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FLAVIA LANZARA (*)

# Optimal $L^2$ Estimate for Harmonic Functions (**)

SUMMARY. — A method to compute the best constant for a $L^2$ estimate for harmonic functions is given.

## Maggiorazione $L^2$ ottimale per funzioni armoniche

SUNTO. — Viene esposto un metodo per il calcolo esplicito della migliore costante relativa a una maggiorazione $L^2$ per funzioni armoniche.

If $\Omega$ is a bounded domain of $\mathbb{R}^m$ with a smooth boundary $\partial\Omega$, for every function $u$ harmonic in $\Omega$ and continuous in $\overline{\Omega}$, the following $L^2$ estimate holds:

$$\int_{\Omega} u^2 dx \leq c_{\Omega} \int_{\Sigma} u^2 d\sigma, \tag{1}$$

where $c_{\Omega}$ is a constant only depending on $\Omega$.

Three problems are connected with estimate (1). They have an increasing degree of difficulty.

i) To prove that some $c_{\Omega}$ exists such that, for every $u$, estimate (1) holds.

ii) To explicitly compute some $c_{\Omega}$ such that, for every $u$, estimate (1) holds.

iii) To compute the minimum constant $c_{\Omega}$ such that, for every $u$, estimate (1) holds.

Solving problem iii) implies an *explicit construction* (¹) of two sequences $\{c_i'\}$ and

(*) Indirizzo dell'Autore: Dipartimento di Matematica, Università degli Studi di Roma «La Sapienza», Piazzale A. Moro 5, 00185 Roma, Italy.
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(¹) To construct explicitly a certain constant $c$ means representing $c$ by a formula such that from this, the numerical value of $c$ can be computed.

$\{c_l''\}$ such that

$$c_l' \leq c_\Omega \leq c_l'', \qquad \lim_{l\to\infty} c_l' = \lim_{l\to\infty} c_l'' = c_\Omega .$$

In explicitly constructing $\{c_l'\}$ and $\{c_l''\}$ the domain $\Omega$ must be considered as the only *datum*.

Problem i) is easily solved by using simple arguments of Functional Analysis. Several Authors have considered problem ii) (see [3], [4], [5], [1], [2], [12], [14, pp. 19-31]) presenting various solutions.

The aim of the present paper is to propose a solution for problem iii).

## 1. - Position of the problem

Let $\Omega$ be a bounded domain (*i.e.* an open set) of the real cartesian space $R^m$ such that $R^m - \overline{\Omega}$ is connected and its boundary $\Sigma = \partial\Omega$ is a Lyapounov hypersurface, *i.e.* $\Sigma$ has a uniformly Hölder continuous field $\nu_x$ of some exponent $\alpha$, $0 < \alpha \leq 1$; $\nu_x = \{\nu_1(x), \nu_2(x), \ldots, \nu_m(x)\}$ is the inward unit normal on $\Sigma$.

We denote by $L^2(\Omega)[L^2(\Sigma)]$ the space of all measurable real functions $u$ such that $u^2$ is integrable over $\Omega[\Sigma]$. Let $(u,v)_\Omega[(u,v)]$ and $\|u\|_\Omega[\|u\|]$ be the scalar product and the norm, respectively, in $L^2(\Omega)[L^2(\Sigma)]$.

Let $s(x,\xi)$ be the fundamental solution for the Laplace equation:

$$s(x,\xi) = \begin{cases} \dfrac{-1}{2\pi} \log|x-\xi| & m = 2 \\[2ex] \dfrac{1}{\omega_m (m-2)} |x-\xi|^{2-m} & m > 2 \end{cases}$$

where $\omega_m$ represents the hypersurface area $\omega_m = 2\pi^{m/2} / \Gamma(m/2)$ of the unit sphere of $R^m$.

Define the space $\mathcal{U}$ formed of all functions $u$ of the form

$$u(x) = \int_\Sigma \varphi(\xi) \frac{\partial}{\partial \nu_\xi} s(x,\xi)\, d\sigma_\xi , \qquad x \in \Omega , \tag{1.1}$$

where $\varphi \in L^2(\Sigma)$.

We consider the following estimate:

$$\int_\Omega u^2 dx \leq c_\Omega \int_\Sigma u^2 d\sigma , \qquad \forall u \in \mathcal{U} . \tag{1.2}$$

If $\Omega \subset R^m$ is a ball of radius $R$ then $c_\Omega = R/m$. We shall give , in a more general bounded domain $\Omega$, a method to compute the optimal constant $c_\Omega$ in (1.2).

Setting

$$K(x, \xi) = 2 \frac{\partial}{\partial \nu_\xi} s(x, \xi) = \frac{2(x - \xi) \cdot \nu_\xi}{\omega_m |x - \xi|^m} \text{ (}^2\text{)},$$

it is known that

$$K(x, \xi) = \mathcal{O}(|x - \xi|^{1-m+\alpha}), \quad x, \xi \in \Sigma.$$

Define the operators

(1.3)
$$(K\varphi)(x) = \int_\Sigma K(x, \xi) \varphi(\xi) d\sigma_\xi$$

and

$$(K^* \varphi)(x) = \int_\Sigma K(\xi, x) \varphi(\xi) d\sigma_\xi .$$

The operators $K$ and $K^*$ are linear and compact operators of the space $L^2(\Sigma)$ into $L^2(\Sigma)$ (see [16, p. 329]).

If $u \in \mathcal{U}$, that is there exists $\varphi \in L^2(\Sigma)$ such that $u = (1/2) K\varphi$ in $\Omega$, it is well known that $\varphi$ is solution of the following integral equation on $\Sigma$:

(1.4)
$$\varphi + K\varphi = 2u .$$

Observe that if $u \in \mathcal{U}$, from (1.4) and (1.2), it follows that $u \in L^2(\Omega)$.

