: *Let u be a solution of (2.5); then $u \in C(\Omega)$. Moreover if $\mu(B(x,r)) \leqslant C r^{n-p+\varepsilon}$, $\varepsilon > 0$, $200r < \overline{R}'$, $B(x,r) \subsetneq B(x,2r) \subsetneq \Omega$, and $20r < \overline{R}$; then u is localy Hölder continuous in Ω.*

We prove now a fundamental embedding result.

LEMMA 2.13: *Let μ be in $K^{q}(\Omega)$, $\overline{B(x,40t)} \subsetneq \Omega$ and $\varepsilon > 0$, there exists $\overline{R}$ such that for every $\varepsilon > 0$ and $0 < s < t \leqslant \overline{R}$*

$$\int\limits_{B(x,t)} |u|^{p}\, d|\mu| \leqslant \varepsilon \int\limits_{B(x,t)} |Xu|^{p}\,dx + \frac{C_{\varepsilon}}{(t-s)^{p}} \int\limits_{B(x,t)-B(x,s)} |u|^{p}\,dx$$

where $0 < s < t \leqslant \overline{R}$, $u \in W^{1,p}(\Omega, X)$ and C_{ε} is a constant depending only on ε.

PROOF: We use the same methods as in [2].
We can assume, without loss of generality $\mu \geqslant 0$, $u \in L^{\infty}(\Omega)$.
Let w be the weak solution of the problem

$$\sum_{i=1}^{m} X_i^{*}(|Xw|^{p-2} X_i w) = \mu \quad \text{in } B(x, 400t),$$

$$w = 0 \quad \text{on } \partial B(x, 40t)$$

and ϕ be the cut-off function between the balls $B(t,x)$ and $B(s,x)$; we have

$$|X\phi| \leqslant \frac{C_1}{(t-s)} \quad \text{a.e. in } B(x,t)$$

(see [2] for the existence of cut-off functions between balls and for the estimate on $|X\phi|$).

We have

$$(2.10)\quad \int_{B(x,t)} |u|^p\, d\mu \le \int_{B(x,t)} |u|^p \phi^p\, d\mu = \sum_{i=1}^{m} \int_{B(x,t)} |Xw|^{p-2} X_i w X_i(|u|^p \phi^p)\, dx \le$$

$$\le p \sum_{i=1}^{m} \int_{B(x,t)} u^{p-1}\phi^p |Xw|^{p-2} X_i w X_i u\, dx + p \sum_{i=1}^{m} \int_{B(x,t)} u^p \phi^{p-1} |Xw|^{p-2} X_i w X_i \phi\, dx \le$$

$$\le \frac{\varepsilon}{2} \int_{B(x,t)} |Xu|^p \phi^p\, dx + \frac{C_2}{\varepsilon}\left[\int_{B(x,t)} u^p |X\phi|^p\, dx + \int_{B(x,t)} u^p \phi^p |Xw|^p\, dx \le\right.$$

$$\le \frac{\varepsilon}{2} \int_{B(x,t)} |Xu|^p \phi^p\, dx + \frac{C_3}{\varepsilon}\left[\frac{1}{(t-s)^p} \int_{B(x,t)} u^p\, dx + \int_{B(x,t)} u^p \phi^p |Xw|^p\, dx\right..$$

We now estimate the term

$$\int_{B(x,t)} u^p \phi^p |Xw|^p\, dx .$$

We have

$$\int_{B(x,t)} u^p \phi^p |Xw|^p\, dx = \sum_{i=1}^{m} \int_{B(x,t)} |Xw|^{p-2} X_i w X_i (w u^p \phi^p)\, dx +$$

$$+ p \sum_{i=1}^{m} \int_{B(x,t)} w u^{p-1} \phi^{p-1} |Xw|^{p-2} X_i w X_i (u\phi)\, dx \le$$

$$\le \int_{B(x,t)} w u^p \phi^p\, dx - p \sum_{i=1}^{m} \int_{B(x,t)} w u^p \phi^{p-1} |Xw|^{p-2} X_i w X_i \phi\, dx -$$

