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Differential Forms in C^1-Domains (**)(***)

SUMMARY. — Some results related to differential forms of L^p-class on the boundary of a C^1-domain are here investigated. The paper prepares the background for [9] and [10] too.

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Forme differenziali in domini di classe C^1

SOMMARIO. — Si studiano alcune questioni relative a forme differenziali di classe L^p sulla frontiera di un dominio di classe C^1. Il lavoro fornisce anche le premesse ai lavori [9] e [10].

INTRODUCTION

This work presents some results obtained for differential forms of class $L^p(\partial\Omega)$, where Ω is a bounded and connected C^1-domain of R^n.

To this purpose the definitions of the *interior and exterior nontangential trace of a differential form* are first introduced and *Stokes' formula* is proved. The spaces of boundary forms here analyzed are the $W_s^{1,p}(\partial\Omega)$ spaces of differential forms of class L^p with distributional exterior derivative of the same class. As a consequence both of the Poincaré duality (see [2]: Chap. VIII, 8.1) and of the de Rham theorem (see [14]: Chap. IV, 29) the *cycles associated with a differential form* are next defined. Through this notion the *integral of a closed s-form extended to an s-cycle of* $\partial\Omega$ is defined. Then we give the defini-

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tion of *fundamental system of cycles* of $\partial\Omega$ and we prove that the dual base of a fundamental system of cycles of $\partial\Omega$ is also a fundamental system of cycles of $\partial\Omega$. Finally some relations between the Betti numbers of $\partial\Omega$, $\bar{\Omega}$ and $\bar{R}^n \setminus \Omega$, being $\bar{R}^n = R^n \cup \{\infty\}$, are investigated. Through these findings and by using some properties of the intersection number of two cycles, we extend a theorem of C. Miranda (see [5]: Theorem VII) to open set Ω of class C^1 and to forms of class $W_r^{1,p}(\partial\Omega)$. This theorem yields a reduction formula for the integral of a differential form on $\partial\Omega$ by means of integrals on C^1-differentiable cycles of $\partial\Omega$.

The results established here are the background for to approach Dirichlet and Neumann problems, already solved in [5] in open sets of class $C^{2,\alpha}$ and hölderian boundary data, in the same context of this work. These extensions will be reported elsewhere.

1. - Notations and definitions

Throughout this work Ω will denote a bounded connected C^1-domain of R^n. Thus $\delta > 0$ exists such that corresponding to each point Q on the boundary $\partial\Omega$ of Ω there is a system of coordinates of R^n with origin Q and a sphere, $B(Q, \delta)$, with center Q and radius δ, such that with respect to this coordinate system

$$\Omega \cap B(Q, \delta) = \{(x, x_n) \in R^{n-1} \times R : x_n > \xi(x)\} \cap B(Q, \delta)$$

where $\xi \in C_0^1(R^{n-1})$, $\xi(0) = (\partial\xi / \partial x_l)(0) = 0$ $(l = 1, \ldots, n-1)$. For any $Q \in \partial\Omega$ and $r \in (0, \delta)$, we define $B(Q, r)$ a *coordinate neighborhood* of Q, while the function

$$\bar{x} = (x, \xi(x)) \in \partial\Omega \cap B(Q, r) \to x \in R^{n-1}$$

is called the *coordinate function* of $B(Q, r)$. The pair $(B(Q, r), \bar{x})$ is named a *coordinate pair*. Furthermore, (see [13]) there exists an increasing sequence of C^∞ domains, $\Omega_k \subset \Omega$, such that $\Omega_k \to \Omega$ in C^1 according to Nečas (see [7]: pag. 85) and a sequence of diffeomorphisms, $\Lambda_k : \partial\Omega \to \partial\Omega_k$ such that

$$\lim_k \sup_{Q \in \partial\Omega} |Q - \Lambda_k(Q)| = 0. \tag{1.1}$$

There is a finite covering $(B_r)_{1 \le r \le m}$ of $\partial\Omega$ by open spheres $B_r = B(Q_r, \delta)$ with center $Q_r \in \partial\Omega$ and radius δ, such that

$$B_r \cap \partial\Omega = \{(x, x_n) \in R^{n-1} \times R : x_n = \xi_r(x)\} \cap B_r \tag{1.2}$$

and

$$B_r \cap \partial\Omega_k = \{(x, x_n) \in R^{n-1} \times R : x_n = \xi_{rk}(x)\} \cap B_r \tag{1.3}$$

where $\xi_r \in C_0^1(R^{n-1})$, $\xi_r(0) = (\partial \xi_r / \partial x_l)(0) = 0$ $(l = 1, \dots, n-1)$, $\xi_{rh} \in C_0^\infty(R^{n-1})$ and

(1.4) $$\lim_h \|\xi_{rh} - \xi_r\|_{C_0^1(R^{n-1})} = 0 .$$

Let

(1.5) $$\tilde{x}_r = (x, \xi_r(x)) \in \partial\Omega \cap B_r \to x \in R^{n-1}$$

and

(1.6) $$\tilde{x}_{rh} = (x, \xi_{rh}(x)) \in \partial\Omega_h \cap B_r \to x \in R^{n-1} .$$

For $l, i = 1, \dots, n-1$ it results that

(1.7) $$\lim_h \frac{\partial (\tilde{x}_{rh} \circ \Lambda_h \circ \tilde{x}_r^{-1})_i}{\partial x_l}(x) = \delta_{il}$$

uniformly in $U_r = \tilde{x}_r(\partial\Omega \cap B_r)$, where $(\tilde{x}_{rh} \circ \Lambda_h \circ \tilde{x}_r^{-1})_i$ is the i-th coordinate of the function $\tilde{x}_{rh} \circ \Lambda_h \circ \tilde{x}_r^{-1}$.

For any $Q \in \partial\Omega$, $N(Q)$ will denote the unit inner normal to $\partial\Omega$ at Q and $N_h(Q)$ the unit inner normal to $\partial\Omega_h$ at $Q_h = \Lambda_h(Q)$. It is easy to see that $N_h(Q) \to N(Q)$ uniformly in $\partial\Omega$.

Throughout this work, if $s = 1, \dots, n$,

(1.8) $$N_s^n = \{ i = (i_1, \dots, i_s) \in N^s : 1 \le i_1 < \dots < i_s \le n \} .$$

Further if $i = (i_1, \dots, i_s) \in N_s^n$ then i' will denote the unique element $(i_{s+1}, \dots, i_n) \in N_{n-s}^n$ such that $(i_1, \dots, i_s, i_{s+1}, \dots, i_n)$ is a permutation of $(1, \dots, n)$.