Consider the eigenvalue problems

(1.5)
$$Kz - \lambda z = 0 , \quad z \in L^2(\Sigma);$$

(1.6)
$$K^* z - \lambda z = 0 , \quad z \in L^2(\Sigma).$$

The spectrum of $K$ contains the value zero and all the eigenvalues of (1.5). They are real and not bigger than one in absolute value. Equation (1.6) has the same eigenvalues of (1.5) with the same geometric multiplicity. Moreover $\lambda = 1$ is an eigenvalue of (1.5) with geometric multiplicity one; $\lambda = -1$ is not an eigenvalue for (1.5) (see [9, pp. 510-513], [10, pp. 309-311], [11, pp. 362-364]). It follows that, for any $u \in L^2(\Sigma)$, there exists one and only one solution $\varphi \in L^2(\Sigma)$ of equation (1.4). Denote by

(1.7)
$$\varphi = 2Su$$

this solution. $S = (I + K)^{-1}$ is a linear and continuous operator of $L^2(\Sigma)$ into itself.

(2) We denote by $(x - \xi) \cdot \nu_\xi = \sum_{i=1}^{m} (x_i - \xi_i) \nu_i(\xi)$.

Setting

$$T(x, \xi) = \int_{\Omega} K(\eta, x) K(\eta, \xi) d\eta, \tag{1.8}$$

we consider the operator

$$(T\varphi)(x) = \int_{\Sigma} T(x, \xi) \varphi(\xi) d\sigma_{\xi}, \qquad \varphi \in L^2(\Sigma). \tag{1.9}$$

$T$ is a linear, self-adjoint and positive operator of the space $L^2(\Sigma)$. Since, if $x, \xi \in \Sigma$,

$$T(x, \xi) = \begin{cases} \mathcal{O}(1 + |\log|x - \xi||) & m = 2, \\ \mathcal{O}(|x - \xi|^{2-m}) & m > 2, \end{cases}$$

(see [13, p. 806]) the operator $T$ is a PCO (Positive Compact Operator) of the space $L^2(\Sigma)$.

If $u \in \mathcal{U}$, from (1.1), (1.8), (1.9) and (1.7) we obtain:

$$(u, u)_{\Omega} = \frac{1}{4}(K\varphi, K\varphi)_{\Omega} = \frac{1}{4}(T\varphi, \varphi) = (TSu, Su) = (S^* TSu, u), \tag{1.10}$$

where $S^*$ denotes the adjoint operator of $S$. Hence inequality (1. 2) is equivalent to the following one:

$$(S^* TSu, u) \leq c_{\Omega}(u, u), \qquad u \in L^2(\Sigma). \tag{1.11}$$

Consider the eigenvalue problem:

$$S^* TSu = \mu u, \qquad u \in L^2(\Sigma). \tag{1.12}$$

Since $S^* TS$ is a PCO of the space $L^2(\Sigma)$, (1.12) has a decreasing sequence of positive eigenvalues tending to zero. If $\mu_1$ is the greatest eigenvalue, from (1.11) we have: $c_{\Omega} = \mu_1$.

Therefore: *the optimal constant $c_{\Omega}$ in (1.2) is the greatest eigenvalue of the PCO of the space $L^2(\Sigma)$: $S^* TS$.*

Lower bounds of $c_{\Omega}$ can be easily obtained by applying the classical Rayleigh-Ritz method (see [6, pp. 112-119], [8, pp. 11-12]). To this end consider a complete system of homogeneous harmonic polynomials $\{\omega_k(x)\}_{k \geq 1}$. For a fixed $l \geq 1$, since $(S^* TS\omega_h, \omega_k) = (\omega_h, \omega_k)_{\Omega}$, the relevant determinant equation is

$$\det\{(\omega_h, \omega_k)_{\Omega} - \mu(\omega_h, \omega_k)\}_{h,k=1,\dots,l} = 0. \tag{1.13}$$

Denote by $c_l'$ its greatest root. Then: $c_l' \leq c_{l+1}' \leq c_{\Omega}$ and $\lim_{l \to \infty} c_l' = c_{\Omega}$.

Of course, we are interested in upper bounds for $c_{\Omega}$ arbitrarily close to $c_{\Omega}$. In order to obtain that we shall construct the operator $S$.

## 2. - The operator $S$

In order to construct $S$, we shall use a method proposed by E. Schmidt and extended by M. Picone in [13, pp. 582-591]. This method consists in solving equation (1.4) by reducing it to a system of linear algebraic equations. This method has also been used by F. Tricomi in [15, pp. 64-66]. In this paper we shall follow the exposition given in [7, pp. 152-182].

Let us fix an arbitrary constant $\varepsilon$: $0 < \varepsilon \leq 1/2$. Since the operator $K$ is compact, it can be approximated by a sequence of finite rank operators uniformly converging to $K$. Let $\{u_h\}$ be a complete system of linearly independent functions in $L^2(\Sigma)$ and let $P_s$ be the orthogonal projector of $L^2(\Sigma)$ onto the $s$-dimensional manifold spanned by $\{u_1, \ldots, u_s\}$ $(s \geq 1)$. The operators sequence $\{P_s K P_s\}$ uniformly converges to $K$. Then there exists an index $n = n(\varepsilon)$ such that

(2.1) $$\|K - K_n\| < \varepsilon\,(^3),$$

where

(2.2) $$K_n = P_n K P_n .$$

If $\{\beta_{ij}\}_{i,j=1,\ldots,n}$ denotes the inverse matrix of the non singular matrix $\{(u_i, u_j)\}_{i,j=1,\ldots,n}$, we have

(2.3) $$P_n \varphi = \sum_{h=1}^{n} (\varphi, u_h)\, \tilde{u}_h ,$$

where

(2.4) $$\tilde{u}_h = \sum_{k=1}^{n} \beta_{hk} u_k , \qquad h = 1, \ldots, n .$$