$$- \sum_{i=1}^{m} \int_{B(x,t)} w u^{p-1} \phi^p |Xw|^{p-2} X_i w X_i u\, dx \le$$

$$\le \eta(t) \int_{B(x,t)} u^p \phi^p\, d\mu + \eta(t) \int_{B(x,t)} u^p \phi^p |Xw|^p\, dx +$$

$$+ \frac{C_4}{\eta(t)} \int_{B(x,t)} w^p (|Xu|^p \phi^p + |X\phi|^p u^p)\, dx .$$

where $\eta(t) = \sup_{B(x,\,8t)} w$. Then taking into account the results in Lemma 2.11, Corollary 2.12 we choose $\bar{R}$ such that for $t \le \bar{R}$ we have $\eta(t) \le \inf(1/2, (\varepsilon^2/8C_3C_4)^{1/(p-1)}$ (we assume $C_3, C_4 \geq 1$, $\varepsilon < 1$), we obtain

$$\int_{B(x,t)} u^p \phi^p |Xw|^p \, dx \leq$$

$$\leq \frac{\varepsilon}{2C_3} \left[\int_{B(x,t)} u^p \phi^p \, d\mu + \frac{\varepsilon^2}{4C_3} \int_{B(x,t)} |Xu|^p \, dx + C_5 \frac{1}{(t-s)^p} \int_{B(x,t)-B(x,s)} u^p \, dx \right.$$

using in (2.10) the above estimate we obtain the result. ■

3. - Proof of Theorems 1.1 and 1.2

We are now ready for the proof of Theorem 1.1:

Proof of Theorem 1.1: The first step in the proof uses the Moser iteration method to prove that for a positive subsolution of the problem (1.4) the following estimate holds

$$\sup_{B(x,r)} u \leq C_1 \left(\frac{1}{m(B(x,2r))} \int_{B(x,2r)} u^q \right)^{1/q} \tag{3.1}$$

where $q > 0$, $B(x, 2r) \subseteq \Omega$, $r \leq \bar{R}$ ($\bar{R}$ suitable) and C_1 depends on p.

Consider now the function $\log u$; by the same methods in [2] we prove that for every positive local solution of (1.4) the following estimate hold

$$\int_{B(x,r)} X(\log u) \, dx \leq C_2 \left(1 + \mu((B(x,2r)) \frac{r^p}{m(B(x,r))} \right) \frac{m(B(x,r))}{r^p} \tag{3.2}$$

where $B(x, 2r) \subseteq \Omega$ and C_2 depends unly on p.

We observe that the term $\mu(B(x,2r)) \dfrac{r^p}{m(B(x,r))}$ is bounded (we use the assumption $\mu \in K^{p'}(\Omega)$); then, using the Poincaré inequality we have

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |\log u - (\log u)_r|^p \, dx \leq C_4 \,. \tag{3.3}$$

As in [2] using (3.3) we prove that there exists $\gamma > 0$ such that

$$\left(\frac{1}{m(B(x,r))} \int_{B(x,r)} u^\gamma \right) \left(\frac{1}{m(B(x,r))} \int_{B(x,r)} u^{-\gamma} \right) \leq C_5 \tag{3.4}$$

where $B(x, 2r) \subseteq \Omega$, $r \leq \bar{R}$, and C_5 depends only on C_4.

Let now $v = u + \varepsilon$, $\varepsilon > 0$, where u is a positive local solution of (1.4); then also v is a positive local solution of (1.4) and $v \geqslant \varepsilon$.

We observe thar $1/v$ is a positive subsolution of (1.4); then by (3.1), (3.4)

$$\inf_{B(x,r)} v \geqslant C_1 \left(\frac{1}{m(B(x,2r))} \int_{B(x,2r)} v^{-\gamma} \right)^{-1/\gamma} \geqslant C_6 \int_{B(x,2r)} v^{\gamma})^{1/\gamma} \geqslant C_7 \sup_{B(x,r)} v$$

where C_7 depends on C_1 and C_5 but does not depend on ε.