Let U be an open set of R^n. We will denote by $C_s^k(U)$, $C_{0,s}^k(U)$ and $L_s^p(U)$ the spaces of all differential forms of degree s whose components belong to $C^k(U)$, $C_0^k(U)$ and $L^p(U)$ respectively. If $\omega_s = \sum_{i \in N_s^n} a_i dX_i$ is a differential form defined in U, where, for $i = (i_1, \dots, i_s) \in N_s^n$, $dX_i = dX_{i_1} \wedge \dots \wedge dX_{i_s}$, then we set

(1.9) $$\|\omega_s\|_{L_s^p(U)} = \sum_{i \in N_s^n} \|a_i\|_{L^p(U)} .$$

We will denote by $\tilde{C}_s^1(U)$ the space of the *regular forms* in U, according to Whitney (see [14]: Chap. III, 16) and by $\tilde{C}_{0,s}^1(U)$ the space of the regular forms with compact support in U. Furthermore, for any $\omega_s \in \tilde{C}_s^1(U)$, $d\omega_s$ is the exterior derivative of ω_s. $C_s^0(\partial\Omega)$ and $L_s^p(\partial\Omega)$) will denote the spaces of all differential forms in $\partial\Omega$ such that for any coordinate function $\tilde{x}, \tilde{x}^{-1*}(\omega_s)$ belongs to $C_s^0(U)$ and $L_s^p(U)$) respectively (see [14]: Chap. II, 12), where U is the range of $\tilde{x}$. Finally, according to Whitney (see [14]: Chap. III, 17), the space of the regular forms in $\partial\Omega$ will be denoted by $\tilde{C}_s^1(\partial\Omega)$ and for any $\omega_s \in \widetilde{C}_s^1(\partial\Omega)$, $d\omega_s$ is the exterior derivative of ω_s. We set

$$\|\omega_s\|_{L_s^p(\partial\Omega)} = \sum_{l=1}^{m} \|\bar{x}_l^{-1*}(\varphi_l \omega_s)\|_{L_s^p(R^{n-1})} \tag{1.10}$$

where $(\varphi_l)_{1 \leq l \leq m}$ is a partition of the unity of class C^1, subordinate to a fixed cover $(B_l)_{1 \leq l \leq m}$ of $\partial\Omega$ and $(B_l, \bar{x}_l)$ is a coordinate pair. It is not difficult to see that using a different covering and a different partition of unity subordinate to the cover will give rise to a norm equivalent to the one we have defined. $L_s^p(\partial\Omega)$ is a Banach space.

2. - Nontangential traces and Stokes' formula

By the definitions of interior and exterior nontangential traces of a function, (see [8]), we give the following

Definition 2.1: *Let $\omega_s = \sum_{i \in N_s^n} a_i dX_i$ be a form defined in Ω (or in $R^n \setminus \overline{\Omega}$). We say that ω_s has interior nontangential trace* (*exterior* respectively) *in $L_s^p(\partial\Omega)$ if, for any $i \in N_s^n$, a_i has interior nontangential trace a_i^-* (*exterior a_i^+* respectively) *in $L^p(\partial\Omega)$. The form* (¹)

$$\omega_s^- = \sum_{i \in N_s^n} a_i^- dX_i(Q) \tag{2.1}$$

(the form

$$\omega_s^+ = \sum_{i \in N_s^n} a_i^+ dX_i(Q) \tag{2.2}$$

respectively) is called the interior nontangential trace (*exterior* respectively) *of ω_s.*

Theorem 2.1: *Let $\omega_s = \sum_{i \in N_s^n} a_i dX_i \in C_s^0(\Omega)$. If ω_s has the interior nontangential trace in $L_s^p(\partial\Omega)$, then $\omega_s \in L_s^p(\Omega)$.*

Proof: It is sufficient to observe that for any coordinate neighborhood $B = B(Q, \delta)$,

$$\|\omega_s\|_{L_s^p(B \cap \Omega)}^p \leq \sum_{i \in N_s^n} \int_0^{2\delta} dt \int_{|x| < \delta} |a_i(x, \xi(x) + t)|^p dx \leq$$

$$\leq \sum_{i \in N_s^n} \int_0^{2\delta} dt \int_{|x| < \delta} (M(a_i)(x, \xi(x)))^p dx$$

where $M(a_i)$ is the interior nontangential maximal function of a_i (see [8]). ■

(¹) $dX_i(Q)$ is the restriction to $\partial\Omega$ of dX_i, hence $dX_i(Q) = j^* dX_i$, where $j: \partial\Omega \to R^n$ is the inclusion map.

THEOREM (of STOKES) 2.2: *Let* $\omega_{n-1} = \sum_{l=1}^{n} a_l dX_{l'} \in C^1_{n-1}(\Omega)$. *If* ω_{n-1} *has interior nontangential trace in* $L^1_{n-1}(\partial\Omega)$ *and, for any* $l = 1, \ldots, n$, $\partial a_l / \partial X_l \in L^1(\Omega)$, *then*

$$\int_{\Omega} d\omega_{n-1} = \int_{\partial\Omega} \omega^-_{n-1}. \tag{2.3}$$

PROOF: If $V = \sum_{l=1}^{n} (-1)^{l-1} a_l (\partial / \partial X_l)$, then $\int_{\Omega} d\omega_{n-1} = \int_{\Omega} \operatorname{div} V \, dX$. From Theorem 2.1 in [8] it follows

$$\int_{\Omega} \operatorname{div} V \, dX = \sum_{l=1}^{n} \int_{\partial\Omega} (-1)^{l-1} a_l^-(Q) N_l(Q) \, d\sigma_Q$$

where $N_l(Q)$ is the l-th coordinate of $N(Q)$. Since $(-1)^{l-1} N_l(Q) \, d\sigma_Q = dX_{l'}(Q)$, the proof is complete. ■

THEOREM 2.3: *Let* $\omega_s \in C^0_s(\Omega)$. *If* ω_s *has interior nontangential trace in* $L^p_s(\partial\Omega)$, *then*

$$\lim_h \Lambda^*_h \omega_{sh} = \omega^-_s \quad \text{in } L^p_s(\partial\Omega) \tag{2.4}$$

where ω_{sh} *is the restriction to* $\partial\Omega_h$ *of* ω_s.

PROOF: Let $\omega_s = \sum_{i \in N^n_s} a_i dX_i$. Since a_i belongs to $C^0(\Omega)$ and has interior nontangential trace in $L^p(\partial\Omega)$, from (1.1) it follows that

$$\lim_h a_i(\Lambda_h(Q)) = a_i^-(Q) \quad \text{in } L^p(\partial\Omega).$$

Hence by (1.7) the theorem follows.

3. - THE SPACE $W^{1,p}_s(R^n)$

Let U be an open set of R^n. From Stokes' theorem it follows that for any $\omega_s \in \bar{C}^1_s(U)$ and $\Phi_{n-s-1} \in \bar{C}^1_{0,n-s-1}(U)$

$$\int_U d\omega_s \wedge \Phi_{n-s-1} = (-1)^{s+1} \int_U \omega_s \wedge d\Phi_{n-s-1}. \tag{3.1}$$

By (3.1) we will extend the definition of exterior derivative to forms of class L^p. For this we prove the following

LEMMA 3.1: *Let $\omega_r \in L_r^p(U)$. If for any $\Phi_{n-r} \in C_{0,n-r}^0(U)$ it is*

$$\int_U \omega_r \wedge \Phi_{n-r} = 0,$$

then $\omega_r = 0$ a.e. in U.