Setting $w_h = P_n K^* u_h$, $h = 1, \ldots, n$, from (2.2) and (2.3) it follows that

(2.5) $$K_n \varphi = \sum_{h=1}^{n} (\varphi, w_h)\, \tilde{u}_h .$$

Equation (1.4) can be rewritten in equivalent way

(2.6) $$\varphi + (K - K_n)\varphi = v - K_n \varphi$$

where $v = 2u$. Then, if we consider $v - K_n\varphi$ as the known term in (2.6), equation (2.6) becomes

$$\varphi = \sum_{s=0}^{\infty} (-1)^s (K - K_n)^s (v - K_n \varphi).$$

(3) $\|K\|$ denotes the norm of $K$: $L^2(\Sigma) \rightarrow L^2(\Sigma)$.

From (2. 1) it follows that

$$\sum_{s=0}^{\infty} \|(-1)^s (K-K_*)^s\| \le \sum_{s=0}^{\infty} \varepsilon^s = \frac{1}{1-\varepsilon}.$$

Therefore the series $\sum_{s=0}^{\infty}(-1)^s (K-K_*)^s$ uniformly converges to the operator

(2.7) $$M = \sum_{s=0}^{\infty}(-1)^s (K-K_*)^s = [I + (K-K_*)]^{-1}.$$

$M$ [4] is a linear and continuous operator of the space $L^2(\Sigma)$ such that

(2.8) $$\|M\| \le \frac{1}{1-\varepsilon} \le 2.$$

Hence equation (2.6) can be transformed into the equation: $\varphi = Mv - MK_*\varphi$. Inserting (2.5) we deduce:

(2.9) $$\varphi = Mv - \sum_{h=1}^{n} (\varphi, w_h) M\bar{u}_h$$

which is equivalent to (2.6) in the sense that $\varphi$ is solution of (2.6) if and only if $\varphi$ is solution of (2.9).

Scalar multiplication of (2.9) by $w_k$ in $L^2(\Sigma)$ gives the system of algebraic equations

(2.10) $$\sum_{h=1}^{n} [\delta_{hk} + (M\bar{u}_h, w_k)]\xi_h = (Mv, w_k), \qquad k = 1, \ldots, n \text{ (}^5\text{)},$$

in which

(2.11) $$\xi_h = (\varphi, w_h), \qquad h = 1, \ldots, n.$$

Consider the matrix $Q = \{q_{hk}\}_{h,k=1,\ldots,n}$ whose elements are

(2.12) $$q_{hk} = \delta_{hk} + (M\bar{u}_h, w_k), \qquad h, k = 1, \ldots, n.$$

Setting

(2.13) $$\gamma_h = (Mv, w_h), \qquad h = 1, \ldots, n$$

the system (2.10), by means of the substitution (2.12), becomes

(2.14) $$\sum_{h=1}^{n} q_{hk}\xi_h = \gamma_k, \qquad k = 1, \ldots, n.$$

Then, if $\varphi$ is solution of (2.6), the $n$-vector $\xi = \{\xi_1, \xi_2, \ldots, \xi_n\}$, whose components are defined in (2.11), is solution of (2.14).

(4) Since $\varepsilon$ is a fixed constant we don't explicitly write $M = M(\varepsilon)$.
(5) $\delta_{hk}$ is the Kronecker delta.

Conversely, let $\xi$ be solution of (2.14). Set

$$\varphi = Mv - \sum_{h=1}^{n} \xi_h M \bar{u}_h . \tag{2.15}$$

Scalar multiplication of (2.15) by $w_k$ in $L^2(\Sigma)$ gives

$$(\varphi, w_k) + \sum_{h=1}^{n} \xi_h (M\bar{u}_h, w_k) = \gamma_k , \qquad k = 1, \ldots, n . \tag{2.16}$$

Subtracting (2.16) from (2.10), by using (2.13), we obtain (2.11). Substituting (2.11) in (2.15) we deduce that $\varphi$ is solution of (2.9), that is (2.6).

Thus we have proved that, if $\xi$ is solution of system (2.14), then $\varphi$, defined in (2.15), is solution of (2.6).

It follows that the matrix $Q$ is non singular. Otherwise there exists a $n$-vector $\{\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_n\}$, $\bar{\xi}_1^2 + \bar{\xi}_2^2 + \ldots + \bar{\xi}_n^2 > 0$, solution of the system $\sum_{h=1}^{n} q_{kh} \bar{\xi}_h = 0$, $k = 1, \ldots, n$. Set $\bar{z} = \sum_{h=1}^{n} \bar{\xi}_h M \bar{u}_h$. Since $\{u_1, u_2, \ldots, u_n\}$ are linearly independent functions, $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n\}$ and $\{M\bar{u}_1, M\bar{u}_2, \ldots, M\bar{u}_n\}$ are linearly independent too. Then $\bar{z} \neq 0$. Moreover $\bar{z}$ is solution of (1.5) by assuming $\lambda = -1$. This is impossible because $\lambda = -1$ is not an eigenvalue for problem (1.5).

Denote by $Q^{-1} = \{a_{ij}\}_{i,j=1,\ldots,n}$ the inverse matrix of $Q$:

$$\sum_{k=1}^{n} a_{ik} q_{kh} = \delta_{ih} , \qquad i, h = 1, \ldots, n . \tag{2.17}$$

Hence, in view of (2.14) and (2.17), we obtain:

$$\xi_h = \sum_{k=1}^{n} a_{hk} \gamma_k , \qquad h = 1, \ldots, n . \tag{2.18}$$

Inserting (2.18) into (2.15), using (2.13), we have that $\varphi$ *is solution of equation* (2.6), i.e. (1.4), *if and only if*

$$\varphi = Sv = Mv - \sum_{h,k}^{1,n} a_{hk} (Mv, w_k) M \bar{u}_h ,$$

*where* $v = 2u$.