Taking $\varepsilon \to 0$ we obtain our result. ■

Proof of Theorem 1.2: Let u be a local solution of (3.4). From (3.1) applied to u^+ and u^- we obtain that

$$\sup_{B(x,r)} u \leqslant C_1 \left(\frac{1}{m(B(x,2r))} \int_{B(x,2r)} u^p \right)^{1/p}$$

so u is locally bounded in Ω. Then $u\mu$ is a measure in $K^{p'}_{\mathrm{loc}}(\Omega)$, so from Corollary 2.12 we obtain the result.

REFERENCES

[1] D. R. Adams - L. I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren für Mathematik, 314, Springer, New York, Heidelberg, Tokyo (1996).

[2] M. Biroli, *An embedding property of Schechter type for Dirichlet forms on an homogeneous space*, Rend. Ist. Lombardo A, 128 (1994), 247-255.

[3] M. Biroli - U. Mosco, *Formes de Dirichlet et estimations structurelles dans les milieux discontinus*, C.R. Acad. Sciences Paris, 313 (1991), 593-598.

[4] M. Biroli - U. Mosco, *A Saint Venant type principle for Diriclet forms on discontinuous media*, Ann. Mat. Pura Appl., 169 (IV) (1995), 125-181.

[5] M. Biroli - U. Mosco, *Sobolev inequalities on homogeneous spaces*, Potential Analysis, 4 (1995), 311-324.

[6] M. Biroli - U. Mosco, *Kato spaces for Dirichlet formes*, Potential Analysis, in print.

[7] N. Burger, *Espace de fonctions à variation moyenne bornée sur un espace de nature homogène*, C.R. Acad. Sciences Paris, 286 (1978), 139-142.

[8] L. Capogna - D. Danielli - N. Garofalo, *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations*, Commun. in P.D.E., 18 (1993), 1765-1794.

[9] R. R. Coifman - G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math., 242, Springer-Verlag, Berlin, Heidelberg, New York (1971).

[10] C. L. Fefferman - A. Sanchez Calle, *Fundamental solution for second order subelliptic operators*, Ann. Math., 124 (1986), 247-272.

[11] B. Franchi - G. Lu - R. L. Wheeden, *Weighted Poincaré inequalities for Hörmander vector fields and local regularity for a class of degenerate elliptic equations*, Potential Analysis, 4 (1995), 361-377.

[12] M. Fukushima - Y. Oshima - M. Takeda, *Dirichlet Forms and Markov Processes*, W. de Gruyter & Co., Berlin, Heidelberg, New York (1994).

[13] D. Jerison, *The Poincaré inequality for vector fields satisfying an Hörmander's condition*, Duke Math. J., 53 (1986), 503-523.
[14] D. Jerison - A. Sanchez Calle, *Subelliptic Second Order Differential Operators*, Lecture Notes in Math., 1277, Springer, New York, Heidelberg, Tokyo (1987), 46-77.
[15] J. Malý, *Pointwise estimates of nonnegative subsolutions of quasilinear elliptic equations at irregular points*, Comm. Math. Univ. Carolinae, 37 (1996), 23-42.
[16] A. Nagel - E. Stein - S. Weinger, *Balls and metrics defined by vector fields I: Basic properties*, Acta Math., 155 (1985), 103-147.
[17] J. M. Rakotoson - W. P. Ziemer, *Local behavior of solutions of quasilinear elliptic equations with general structure*, Trans. A.M.S., 319 (1990), 747-764.
[18] A. Sanchez Calle, *Fundamental solutions and geometry of square of vector fields*, Inv. Math., 78 (1984), 143-160.
[19] E. Sartori, *Local regularity for solutions to Schrödinger equations relative to Dirichlet forms*, Rend. Accad. Naz. Sci. XL, Mem. Mat. Appl., XIX (1) (1995), 169-189.
[20] S. K. Vodop'yanov, *Potential theory on homogeneous groups*, Math. USSR Sbornik, 66 (1990), 59-81.
[21] N. S. Trudinger, *On Harnack inequalities and their applications to quasilinear elliptic equations*, Comm. Pure Appl. Math., 20 (1967), pp. 721-747.

Direttore responsabile: Prof. A. Ballio - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5/L - Bologna