PROOF: Let $\omega_r = \sum_{i \in N_r^n} a_i dX_i$. For $\bar{i} \in N_r^n$ and $\Phi \in C_0^0(U)$ set $\Phi_{n-r} = \Phi dX_{\bar{i}'}$. Then

$$\int_U \omega_r \wedge \Phi_{n-r} = \int_U a_{\bar{i}} \Phi = 0 .$$

Hence $a_{\bar{i}} = 0$ a.e. in U. ■

DEFINITION 3.1: *Let $\omega_r \in L_r^p(R^n)$. We say that ω_r has distributional exterior derivative $d\omega_r \in L_{r+1}^p(R^n)$ iff. for any $\Phi_{n-r-1} \in \bar{C}_{0,n-r-1}^1(R^n)$, (3.1) is verified. $W_r^{1,p}(R^n)$ is the space of all forms $\omega_r \in L_r^p(R^n)$ with distributional exterior derivative $d\omega_r \in L_{r+1}^p(R^n)$. We assume*

$$\|\omega_r\|_{W_r^{1,p}(R^n)} = \|\omega_r\|_{L_r^p(R^n)} + \|d\omega_r\|_{L_{r+1}^p(R^n)} . \tag{3.2}$$

THEOREM 3.1: *Let $\omega_r \in L_r^p(R^n)$. Then $\omega_r \in W_r^{1,p}(R^n)$ iff there exists a sequence $(\Phi_{rh})_{h\in N}$, $\Phi_{rh} \in C_{0,r}^\infty(R^n)$, such that $(d\Phi_{rh})_{h\in N}$ is a Cauchy sequence in $L_{r+1}^p(R^n)$ and*

$$\lim_h \Phi_{rh} = \omega_r \quad \text{in } L_r^p(R^n) .$$

PROOF: Let $\omega_r = \sum_{i \in N_r^n} a_i dX_i \in W_r^{1,p}(R^n)$ and $\omega_{rh} = \sum_{i \in N_r^n} a_{ih} dX_i$, where

$$a_{ih}(X) = \begin{cases} a_i(X) & \text{if } |X| \leqslant h ; \\ 0 & \text{if } |X| > h . \end{cases}$$

Furthermore let (χ_h) be a sequence of mollifiers and

$$\Phi_{rh} = \chi_h * \omega_{rh} = \sum_{i \in N_r^n} \chi_h * a_{ih}$$

where $*$ is the usual convolution product between functions. Then

$$\Phi_{rh} \in C_{0,r}^\infty(R^n) \quad \text{and} \quad \Phi_{rh} \to \omega_r \text{ in } L_r^p(R^n) .$$

By Lemma 16b in [14],

$$d\Phi_{rh} = \chi_h * d\omega_{rh} ;$$

hence $d\Phi_{rh} \in C_{0,r+1}^\infty(R^n)$ and $d\Phi_{rh} \to d\omega_r$ in $L_{r+1}^p(R^n)$.

The proof of the sufficient condition is trivial. ■

THEOREM 3.2: *If* $\omega_s = \sum_{i \in N_s^n} a_i dX_i \in W_s^{1,p}(R^n)$, *then for any* $i \in N_s^n$ *and* $j = s + 1, \ldots, n$, *the partial derivative in the sense of distributions* $\partial a_i / \partial X_{i_j}$ *belongs to* $L^p(R^n)$ *and*

$$d\omega_s = \sum_{i \in N_s^n} \sum_{j=s+1}^{n} \frac{\partial a_i}{\partial X_{i_j}} dX_{i_j} \wedge dX_i. \tag{3.3}$$

PROOF: From Theorem 3.1 it follows that there exists a sequence $(\Phi_{sh})_{h \in N}$, $\Phi_{sh} \in C^1_{0,s}(R^n)$, such that

$$\Phi_{sh} \to \omega_s \quad \text{in } L^p_s(R^n) \quad \text{and} \quad d\Phi_{sh} \to d\omega_s \quad \text{in} \quad L^p_{s+1}(R^n). \tag{3.4}$$

If

$$d\omega_s = \sum_{i \in N_s^n} \sum_{j=s+1}^{n} \alpha_{i,j} dX_{i_j} \wedge dX_i$$

and

$$\Phi_{sh} = \sum_{i \in N_s^n} a_{ih} dX_i,$$

then, from (3.4), it follows that $a_{ih} \to a_i$ in $L^p(R^n)$ and $(\partial a_{ih} / \partial X_{i_j}) \to \alpha_{i,j}$ in $L^p(R^n)$. Hence $\alpha_{i,j} = (\partial a_i / \partial X_{i_j})$. ∎

From properties of the regular forms and Theorem 3.1 further theorems follow:

THEOREM 3.3: *If* $\omega_s \in W_s^{1,p}(R^n)$ *and* $\omega_r \in W_r^{1,p'}(R^n)$ *with* p' *the conjugate exponent of* p, *then* $\omega_s \wedge \omega_r \in W_{s+r}^{1,1}(R^n)$ *and*

$$d(\omega_s \wedge \omega_r) = d\omega_s \wedge \omega_r + (-1)^s \omega_s \wedge d\omega_r.$$

THEOREM 3.4: *If* $\omega_s \in W_s^{1,p}(R^n)$, *then* $d\omega_s \in W_{s+1}^{1,p}(R^n)$ *and*

$$dd\omega_s = 0.$$

4. - THE SPACE $W_s^{1,p}(\partial\Omega)$

LEMMA 4.1: *Let* $\omega_s \in L_s^p(\partial\Omega)$. *If for any* $\Phi_{n-s-1} \in C^0_{n-s-1}(\partial\Omega)$

$$\int_{\partial\Omega} \omega_s \wedge \Phi_{n-s-1} = 0,$$

then $\omega_s = 0$ *a.e. on* $\partial\Omega$.

PROOF: It is sufficient to show that if $(B,\tilde{x})$ is a coordinate pair and $\varphi \in C^1(\partial\Omega)$ with $\mathrm{supp}(\varphi) \subset B$, then $\tilde{x}^{-1*}(\varphi\omega_s) = 0$ a.e. in R^{n-1}.

Let $G_{n-s-1} \in C^0_{0,n-s-1}(R^{n-1})$ and $\Phi_{n-s-1} = \tilde{x}^*((\varphi \circ \tilde{x}^{-1})G_{n-s-1})$. Since $\Phi_{n-s-1} \in C^0_{n-s-1}(\partial\Omega)$, then it results

$$0 = \int_{\partial\Omega} \omega_s \wedge \Phi_{n-s-1} = \int_{R^{n-1}} \tilde{x}^{-1*}(\omega_s \wedge \Phi_{n-s-1}) = \int_{R^{n-1}} \tilde{x}^{-1*}(\varphi\omega_s) \wedge G_{n-s-1},$$

and, from Lemma 3.1, the proof is complete. ■

LEMMA 4.2: *If $\omega_s \in \tilde{C}^1_s(\partial\Omega)$, then for any $\Phi_{n-s-2} \in \tilde{C}^1_{n-s-2}(\partial\Omega)$*

$$\int_{\partial\Omega} d\omega_s \wedge \Phi_{n-s-2} = (-1)^{s+1} \int_{\partial\Omega} \omega_s \wedge d\Phi_{n-s-2}. \tag{4.1}$$

PROOF: It is sufficient to use (3.1) and to observe that for any coordinate function $\tilde{x}$

$$d(\tilde{x}^{-1*}\omega_s) = \tilde{x}^{-1*}\, d\omega_s \tag{4.2}$$

(see. [14]: Chap. III, 17). ■

It is now justified the following

DEFINITION 4.1: *Let $\omega_s \in L^p_s(\partial\Omega)$. We say that ω_s has distributional exterior derivative $d\omega_s \in L^p_{s+1}(\partial\Omega)$ iff, for any $\Phi_{n-s-2} \in \tilde{C}^1_{n-s-2}(\partial\Omega)$, (4.1) is verified. $W^{1,p}_s(\partial\Omega)$ is the space of all forms $\omega_s \in L^p_s(\partial\Omega)$ with distributional exterior derivative $d\omega_s \in$ $\in L^p_{s+1}(\partial\Omega)$. We assume*

$$\|\omega_s\|_{W^{1,p}_s(\partial\Omega)} = \|\omega_s\|_{L^p_s(\partial\Omega)} + \|d\omega_s\|_{L^p_{s+1}(\partial\Omega)}. \tag{4.3}$$

REMARK 4.1: $W^{1,p}_s(\partial\Omega)$ is a Banach space.