Thus we have constructed the operator $S$.

## 3. - Approximation of the operator $S$

Setting, for $l \geq 1$

$$M_l = \sum_{s=0}^{l} (-1)^s (K - K_*)^s ; \qquad R_l = \sum_{s=l+1}^{\infty} (-1)^s (K - K_*)^s , \tag{3.1}$$

from (2.7) the operator $M = M_l + R_l$. Moreover, from the inequality (2.1), since

$0 < \varepsilon \leq 1/2$, we deduce that

$$\|M_l\| \leq 2; \tag{3.2}$$

$$\|R_l\| \leq 2\varepsilon^{l+1}. \tag{3.3}$$

Then the elements $\{q_{hk}\}$ of the matrix $Q$ can be written in the form:

$$q_{hk} = q_{hk}^{(l)} + r_{hk}^{(l)}, \qquad h, k = 1, \dots, n$$

where

$$q_{hk}^{(l)} = \delta_{hk} + (KP_n M_l \tilde{u}_k, u_h), \qquad h, k = 1, \dots, n,$$

$$r_{hk}^{(l)} = (KP_n R_l \tilde{u}_k, u_h), \qquad h, k = 1, \dots, n.$$

For any $k = 1, \dots, n$ we have: $\|\tilde{u}_k\| = \sqrt{\beta_{kk}}$. Then, from (3.2), we deduce the following estimate for the elements of the $n \times n$ matrix $\mathcal{R}_l = \{r_{hk}^{(l)}\}_{h,k=1,\dots,n}$:

$$|r_{hk}^{(l)}| \leq 2\|K\|\|u_h\|\sqrt{\beta_{kk}}\varepsilon^{l+1}, \qquad h, k = 1, \dots, n. \tag{3.4}$$

Then, if we denote by $\operatorname{tr} \mathcal{B} = \sum_{k=1}^{n} \beta_{kk}$ the trace of the matrix $\mathcal{B} = \{\beta_{hk}\}_{h,k=1,\dots,n}$, from (3.4) we obtain the following estimate for the Frobenius norm $|\mathcal{R}_l|_F$ of $\mathcal{R}_l$:

$$|\mathcal{R}_l|_F = \left(\sum_{h,k}^{1,n} |r_{hk}^{(l)}|^2\right)^{1/2} \leq 2\|K\|\left(\sum_{h=1}^{n} \|u_h\|^2\right)^{1/2} \sqrt{\operatorname{tr} \mathcal{B}}\, \varepsilon^{l+1}. \tag{3.5}$$

Consider the $n \times n$ matrix $Q^* Q$. It has $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ positive eigenvalues. Denote by $Q_l$ the matrix $\{q_{hk}^{(l)}\}_{h,k=1,\dots,n}$. We have $Q = Q_l + \mathcal{R}_l$. Let $\lambda_1^{(l)} \geq \lambda_2^{(l)} \geq \dots \geq \lambda_n^{(l)}$ be the eigenvalues of $Q_l^* Q_l$. Since

$$|\lambda_n^{(l)} - \lambda_n| \leq |Q_l^* Q_l - Q^* Q| \leq |\mathcal{R}_l|_F [2|Q_l|_F + |\mathcal{R}_l|_F] (^*), \tag{3.6}$$

we have that

$$\lim_{l \to \infty} \lambda_n^{(l)} = \lambda_n.$$

Assume $l_0 \geq 1$ such that, for $l \geq l_0$, we have:

$$\lambda_n^{(l)} > \frac{1}{2}\lambda_n. \tag{3.7}$$

Set

$$\gamma_h = (KP_n Mv, u_h) = \gamma_h^{(l)} + \varrho_h^{(l)}, \qquad h = 1, \dots, n$$

(*) Notice that, for a $n \times n$ matrix $A$, we have $|A| \leq |A|_F$, where

$$|A| = \sup_{x \in \mathbb{R}^n - \{0\}} \frac{|Ax|}{|x|}, \qquad |x| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

where $\gamma_h^{(l)} = (KP_n M_l v, u_h)$ and $\varrho_h^{(l)} = (KP_n R_l v, u_h)$. If $\varrho^{(l)} = \{\varrho_1^{(l)}, \dots, \varrho_n^{(l)}\}$, $\gamma^{(l)} = \{\gamma_1^{(l)}, \dots, \gamma_n^{(l)}\}$ and $\gamma = \{\gamma_1, \dots, \gamma_n\}$, we have $\gamma = \gamma^{(l)} + \varrho^{(l)}$.

From (3.2) and (3.3) it follows that

$$|\varrho^{(l)}| = \sqrt{\sum_{h=1}^{n} (\varrho_h^{(l)})^2} \leq 2\|K\|\left(\sum_{h=1}^{n} \|u_h\|^2\right)^{1/2} \|v\| \varepsilon^{l+1}; \tag{3.8}$$

$$|\gamma^{(l)}| = \sqrt{\sum_{h=1}^{n} (\gamma_h^{(l)})^2} \leq 2\|K\|\left(\sum_{h=1}^{n} \|u_h\|^2\right)^{1/2} \|v\|. \tag{3.9}$$