REMARK 4.2: Let $\omega_s \in W^{1,p}_s(\partial\Omega)$. For any coordinate pair $(B,\tilde{x})$ and $\varphi \in C^1(\partial\Omega)$ with $\mathrm{supp}(\varphi) \subset B$ then $\tilde{x}^{-1*}(\varphi\omega_s) \in W^{1,p}_s(R^{n-1})$, and $d(\tilde{x}^{-1*}(\varphi\omega_s)) =$ $= \tilde{x}^{-1*} d(\varphi\omega_s)$.

DEFINITION 4.2: *Let $\omega_s \in W^{1,p}_s(\partial\Omega)$. Then ω_s is called closed iff $d\omega_s = 0$.*

THEOREM 4.1: *Let $\omega_s \in L^p_s(\partial\Omega)$. Then $\omega_s \in W^{1,p}_s(\partial\Omega)$ iff there exists a sequence $(\Psi_{sh})_{h\in N}$, $\Psi_{sh} \in \tilde{C}^1_s(\partial\Omega)$, such that $(d\Psi_{sh})_{h\in N}$ is a Cauchy sequence in $L^p_{s+1}(\partial\Omega)$ and*

$$\lim_h \Psi_{sh} = \omega_s \quad \text{in} \quad L^p_s(\partial\Omega).$$

PROOF: Let $\omega_r \in W_r^{1,p}(\partial\Omega)$. Let $(B_l)_{1\le l\le m}$ be a cover of $\partial\Omega$ formed by coordinate neighborhoods and $(\varphi_l)_{1\le l\le m}$ a partition of unity subordinate to this cover. Then, from Remark 4.2, we have $\tilde{x}_l^{-1*}(\varphi_l\omega_r) \in W_r^{1,p}(R^{n-1})$. By Theorem 3.1, there exists a sequence $(\Phi_{ib}^l)_{b\in N}$, $\Phi_{ib}^l \in C^1_{0,r}(R^{n-1})$ such that

$$\lim_b \Phi_{ib}^l = \tilde{x}_l^{-1*}(\varphi_l\omega_r) \quad \text{in } L_r^p(R^{n-1})$$

and

$$\lim_b d\Phi_{ib}^l = d\tilde{x}_l^{-1*}(\varphi_l\omega_r) = \tilde{x}_l^{-1*}(\varphi_l d\omega_r + (-1)^r\omega_r \wedge d\varphi_l).$$

Set

$$\Psi_{ib}^l = \begin{cases} \tilde{x}_l^*(\Phi_{ib}^l) & \text{in } B_l \cap \partial\Omega; \\ 0 & \text{in } \partial\Omega \setminus B_l. \end{cases}$$

Then $\Psi_{ib}^l \in \tilde{C}_r^1(\partial\Omega)$. Finally, set

$$\Psi_{rb} = \sum_{l=1}^m \Psi_{ib}^l,$$

the necessity of the condition is established. It is easy to prove the sufficiency. ■

From Theorem 4.1 and properties of the regular forms on $\partial\Omega$ the following theorems can be deduced:

THEOREM 4.2: *If $\omega_s \in W_s^{1,p}(\partial\Omega)$ and $\omega_r \in W_r^{1,p'}(\partial\Omega)$ with p' the conjugate exponent of p, then $\omega_s \wedge \omega_r \in W_{s+r}^{1,1}(\partial\Omega)$ and*

$$d(\omega_s \wedge \omega_r) = d\omega_s \wedge \omega_r + (-1)^s \omega_s \wedge d\omega_r.$$

THEOREM 4.3: *If $\omega_r \in W_r^{1,p}(\partial\Omega)$, then $d\omega_r \in W_{r+1}^{1,p}(\partial\Omega)$ and*

$$dd\omega_r = 0.$$

THEOREM 4.4: *If $\omega_{n-2} \in W_{n-2}^{1,1}(\partial\Omega)$, then*

$$\int_{\partial\Omega} d\omega_{n-2} = 0.$$

PROOF: Let $\omega_{n-2} \in W_{n-2}^{1,1}(\partial\Omega)$ and $(\omega_{n-2}^b)_{b\in N}$ a sequence such that $\omega_{n-2}^b \in \tilde{C}_{n-2}^1(\partial\Omega)$ and (see Theorem 4.1)

$$\lim_b \omega_{n-2}^b = \omega_{n-2} \quad \text{in } W_{n-2}^{1,1}(\partial\Omega).$$

Then, since $\partial\Omega$ is a non-bounded C^1-manifold, by Stokes' theorem we have

$$\int_{\partial\Omega} d\omega_{n-2} = \lim_h \int_{\partial\Omega} d\omega^h_{n-2} = 0 . \qquad \blacksquare$$

THEOREM 4.5: *Let $\omega_s \in C^1_s(\Omega)$. If ω_s and $d\omega_s$ have interior nontangential trace in $L^p_s(\partial\Omega)$ and in $L^p_{s+1}(\partial\Omega)$ respectively, then*

$$\omega_s^- \in W^{1,p}_s(\partial\Omega) \quad \textit{and} \quad d(\omega_s^-) = (d\omega_s)^- .$$

PROOF: Let $G_{n-2-s} \in \breve{C}^1_{n-2-s}(\partial\Omega)$. As a consequence of Theorem 2.3 and properties of regular forms it can be obtained that

$$\int_{\partial\Omega} \omega_s^- \wedge dG_{n-s-2} = \lim_h \int_{\partial\Omega} \Lambda_h^* \omega_{sh} \wedge dG_{n-s-2} =$$

$$= \lim_h (-1)^{s+1} \int_{\partial\Omega_h} d(\omega_{sh}) \wedge \Lambda_h^{-1*} G_{n-s-2} = \lim_h (-1)^{s+1} \int_{\partial\Omega} \Lambda_h^* (d\omega_s)_h \wedge G_{n-s-2} =$$

$$= (-1)^{s+1} \int_{\partial\Omega} (d\omega_s)^- \wedge G_{n-s-2}$$

where $(d\omega_s)_h$ is the restriction of $d\omega_s$ to $\partial\Omega_h$. $\blacksquare$

THEOREM 4.6: *Let $\omega_s \in C^1_s(\Omega)$. If ω_s is a closed form with interior nontangential trace in $L^p_s(\partial\Omega)$, then ω_s^- belongs to $\in W^{1,p}_s(\partial\Omega)$ and is closed.*

5. - SOME PROPERTIES OF CLOSED FORMS IN $W^{1,p}_s(\partial\Omega)$

Since Ω is a C^1-domain, in virtue of Theorem 12 A in [14], there exist a finite simplicial complex K and a C^1-triangulation f of K onto $\partial\Omega$ with the following property: for each simplex α of K, $\sigma = f(\alpha)$ is a C^1-differentiable simplex in $\partial\Omega$ and there is a coordinate neighborhood B such that $f(\alpha) \subset B \cap \partial\Omega$ and $\bar{x} \circ f(\alpha)$ is affine in α. The C^1-simplexes $\sigma = f(\alpha)$ form a *finite curvilinear complex L*.