Denote by $\xi = \{\xi_1, \dots, \xi_n\}$ the solution of the system (2.14), that is $Q\xi = \gamma$. Analogously, let $\xi^{(l)} = \{\xi_1^{(l)}, \dots, \xi_n^{(l)}\}$ $(l \geq l_0)$ be the solution of the system

$$Q_l \xi^{(l)} = \gamma^{(l)}. \tag{3.10}$$

Denote by $x \cdot x' = \sum_{i=1}^{n} x_i x_i'$ the scalar product in $\mathbb{R}^n$. We have

$$|Q^{-1}| = \sup_{x \in \mathbb{R}^n - \{0\}} \left(\frac{Q^{-1}x \cdot Q^{-1}x}{x \cdot x}\right)^{1/2} = \inf_{\eta \in \mathbb{R}^n - \{0\}} \left(\frac{Q\eta \cdot Q\eta}{\eta \cdot \eta}\right)^{-1/2} = (\lambda_n)^{-1/2}. \tag{3.11}$$

Analogously, assuming $l \geq l_0$, if $Q_l^{-1}$ is the inverse matrix of $Q_l$, from (3.7),

$$|Q_l^{-1}| = (\lambda_n^{(l)})^{-1/2} < \sqrt{2}(\lambda_n)^{-1/2}. \tag{3.12}$$

Let us write

$$\xi = \xi^{(l)} + (\xi - \xi^{(l)}) = \xi^{(l)} + \Lambda_1^{(l)} + \Lambda_2^{(l)}, \tag{3.13}$$

where $\Lambda_1^{(l)} = Q^{-1}(\gamma - \gamma^{(l)})$ and $\Lambda_2^{(l)} = Q^{-1}(Q_l - Q)Q_l^{-1}\gamma^{(l)}$.

From (3.11) and (3.8) we deduce the following estimate for the $n$-vector $\Lambda_1^{(l)} = \{\Lambda_{1,1}^{(l)}, \dots, \Lambda_{1,n}^{(l)}\}$:

$$|\Lambda_1^{(l)}| = |Q^{-1}\varrho^{(l)}| \leq |Q^{-1}||\varrho^{(l)}| \leq \frac{2\|K\|\|v\|}{\sqrt{\lambda_n}}\left(\sum_{h=1}^{n} \|u_h\|^2\right)^{1/2} \varepsilon^{l+1} \tag{3.14}$$

and for $\Lambda_2^{(l)} = \{\Lambda_{2,1}^{(l)}, \dots, \Lambda_{2,n}^{(l)}\}$, from (3.11), (3.12), (3.9) and (3.5),

$$\begin{aligned} |\Lambda_2^{(l)}| &= |Q^{-1}\mathcal{R}_l Q_l^{-1}\gamma^{(l)}| \leq |Q^{-1}||\mathcal{R}_l||Q_l^{-1}||\gamma^{(l)}| \leq \\ &\leq \frac{4\sqrt{2}\|K\|^2\|v\|}{\lambda_n}\left(\sum_{h=1}^{n} \|u_h\|^2\right)\sqrt{\operatorname{tr}\mathcal{B}}\,\varepsilon^{l+1}. \end{aligned} \tag{3.15}$$

Set $M\bar{U} = \{M\bar{u}_1, \dots, M\bar{u}_n\}$; $M_l\bar{U} = \{M_l\bar{u}_1, \dots, M_l\bar{u}_n\}$ and $R_l\bar{U} = \{R_l\bar{u}_1, \dots, R_l\bar{u}_n\}$. Inserting (3.13) in (2.15), since $M = M_l + R_l$, we obtain

$$\varphi = Sv = M_l v + R_l v - \xi^{(l)} \cdot M_l\bar{U} - \xi^{(l)} \cdot R_l\bar{U} - \Lambda_1^{(l)} \cdot M\bar{U} - \Lambda_2^{(l)} \cdot M\bar{U}.$$

Setting

$$S_l v = M_l v - \xi^{(l)} \cdot M_l \bar{U} \tag{3.16}$$

and

$$N_l v = R_l v - \xi^{(l)} \cdot R_l \bar{U} - \Lambda_1^{(l)} \cdot M \bar{U} - \Lambda_2^{(l)} \cdot M \bar{U} \tag{3.17}$$

we have: $S = S_l + N_l$. $N_l$ is a linear and continuous operator of the space $L^2(\Sigma)$.

From (3.3), (3.10), (3.12), (3.9) and (2.8) we obtain the following inequalities:

$$\|R_l v\| \leqslant 2\|v\|\varepsilon^{l+1};$$

$$\|\xi^{(l)} \cdot R_l \bar{U}\| \leqslant \sum_{i=1}^{n} |\xi_i^{(l)}| \|R_l \bar{u}_i\| \leqslant |\xi^{(l)}| \Big(\sum_{i=1}^{n} \|R_l \bar{u}_i\|^2\Big)^{1/2} = |Q_l^{-1} \gamma^{(l)}| \Big(\sum_{i=1}^{n} \|R_l \bar{u}_i\|^2\Big)^{1/2} \leqslant$$

$$\leqslant 2|Q_l^{-1}| |\gamma^{(l)}| \sqrt{\operatorname{tr} \mathcal{B}}\, \varepsilon^{l+1} \leqslant \frac{4\sqrt{2}\|K\|\|v\|}{\sqrt{\lambda_n}} \Big(\sum_{k=1}^{n} \|u_k\|^2\Big)^{1/2} \sqrt{\operatorname{tr} \mathcal{B}}\, \varepsilon^{l+1};$$

$$\|\Lambda_1^{(l)} \cdot M \bar{U}\| \leqslant \sum_{i=1}^{n} |\Lambda_{1,i}^{(l)}| \|M \bar{u}_i\| \leqslant |\Lambda_1^{(l)}| \Big(\sum_{i=1}^{n} \|M \bar{u}_i\|^2\Big)^{1/2} \leqslant 2|\Lambda_1^{(l)}| \sqrt{\operatorname{tr} \mathcal{B}};$$

$$\|\Lambda_2^{(l)} \cdot M \bar{U}\| \leqslant \sum_{i=1}^{n} |\Lambda_{2,i}^{(l)}| \|M \bar{u}_i\| \leqslant |\Lambda_2^{(l)}| \Big(\sum_{i=1}^{n} \|M \bar{u}_i\|^2\Big)^{1/2} \leqslant 2|\Lambda_2^{(l)}| \sqrt{\operatorname{tr} \mathcal{B}}.$$