We will denote by $\mathcal{K}_s(\partial\Omega)$ ($\mathcal{K}^s(\partial\Omega)$) the s-th C^1-differentiable singular homology (cohomology respectively) space of $\partial\Omega$ with real coefficients. In $\mathcal{K}_s(\partial\Omega) \times \mathcal{K}^s(\partial\Omega)$ it can be defined the *Kronecker product*

$$\langle [\sigma_s], [\sigma^s] \rangle = \langle \sigma_s, \sigma^s \rangle = \sigma^s(\sigma_s)$$

while in $\mathcal{K}^r(\partial\Omega) \times \mathcal{K}^s(\partial\Omega)$ is defined the bilinear map

$$[\sigma^r] \cup [\sigma^s] = [\sigma^r \cup \sigma^s]$$

where $\sigma^r \cup \sigma^s$ is the *cup-product* of σ^r and σ^s (see [2]: Chap. VII, 8) and in $\mathcal{K}^s(\partial\Omega) \times$

$\times \mathcal{H}_{s+l}(\partial\Omega)$ is defined the bilinear map

$$[\sigma^s] \cap [\sigma_{s+l}] = [\sigma^s \cap \sigma_{s+l}]$$

where $\sigma^s \cap \sigma_{s+l}$ is the *cap-product* of σ^s and σ_{s+l} (see [2]: Chap. VII, 12).

By the definition we have

$$\langle \sigma^s \cap \sigma_{s+l}, \sigma^l \rangle = \langle \sigma_{s+l}, \sigma^s \cup \sigma^l \rangle. \tag{5.1}$$

Given the s-form $\omega_s \in \widetilde{C}^1_s(\partial\Omega)$, the function $\int_{\sigma_s} \omega_s$ of C^1-differentiable s-chain σ_s, defines an s-cochain $\psi_s(\omega_s)$. The linear transformation ψ_s defines the following linear transformation

$$\Psi_s : [\omega_s] \in H^s(\partial\Omega) \to [\psi_s(\omega_s)] \in \mathcal{H}^s(\partial\Omega)$$

where $H^s(\partial\Omega)$ denotes the s-regular differential cohomology space of $\partial\Omega$. As a consequence of de Rham's theorem (see [14]: Chap. IV, 29) the functions Ψ_s are isomorphisms and defines a ring-isomorphism Ψ of $H^*(\partial\Omega)$ onto $\mathcal{H}^*(\partial\Omega)$, where $H^*(\partial\Omega)$ and $\mathcal{H}^*(\partial\Omega)$ are direct sum of the spaces $H^s(\partial\Omega)$ and $\mathcal{H}^s(\partial\Omega)$ respectively. Since there exists a ring-isomorphism of $\mathcal{H}^*(\partial\Omega)$ onto the direct sum of s-th singular cohomology spaces of $\partial\Omega$, from the Poincaré duality (see. [2]: Chap VIII, 8) it follows that

$$\overline{\Psi}_s : [\sigma^s] \in \mathcal{H}^s(\partial\Omega) \to [\sigma^s \cap \partial\Omega] \in \mathcal{H}_{n-1-s}(\partial\Omega)$$

is an isomorphism.

Setting $\Theta_s = \overline{\Psi}_s \circ \Psi_s$, it is immediate that Θ_s is an isomorphism of $H^s(\partial\Omega)$ onto $\mathcal{H}_{n-1-s}(\partial\Omega)$.

DEFINITION 5.1: *A closed regular form* $\omega_s \in \widetilde{C}^1_s(\partial\Omega)$ *and a* C^1*-differentiable cycle* σ_{n-1-s} *in* $\partial\Omega$ *are associated iff*

$$\Theta_s([\omega_s]) = [\sigma_{n-1-s}].$$

DEFINITION 5.2: *If* σ_s *and* σ_{n-1-s} *are* C^1*-differentiable cycles of* $\partial\Omega$, *then we assume*

$$I(\sigma_s, \sigma_{n-1-s}) = \int_{\sigma_s} \omega_s$$

where ω_s *is a* s*-form associated with* σ_{n-1-s}. *Then the integral* $\int_{\sigma_s} \omega_s$ *is called the intersection number of* σ_s *and* σ_{n-1-s}.

It is immediate that

$$I(\sigma_s, \sigma_{n-1-s}) = \langle \sigma_s, \psi_s(\omega_s) \rangle. \tag{5.2}$$

THEOREM 5.1: *If ω_s and $\overline{\omega}_{n-1-s}$ are closed regular forms in $\partial\Omega$ associated with σ_{n-1-s} and $\overline{\sigma}_s$ respectively, then*

$$\int_{\partial\Omega} \omega_s \wedge \overline{\omega}_{n-1-s} = I(\overline{\sigma}_s, \sigma_{n-1-s}) = \int_{\overline{\sigma}_s} \omega_s . \tag{5.3}$$

PROOF: It is enough to observe that

$$\int_{\partial\Omega} \omega_s \wedge \overline{\omega}_{n-1-s} = \langle \partial\Omega, \psi_{n-1}(\omega_s \wedge \overline{\omega}_{n-1-s}) \rangle =$$

$$= \langle \partial\Omega, \psi_s(\omega_s) \cup \psi_{n-1-s}(\overline{\omega}_{n-1-s}) \rangle = \langle \psi_{n-1-s}(\overline{\omega}_{n-1-s}) \cap \partial\Omega, \psi_s(\omega_s) \rangle =$$

$$= \langle \Theta_{n-1-s}([\overline{\omega}_{n-1-s}]), [\psi_s(\omega_s)] \rangle = \langle \overline{\sigma}_s, \psi_s(\omega_s) \rangle . \quad \blacksquare$$

REMARK 5.1: It is easy to see that

$$I(\overline{\sigma}_s, \sigma_{n-1-s}) = (-1)^{s(n-1-s)} I(\sigma_{n-1-s}, \overline{\sigma}_s) .$$

Theorem 5.1 suggests a way to define $\int_{\sigma_s} \omega_s$ for any closed form $\omega_s \in W_s^{1,p}(\partial\Omega)$. A preliminary Lemma will thus be proved

LEMMA 5.1: *Let ω_s a closed form in $W_s^{1,p}(\partial\Omega)$. If ω^1_{n-1-s} and ω^2_{n-1-s} are cohomologous regular closed forms in $\partial\Omega$, then*

$$\int_{\partial\Omega} \omega_s \wedge \omega^1_{n-1-s} = \int_{\partial\Omega} \omega_s \wedge \omega^2_{n-1-s} .$$

PROOF: It is enough to observe that from Theorem 4.4, if $\omega^1_{n-1-s} - \omega^2_{n-1-s} = d\omega_{n-2-s}$, then

$$\int_{\partial\Omega} \omega_s \wedge (\omega^1_{n-1-s} - \omega^2_{n-1-s}) = (-1)^s \int_{\partial\Omega} d(\omega_s \wedge \omega_{n-2-s}) . \quad \blacksquare$$

It is now justified the following

DEFINITION 5.3: *If ω_s is a closed form in $W_s^{1,p}(\partial\Omega)$ and if σ_s is a C^1-differentiable cycle in $\partial\Omega$, then we assume*

$$\int_{\sigma_s} \omega_s = \int_{\partial\Omega} \omega_s \wedge \overline{\omega}_{n-1-s}$$

where $\overline{\omega}_{n-1-s}$ is a form associated with σ_s.