Because of these inequalities, from (3.17), and keeping in mind (3.14), (3.15), we have, for $l \geqslant l_0$,

$$\|N_l\| \leqslant c_n \varepsilon^{l+1} \tag{3.18}$$

where

$$c_n = 2 + 4(1+\sqrt{2}) \frac{\|K\|}{\sqrt{\lambda_n}} \Big(\sum_{k=1}^{n} \|u_k\|^2\Big)^{1/2} \sqrt{\operatorname{tr} \mathcal{B}} + \frac{8\sqrt{2}\|K\|^2}{\lambda_n} \Big(\sum_{k=1}^{n} \|u_k\|^2\Big) \operatorname{tr} \mathcal{B}. \tag{8.19}$$

## 4. - The optimal constant $c_Q$ in (1.2)

The operator $K_n$ admits the integral representation:

$$(K_n \varphi)(x) = \int_{\Sigma} K_n(x, \xi) \varphi(\xi) d\sigma_\xi,$$

where, in view of (2.2), (2.3), (2.4) and (1.3), the kernel $K_n(x, \xi)$ has the form:

$$K_n(x, \xi) = \sum_{h,k}^{1,n} \beta_{hk} \sum_{i,j}^{1,n} \beta_{ij} u_j(\xi) u_k(x) \int_{\Sigma} u_h(\eta) d\sigma_\eta \int_{\Sigma} K(\eta, y) u_i(y) d\sigma_y .$$

Define

$$H_s^{(n)}(x, \xi) = \int_{\Sigma} H_{s-1}^{(n)}(x, \eta) H^{(n)}(\eta, \xi) d\sigma_\eta , \qquad s \geq 1;$$

$$H_1^{(n)}(x, \xi) = K(x, \xi) - K_n(x, \xi).$$

$H_s^{(n)}(x, \xi)$ is the $s$-th iterated kernel that is the kernel of the operator $(K - K_n)^s$.

Setting

$$\Gamma_l(x, \xi) = \sum_{s=1}^{l} (-1)^s H_s^{(n)}(x, \xi),$$

the operator $M_l$ in (3.1) can be written in the form:

$$(M_l v)(x) = v(x) + \int_{\Sigma} \Gamma_l(x, \xi) v(\xi) d\sigma_\xi . \tag{4.1}$$

Denote by $\{a_{hk}^{(l)}\}$ the elements of the matrix $Q_l^{-1}$. The operator $S_l$ in (3.16), by means of the substitution $\xi^{(l)} = Q_l^{-1} \gamma^{(l)}$, is the following:

$$S_l v = M_l v - \sum_{h,k}^{1,n} a_{hk}^{(l)} (M_l v, w_k) M_l \bar{u}_h . \tag{4.2}$$

Set, for $h = 1, \dots, n$,

$$\begin{aligned} \alpha_h^{(l)}(x) &= (M_l \bar{u}_h)(x) = \bar{u}_h(x) + \int_{\Sigma} \Gamma_l(x, \xi) \bar{u}_h(\xi) d\sigma_\xi ; \\ \eta_h^{(l)}(x) &= (M_l^* w_h)(x) = w_h(x) + \int_{\Sigma} \Gamma_l(\xi, x) w_h(\xi) d\sigma_\xi . \end{aligned} \tag{4.3}$$

Consider the integral operator

$$(\Omega_l v)(x) = \int_{\Sigma} \Omega_l(x, \xi) v(\xi) d\sigma_\xi ,$$

where

$$\Omega_l(x, \xi) = \Gamma_l(x, \xi) - \sum_{h,k}^{1,n} a_{hk}^{(l)} \alpha_h^{(l)}(x) \eta_k^{(l)}(\xi).$$

The operator $S_l$ in (4.2), by means of (4.1) and (4.3), is given by $S_l = I + \Omega_l$, where $I$

denotes the identity of the space $L^2(\Sigma)$. Then

$$S = I + \mathcal{Q}_l + N_l. \tag{4.4}$$

Hence we have:

$$S^* TS = (I + \mathcal{Q}_l^*)\, T(I + \mathcal{Q}_l) + N_l^*\, T(I + \mathcal{Q}_l) + (I + \mathcal{Q}_l^*)\, TN_l + N_l^*\, TN_l\,.$$

From (3.16), (3.10), (3.2), (3.12) and (3.9) we deduce the following estimate:

$$\|I + \mathcal{Q}_l\| = \|S_l\| \leq \|M_l\| + \sup_{\|v\|=1} \|Q_l^{-1} \gamma^{(l)} \cdot M_l \bar{U}\| \leq d_n\,, \tag{4.5}$$

where

$$d_n = 2 + \frac{4\sqrt{2}\|K\|}{\sqrt{\lambda_n}} \sqrt{\operatorname{tr} \mathcal{B}} \left( \sum_{k=1}^{n} \|u_k\|^2 \right)^{1/2}. \tag{4.6}$$

The first eigenvalue of (1.12) is characterized by

$$\mu_1 = \sup_{\substack{u \in L^2(\Sigma) \\ \|u\|=1}} (S^* TSu, u). \tag{4.7}$$

Let $\mu_1^{(l)}$ be the greatest eigenvalue of the PCO: $S_l^* TS_l$ that is

$$\mu_1^{(l)} = \sup_{\substack{u \in L^2(\Sigma) \\ \|u\|=1}} (S_l^* TS_l u, u). \tag{4.8}$$

From (4.7), (4.4) and (4.8) we deduce

$$\mu_1 = \sup_{\substack{u \in L^2(\Sigma) \\ \|u\|=1}} ((I + \mathcal{Q}_l^* + N_l^*)\, T(I + \mathcal{Q}_l + N_l)u, u) \leq$$

$$\leq \mu_1^{(l)} + 2\|N_l\|\|T\|\|I + \mathcal{Q}_l\| + \|N_l\|^2 \|T\|.$$

Then, from (3.18) and (4.5),

$$\mu_1 \leq \chi^{(l)} \tag{4.9}$$

where

$$\chi^{(l)} = \mu_1^{(l)} + 2\|T\| d_n c_n \varepsilon^{l+1} + c_n^2 \|T\| \varepsilon^{2l+2}. \tag{4.10}$$

Since $\lim_{l \to \infty} \mu_1^{(l)} = \mu_1$, we have

$$\lim_{l \to \infty} \chi^{(l)} = \mu_1\,.$$

Provided $l$ large enough, taking into account (4.10), (3.19) and (4.6), (4.9) gives upper approximation of $\mu_1 = c_\Omega$ arbitrarily close to the optimal $c_\Omega$.

We remark that all the terms in the right-hand side of (4.10) are known or they can be explicitly computed. In order to do that let us recall some definitions.

A PCO $G$ of the space $L^2(\Sigma)$ belongs to the class $\mathcal{C}$ if and only if $G'$ admits the following integral representation:

$$G'u = \int_\Sigma \mathcal{G}(x,\xi)u(\xi)\,d\sigma_\xi; \qquad \mathcal{G}(x,\xi) = \int_\Sigma \bar{\mathcal{G}}(x,\eta)\,\bar{\mathcal{G}}(\eta,\xi)\,d\sigma_\eta$$

and $\bar{\mathcal{G}}(x,\xi)$ is a hermitian kernel belonging to $L^2(\Sigma\times\Sigma)$. Then every Orthogonal Invariant $\mathfrak{I}_1^p(G)$ (see [6], [8]) is finite for $p \geq r$. Moreover $\mathfrak{I}_1^r(G)$ admits the following integral representation

$$\mathfrak{I}_1^r(G) = \int_\Sigma \mathcal{G}(x,x)\,d\sigma_x = \int_\Sigma d\sigma_x \int_\Sigma |\bar{\mathcal{G}}(x,\xi)|^2 d\sigma_\xi .$$

The Method of Orthogonal Invariants permits to compute upper approximations of the eigenvalues of a PCO $G$ provided some Orthogonal Invariant $\mathfrak{I}_1^p(G)$ of $G$ is known (for details on the method we refer to [6, pp. 139-163], [8, pp. 27-35]).

Consider the right-hand side in (4.10), together with (3.19) and (4.6).

$\|T\|$ is the greatest eigenvalue of the PCO $T$ of the space $L^2(\Sigma)$. Upper bounds to $\|T\|$ can be computed by applying the Method of Orthogonal Invariants. In fact, if $T_r(x,\xi)$ denotes the $r$-th iterated kernel of $T_1(x,\xi) = T(x,\xi)$, it is well known (see [13, p. 806]) that, for $x, \xi \in \Sigma$,

$$T_r(x,\xi) = \begin{cases} \mathcal{O}(|x-\xi|^{r+1-m}) & m > r+1, \\ \mathcal{O}(1 + |\log|x-\xi||) & m = r+1, \\ \mathcal{O}(1) & m < r+1. \end{cases}$$

If we assume $r > (m-1)/2$ then $T_r(x,\xi) \in L^2(\Sigma\times\Sigma)$. We deduce that $T^2$ belongs to the class $\mathcal{C}$ for $r > (m-1)/2$. For $s \geq 1$, let $\bar{\delta}_1^{(s)} \geq \ldots \geq \bar{\delta}_s^{(s)}$ be the roots of the determinant equation $\det\{(T\omega_h, \omega_k) - \bar{\delta}(\omega_h, \omega_k)\}_{h,k=1,\ldots,s} = 0$. $\{\omega_h\}$ have the same meaning as in (1.13). Fix $r > (m-1)/(4\alpha)$ and set

$$\bar{\chi}^{(s)} = \left[\int_\Sigma d\sigma_x \int_\Sigma |T_r(x,\xi)|^2 d\sigma_\xi - \sum_{h=2}^{s} [\bar{\delta}_h^{(s)}]^{2r}\right]^{1/2r}.$$

We have $\bar{\delta}_1^{(s)} \leq \|T\| \leq \bar{\chi}^{(s)}$ and $\lim_{s\to\infty} \bar{\delta}_1^{(s)} = \lim_{s\to\infty} \bar{\chi}^{(s)} = \|T\|$.

$\|K\|^2$ is the greatest eigenvalue of the PCO of the space $L^2(\Sigma)$: $K^*K$. Denote by

$$\mathcal{K}(x,\xi) = \int_\Sigma K(\eta,x)K(\eta,\xi)\,d\sigma_\eta$$

the kernel of $K^*K$. If $\mathcal{K}_q(x,\xi)$ is $q$-th iterated kernel of $\mathcal{K}_1(x,\xi) = \mathcal{K}(x,\xi)$ we have, for $x, \xi \in \Sigma$, $\mathcal{K}_q(x,\xi) = \mathcal{O}(|x-\xi|^{1+2q\alpha-m})$, $(1 \leq q < (m-1)/(2\alpha))$. Then $(K^*K)^2$

belongs to the class $\mathcal{C}^q$ for $q > (m-1)/(4\alpha)$ and upper approximations of $\|K\|^2$ can be computed by applying again the Method of Orthogonal Invariants. Assume $q > (m-1)/(4\alpha)$ and $p \geq 1$. Set

$$\tilde{\sigma}^{(p)} = \left[ \int_\Sigma d\sigma_x \int_\Sigma |\mathcal{K}_q(x, \xi)|^2 d\sigma_\xi - \sum_{h=2}^{p} [v_h^{(p)}]^{2q} \right]^{1/2q},$$

where $v_1^{(p)} \geq \ldots \geq v_p^{(p)}$ are the roots of the determinant equation $\det\{(K^* K\omega_h, \omega_k) - v(\omega_h, \omega_k)\}_{h,k=1,\ldots,p} = 0$. We have $v_1^{(p)} \leq \|K\|^2 \leq \tilde{\sigma}^{(p)}$; $\lim_{p\to\infty} v_1^{(p)} = \lim_{p\to\infty} \tilde{\sigma}^{(p)} = \|K\|^2$.