Remark 5.2: Since forms associated with homologous cycles are cohomologous, it is easy to see that, for any closed form $\omega_s \in W_s^{1,p}(\partial\Omega)$,

$$\int_{\sigma_s} \omega_s = \int_{\sigma_s'} \omega_s$$

if σ_s, σ_s' are homologous C^1-differentiable cycles in $\partial\Omega$.

Let R_s, R_s^- and R_s^+ denote the s-th *Betti number* of $\partial\Omega$, $\overline{\Omega}$ and $\overline{R}^n \setminus \Omega$ respectively. Since $\overline{\Psi}_s$ is a isomorphism of $\mathcal{H}^s(\partial\Omega)$ in $\mathcal{H}_{n-1-s}(\partial\Omega)$, then

$$R_s = R_{n-1-s}.$$

Let $0 < s < n-1$. From Alexander duality (see.[2]: Chap. VIII, 8.15) it follows that there is a isomorphism of $\mathcal{H}^s(\overline{\Omega})$ onto $\mathcal{H}_{n-1-s}(\overline{R}^n \setminus \overline{\Omega})$. In virtue of Theorem 20 in [11] (see.[11]: Chap. VI, Theorem 20), if j is the convenient inclusion map,

$$j_* : \mathcal{H}_s(\Omega) \to \mathcal{H}_s(\overline{\Omega}) \tag{5.4}$$

and

$$j_* : \mathcal{H}_s(\overline{R}^n \setminus \overline{\Omega}) \to \mathcal{H}_s(\overline{R}^n \setminus \Omega) \tag{5.5}$$

are isomorphisms. Hence

$$R_s^- = R_{n-1-s}^+.$$

Since (see[2]: Chap. VIII, 6.28; Chap. III, 8.6)

$$(j_{1*}, -j_{2*}) : \mathcal{H}_s(\partial\Omega) \to \mathcal{H}_s(\overline{\Omega}) \oplus \mathcal{H}_s(\overline{R}^n \setminus \Omega) \tag{5.6}$$

(where j_i is the canonical injection) is an isomorphism, then we obtain that

$$R_s = R_s^- + R_s^+$$

and there exists a base $([\tau_s^i], [\gamma_s^l])_{\substack{1 \le i \le R_s^- \\ 1 \le l \le R_s^+}}$ of $\mathcal{H}_s(\partial\Omega)$ such that

$$\tau_s^i \sim 0 \quad \text{in } \overline{R}^n \setminus \Omega \quad \text{and} \quad \gamma_s^l \sim 0 \quad \text{in } \overline{\Omega}. \tag{5.7}$$

From (5.4) and (5.5), there are $([t_s^i])_{1 \le i \le R_s^-}$ and $([c_s^l])_{1 \le l \le R_s^+}$ bases of $\mathcal{H}_s(\Omega)$ and $\mathcal{H}_s(\overline{R}^n \setminus \overline{\Omega})$ respectively such that t_s^i and c_s^l are C^∞-differentiable cycles and

$$t_s^i \sim \tau_s^i \quad \text{in } \overline{\Omega} \quad \text{and} \quad c_s^l \sim \gamma_s^l \quad \text{in } \overline{R}^n \setminus \Omega. \tag{5.8}$$

Definition 5.4: *Let* $([\tau_s^i], [\gamma_s^l])_{\substack{1 \le i \le R_s^- \\ 1 \le l \le R_s^+}}$ *be a base of* $\mathcal{H}_s(\partial\Omega)$. *We say that* $(\tau_s^i, \gamma_s^l)_{\substack{1 \le i \le R_s^- \\ 1 \le l \le R_s^+}}$ *is a fundamental system of* $\partial\Omega$ *if the conditions* (5.7) *are satisfied.*

DEFINITION 5.5: *Let* $([\tau_s^i],[\gamma_s^l])_{\substack{1\le i\le R_s^- \\ 1\le l\le R_s^+}}$ *be a base of* $\mathcal{H}_s(\partial\Omega)$ *and let* $([\tau_i^s],[\gamma_l^s])_{\substack{1\le i\le R_s^- \\ 1\le l\le R_s^+}}$ *be the dual base of* $\mathcal{H}^s(\partial\Omega)$. *If*

$$[\overline{\gamma}^l_{n-1-s}]=\overline{\Psi}_s([\tau_l^s]) \quad \text{and} \quad [\overline{\tau}^i_{n-1-s}]=\overline{\Psi}_s([\gamma_i^s]),$$

then

$$([\overline{\tau}^i_{n-1-s}],[\overline{\gamma}^l_{n-1-s}])_{\substack{1\le i\le R^-_{n-1-s} \\ 1\le l\le R^+_{n-1-s}}}$$

is called the dual base of $([\tau_s^i],[\gamma_s^l])_{\substack{1\le i\le R^-_{n-1-s} \\ 1\le l\le R^+_{n-1-s}}}$.

We have the following

THEOREM 5.2: *Let* $([\tau_s^i],[\gamma_s^l])_{\substack{1\le i\le R_s^- \\ 1\le l\le R_s^+}}$ *be a base of* $\mathcal{H}_s(\partial\Omega)$ *and let* $([\overline{\tau}^l_{n-1-s}],[\overline{\gamma}^l_{n-1-s}])_{\substack{1\le l\le R^-_{n-1-s} \\ 1\le i\le R^+_{n-1-s}}}$ *be its dual base. If* $(\tau_s^i,\gamma_s^l)_{\substack{1\le i\le R_s^- \\ 1\le l\le R_s^+}}$ *is a fundamental system, then* $(\overline{\tau}^l_{n-1-s},\overline{\gamma}^l_{n-1-s})_{\substack{1\le l\le R^-_{n-1-s} \\ 1\le i\le R^+_{n-1-s}}}$ *is a fundamental system too.*

PROOF: Since $\overline{\Omega}\subset R^n$ is a C^1-manifold with boundary, it follows from VIII, 9.1 in [2] that the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{H}^s(\overline{\Omega}) & \xrightarrow{j_1^*} & \mathcal{H}^s(\partial\Omega) \\
\downarrow (-1)^s\cap O_1 & & \downarrow \cap\,\partial\Omega \\
\mathcal{H}_{n-s}(\overline{\Omega},\partial\Omega) & \xrightarrow{\delta_{1*}} & \mathcal{H}_{n-1-s}(\partial\Omega).
\end{array}$$

All vertical arrows are isomorphic; $j_1:\partial\Omega\to\overline{\Omega}$ is the inclusion map, δ_{1*} is the connecting homomorphism of $(\overline{\Omega},\partial\Omega)$ and $\delta_{1*}(O_1)=\partial\Omega$, where $O_1\in\mathcal{H}_n(\overline{\Omega},\partial\Omega)$.

Since the inclusion map $\bar{j}_1:(\overline{\Omega},\partial\Omega)\to(\overline{R}^n,\overline{R}^n\setminus\Omega)$ is a map of pairs, (see [2]: Chap. III, 3) it follows that the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{H}_{n-s}(\overline{\Omega},\partial\Omega) & \xrightarrow{\delta_{1*}} & \mathcal{H}_{n-1-s}(\partial\Omega) \\
\downarrow \bar{j}_{1*} & & \downarrow j_{2*} \\
\mathcal{H}_{n-s-1}(\overline{R}^n,R^n\setminus\Omega) & \xrightarrow{\overline{\delta}_{1*}} & \mathcal{H}_{n-1-s}(\overline{R}^n\setminus\Omega).
\end{array}$$

where $j_2:\partial\Omega\to\overline{R}^n\setminus\Omega$ is the inclusion map and $\overline{\delta}_{1*}$ is the connecting homorphism of $(\overline{R}^n,\overline{R}^n\setminus\Omega)$. As a consequence of IV, 2.2 in [2], $\overline{\delta}_{1*}$ is an isomorphism. We remark that $\overline{R}^n$, $\overline{R}^n\setminus\Omega$ and $\overline{\Omega}$ are ENRs and $\partial\Omega$ separates $\overline{R}^n\setminus\Omega$ and $\overline{\Omega}$. This implies that $(\overline{R}^n,\overline{R}^n\setminus\Omega,\overline{\Omega})$ is an excisive triad (see [2]: Chap. VIII, 6.28). From III, 8.1 in [2], it follows that $\bar{j}_{1*}$ is an isomorphism. Consequently the following diagram is commutative

$$\begin{array}{ccc} \mathcal{H}^r(\overline{\Omega}) & \xrightarrow{j_1^*} & \mathcal{H}^r(\partial\Omega) \\ \downarrow f_1 & & \downarrow \cap\partial\Omega \\ \mathcal{H}_{n-r}(\overline{R}^n\setminus\Omega) & \xleftarrow{j_{2*}} & \mathcal{H}_{n-1-r}(\partial\Omega) \end{array}$$

where $f_1 = \overline{\delta}_{1*} \circ \bar{j}_{1*} \circ (-1)^r \cap O_1$; f_1 is an isomorphism.

Applying similar arguments for $\overline{R}^n\setminus\Omega$, we obtain that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{H}^r(\overline{R}^n\setminus\Omega) & \xrightarrow{j_2^*} & \mathcal{H}^r(\partial\Omega) \\ \downarrow f_2 & & \downarrow \cap\partial\Omega \\ \mathcal{H}_{n-r}(\overline{\Omega}) & \xleftarrow{j_{1*}} & \mathcal{H}_{n-1-r}(\partial\Omega) \end{array}$$

where $f_2 = \overline{\delta}_{2*} \circ \bar{j}_{2*} \circ (-1)^r \cap O_2$, $\overline{\delta}_{2*}$ is the connecting homorphism of $(\overline{R}^n, \overline{\Omega})$, $\bar{j}_2 : (\overline{R}^n\setminus\Omega, \partial\Omega) \to (\overline{R}^n, \overline{\Omega})$ is inclusion map and $O_2 \in \mathcal{H}_n(\overline{R}^n\setminus\Omega, \partial\Omega)$; f_2 is an isomorphism.

Hence, in virtue of (5.6), the following diagram is commutative

$$\begin{array}{ccc} \mathcal{H}^r(\overline{\Omega} \oplus \mathcal{H}^r(\overline{R}^n\setminus\Omega) & \xrightarrow{(j_1^*, -j_2^*)} & \mathcal{H}^r(\partial\Omega) \\ \downarrow f_1 \oplus f_2 & & \downarrow \cap\partial\Omega \\ \mathcal{H}_{n-r}(\overline{R}^n\setminus\Omega) \oplus H_{n-r-1}(\overline{\Omega}) & \xleftarrow{(j_{2*}, -j_{1*})} & \mathcal{H}_{n-1-r}(\partial\Omega). \end{array} \tag{5.9}$$

All vertical and horizontal maps are isomorphism.

Finally, we suppose that $(\tau_i^r, \gamma_l^r)_{\substack{1 \le i \le R_r^- \\ 1 \le l \le R_r^+}}$ is a fundamental system of $\partial\Omega$. Then $(j_{1*}([\tau_i^r]))_{1 \le i \le R_r^-}$ and $(j_{2*}([\gamma_l^r]))_{1 \le l \le R_r^+}$ are bases in $\mathcal{H}_r(\overline{\Omega})$ and in $\mathcal{H}_r(\overline{R}^n\setminus\Omega)$ respectively.

Let $([\tau_i^r], [\gamma_l^r])_{\substack{1 \le i \le R_r^- \\ 1 \le l \le R_r^+}}$ be the dual base of $([\tau_i^r], [\gamma_l^r])^{\substack{1 \le i \le R_r^- \\ 1 \le l \le R_r^+}}$ in $\mathcal{H}^r(\partial\Omega)$, $([\tau_i^{*r}])_{1 \le i \le R_r^-}$ the dual base of $(j_{1*}([\tau_i^r]))_{1 \le i \le R_r^-}$ in $\mathcal{H}^r(\overline{\Omega})$ and $([\gamma_l^{*r}])_{1 \le l \le R_r^+}$ the dual base of $(j_{2*}([\gamma_l^r]))_{1 \le l \le R_r^+}$ in $\mathcal{H}^r(\overline{R}^n\setminus\Omega)$.

Since $j_{2*}([\tau_i^r]) = 0$ and $j_{1*}([\gamma_l^r]) = 0$, we obtain that

$$j_1^*([\tau_i^{*r}]) = [\tau_i^r], \quad j_2^*([\gamma_l^{*r}]) = [\gamma_l^r].$$

Then (see Definition 5.5)

$$[\overline{\gamma}_{n-1-r}^i] = \overline{\Psi} \circ (j_1^*, -j_2^*)([\tau_i^{*r}], [0])$$

and, using the commutative diagram (5.9), we obtain

$$(j_{2*}, -j_{1*})([\overline{\gamma}^{l}_{n-1-s}]) = f_1 \oplus f_2([\tau_i^{*s}], [0]) = (f_1([\tau_i^{*s}]), [0]).$$

Hence $\overline{\gamma}^{l}_{n-1-s} \sim 0$ in $\overline{\Omega}$. Similarly we prove that $\overline{\tau}^{l}_{n-1-s} \sim 0$ in $\overline{R}^n \setminus \Omega$. ■

It is now justified the following

DEFINITION 5.6: *If* $(\tau_s^i, \gamma_s^l)_{\substack{1 \le i \le R_s^- \\ 1 \le l \le R_s^+}}$ *is a fundamental system of cycles of* $\partial\Omega$, *then*

$$(\overline{\tau}^{l}_{n-1-s}, \overline{\gamma}^{l}_{n-1-s})_{\substack{1 \le l \le R^-_{n-1-s} \\ 1 \le i \le R^+_{n-1-s}}}$$

is called a dual fundamental system of $(\tau_s^i, \gamma_s^l)_{\substack{1 \le i \le R_s^- \\ 1 \le l \le R_s^+}}$.

THEOREM 5.3: *Let* ω_s *be a closed form of* $C_s^1(\Omega)$. *If* ω_s *has the interior nontangential trace in* $L_s^p(\partial\Omega)$, *then*

$$\int_{\gamma_s} \omega_s^- = 0$$

for all C^1*-differentiable cycle* $\gamma_s \subset \partial\Omega$ *such that* $\gamma_s \sim 0$ *in* $\overline{\Omega}$.

PROOF: Let $\overline{\omega}_{n-1-s}$ be an associated form with γ_s. As a consequence of Theorem 2.3 we have

$$\int_{\gamma_s} \omega_s^- = \int_{\partial\Omega} \omega_s^- \wedge \overline{\omega}_{n-1-s} = \lim_k \int_{\partial\Omega} \Lambda_k^* \omega_{sk} \wedge \overline{\omega}_{n-1-s}.$$

It seen that ω_{sk} is a closed form in $\partial\Omega_k$ because ω_s is a closed form in Ω. Hence $\Lambda_k^* \omega_{sk}$ is a closed form in $\partial\Omega$, since Λ_k is a diffeomorphism of $\partial\Omega$ onto $\partial\Omega_k$. Then

$$\int_{\gamma_s} \omega_s^- = \lim_k \int_{\gamma_s} \Lambda_k^* \omega_{sk} = \lim_k \int_{\gamma_{sk}} \omega_{sk} \tag{5.10}$$

where $\gamma_{sk} = \Lambda_k(\gamma_s)$. Using the Mayer-Vietoris sequence, we see that $\gamma_s \sim 0$ in $\overline{\Omega} \setminus \Omega_k$, since $\gamma_s \sim 0$ in $\overline{\Omega}$ and Ω_k is an open set such that $\overline{\Omega}_k \subset \Omega$.

Further there exists a diffeomorphism $F_k : G' \to G_k$, where G' and G_k are collars of $\partial\Omega$ and $\partial\Omega_k$ respectively, such that $F_k(G' \cap \overline{\Omega}) \subset G_k \cap \overline{\Omega}_k$ and $F_k = \Lambda_k$ on $\partial\Omega$. Hence $\gamma_{sk} \sim 0$ in $\overline{\Omega}_k \cap G_k$. Thus, there is a cycle $\sigma^k_{s+1} \subset \overline{\Omega}_k$ such that $\partial\sigma^k_{s+1} = \gamma_{sk}$ and this implies

$$\int_{\gamma_{sk}} \omega_{sk} = \int_{\partial\sigma^k_{s+1}} \omega_s = \int_{\sigma^k_{s+1}} d\omega_s = 0$$

because ω_s is closed in Ω. ■

Theorem VII of [5] is now extended to forms of class $W_s^{1,p}(\partial\Omega)$.

THEOREM 5.4: *Let* $(\overline{\tau}^i_{n-1-s}, \overline{\gamma}^l_{n-1-s})_{\substack{1 \leq i \leq R^-_{n-1-s} \\ 1 \leq l \leq R^+_{n-1-s}}}$ *be a dual fundamental system of a fundamental system* $(\tau^i_s, \gamma^l_s)_{1 \leq i \leq R_s^-}$ *of* $\partial\Omega$. *Let* $\omega_s \in W_s^{1,p}(\partial\Omega)$ *and* $\Phi_{n-1-s} \in \in \tilde{C}^1_{n-1-s}(\partial\Omega)$. *If* ω_s *and* Φ_{n-1-s} *are closed forms, then*

$$\int_{\partial\Omega} \omega_s \wedge \Phi_{n-1-s} = (-1)^{s(n-1-s)} \left(\sum_{i=1}^{R_s^-} \int_{\tau_s^i} \omega_s \int_{\overline{\gamma}^i_{n-1-s}} \Phi_{n-1-s} + \sum_{l=1}^{R_s^+} \int_{\gamma_s^l} \omega_s \int_{\overline{\tau}^l_{n-1-s}} \Phi_{n-1-s} \right).$$

PROOF: Let $\overline{\omega}^i_{n-1-s}$ and $\overline{w}^l_{n-1-s}$ be forms associated with τ^i_s and γ^l_s respectively. Then

$$([\overline{\omega}^i_{n-1-s}], [\overline{w}^l_{n-1-s}])_{\substack{1 \leq i \leq R^+_{n-1-s} \\ 1 \leq l \leq R^-_{n-1-s}}}$$

is a base of $H^{n-1-s}(\partial\Omega)$. Hence

$$[\Phi_{n-1-s}] = \sum_{i=1}^{R_s^-} c_i [\overline{\omega}^i_{n-1-s}] + \sum_{l=1}^{R_s^+} b_l [\overline{w}^l_{n-1-s}].$$

Thus

$$\int_{\partial\Omega} \omega_s \wedge \Phi_{n-1-s} = \sum_{i=1}^{R_s^-} c_i \int_{\partial\Omega} \omega_s \wedge \overline{\omega}^i_{n-1-s} + \sum_{l=1}^{R_s^+} b_l \int_{\partial\Omega} \omega_s \wedge \overline{w}^l_{n-1-s} =$$

$$= \sum_{i=1}^{R_s^-} c_i \int_{\tau_s^i} \omega_s + \sum_{l=1}^{R_s^+} b_l \int_{\gamma_s^l} \omega_s .$$

Similarly

$$\int_{\overline{\gamma}^i_{n-1-s}} \Phi_{n-1-s} = \sum_{i=1}^{R_s^-} c_i \int_{\overline{\gamma}^i_{n-1-s}} \overline{\omega}^i_{n-1-s} + \sum_{l=1}^{R_s^+} b_l \int_{\overline{\gamma}^i_{n-1-s}} \overline{w}^l_{n-1-s} =$$

$$= \sum_{i=1}^{R_s^-} c_i I(\overline{\gamma}^i_{n-1-s}, \tau^i_s) + \sum_{l=1}^{R_s^+} b_l I(\overline{\gamma}^i_{n-1-s}, \gamma^l_s).$$

Further

$$I(\overline{\gamma}^i_{n-1-s}, \tau^i_s) = (-1)^{s(n-1-s)} I(\tau^i_s, \overline{\gamma}^i_{n-1-s}) = (-1)^{s(n-1-s)} \langle \tau^i_s, \psi_s(\overline{u}^i_s) \rangle$$

where $\overline{u}^i_s$ is a regular form associated with $\overline{\gamma}^i_{n-1-s}$. Since

$$[\overline{\gamma}^i_{n-1-s}] = \Theta_s([\overline{u}^i_s]) = \overline{\Psi}_s([\psi_s(\overline{u}^i_s)])$$

and

$$[\overline{\gamma}^i_{n-1-s}] = \overline{\Psi}_s([\tau^i_s]),$$

it easy be seen that $\Psi_r([\overline{w}^j_r]) = [\tau^j_r]$, where $([\tau^j_r],[\gamma^l_r])_{\substack{1 \leq j \leq R_r^- \\ 1 \leq l \leq R_r^+}}$ is the dual base of $([\tau^j_r],[\gamma^l_r])_{\substack{1 \leq j \leq R_r^- \\ 1 \leq l \leq R_r^+}}$ in $\mathcal{H}^r(\partial\Omega)$. Thus

$$I(\overline{\gamma}^j_{n-1-r}, \tau^i_r) = (-1)^{r(n-1-r)} \langle \tau^i_r, \tau^j_r \rangle = (-1)^{r(n-1-r)} \delta^i_j .$$

Similarly we obtain $I(\overline{\gamma}^j_{n-1-r}, \gamma^l_r) = 0$, hence

$$\int_{\overline{\gamma}^j_{n-1-r}} \Phi_{n-1-r} = (-1)^{r(n-1-r)} c_j .$$

In the same manner we prove that $\int_{\overline{\tau}^l_{n-1-r}} \Phi_{n-1-r} = (-1)^{r(n-1-r)} b_l$. This concludes the proof. ■

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