Assume $s$ and $p$ are fixed and set $\mathfrak{M} = \tilde{\chi}^{(s)}$; $\mathcal{N} = \tilde{\sigma}^{(p)}$. We have

$$\|T\| \leq \mathfrak{M}; \quad \|K\|^2 \leq \mathcal{N}. \tag{4.11}$$

For any fixed positive $\varepsilon$ it is possible to find $n = n(\varepsilon)$ such that (2.1) holds. Assume, for example, $\varepsilon = 1/2$. The operator $H_n = K - K_n$ has the following integral representation

$$(H_n u)(x) = \int_\Sigma H^{(n)}(x, \xi) u(\xi) d\sigma_\xi, \qquad H^{(n)}(x, \xi) = K(x, \xi) - K_n(x, \xi).$$

Then $\|H_n\|^2$ is the greatest eigenvalue of the PCO of the space $L^2(\Sigma)$: $H_n^* H_n$. $H_n^* H_n$ admits the integral representation

$$(H_n^* H_n u)(x) = \int_\Sigma \mathcal{H}^{(n)}(x, \xi) u(\xi) d\sigma_\xi, \qquad \mathcal{H}^{(n)}(x, \xi) = \int_\Sigma H^{(n)}(\eta, x) H^{(n)}(\eta, \xi) d\sigma_\eta$$

and $(H_n^* H_n)^2$ belongs to the same class $\mathcal{C}^q$ of $(K^* K)^2$. Let us fix $q > (m-1)/(4\alpha)$. We have

$$\|K - K_n\|^2 = \|H_n\|^2 \leq [\mathfrak{J}_1^q (H_n^* H_n)]^{1/2q} = \left[ \int_\Sigma d\sigma_x \int_\Sigma |\mathcal{H}_q^{(n)}(x, \xi)|^2 d\sigma_\xi \right]^{1/2q},$$

where $\mathcal{H}_q^{(n)}(x, \xi)$ is the $q$-th iterated kernel of $\mathcal{H}_1^{(n)}(x, \xi) = \mathcal{H}^{(n)}(x, \xi)$. Moreover

$$\lim_{n\to\infty} \int_\Sigma d\sigma_x \int_\Sigma |\mathcal{H}_q^{(n)}(x, \xi)|^2 d\sigma_\xi = 0.$$

Therefore (2.1), assuming $\varepsilon = 1/2$, is certainly satisfied if $n$ is such that

$$\left[ \int_\Sigma d\sigma_x \int_\Sigma |\mathcal{H}_q^{(n)}(x, \xi)|^2 d\sigma_\xi \right]^{1/2q} < \frac{1}{4}.$$

Once we have determined $n$, we consider $\lambda_n$, which is the lowest eigenvalue of the $n \times n$ matrix $Q^* Q$. Because of (3.4), (3.5) and (3.6) it is easy to explicitly compute upper and lower bounds to $\lambda_n$ as well as we wish. Suppose, for sake of simplicity, that $\{u_h\}$, introduced in Section 2, is an orthonormal system in $L^2(\Sigma)$. The $n \times n$ matrix $Q_l$

is such that $|Q_l|_F \leq n(1 + 2\|K\|)$. From (3.6), (3.5) and (4.11) we deduce that

$$|\lambda_n - \lambda_n^{(l)}| \leq n^2 \|K\| 2^{-l} [(4 + 2^{-l})\|K\| + 2] \leq e_n^{(l)};$$

$$e_n^{(l)} = n^2 \sqrt{N} 2^{-l} [(4 + 2^{-l})\sqrt{N} + 2]$$

and $\lim_{l \to \infty} e_n^{(l)} = 0$. Assume $l_0 \geq 1$ such that $\bar{\lambda}_n = \lambda_n^{(l_0)} - e_n^{(l_0)} > 0$. We have $\bar{\lambda}_n < \lambda_n$ and $\bar{\lambda}_n$ can be explicitly computed.

As far as $\mu_1^{(l)}$ is concerned, notice that the operator $S_l^* TS_l = (I + \mathcal{Q}_l^*) T (I + \mathcal{Q}_l)$ admits the integral representation:

$$(S_l^* TS_l v)(x) = \int_\Sigma \mathcal{S}^{(l)}(x, \xi) v(\xi) d\sigma_\xi,$$

where

$$\mathcal{S}^{(l)}(x, \xi) = T(x, \xi) + \int_\Sigma T(x, y) \mathcal{Q}_l(y, \xi) d\sigma_y +$$

$$+ \int_\Sigma T(\xi, y) \mathcal{Q}_l(y, x) d\sigma_y + \int_\Sigma \mathcal{Q}_l(y, x) d\sigma_y \int_\Sigma T(y, \eta) \mathcal{Q}_l(\eta, \xi) d\sigma_\eta.$$

Since $T(x, \xi)$ and $\mathcal{Q}_l(x, \xi)$ are known, the kernel $\mathcal{S}^{(l)}(x, \xi)$, *i.e.* the operator $S_l^* TS_l$, is known. Since $S_l^* TS_l$ belongs to the same class $\mathcal{C}$ of the operator $T$, arbitrarily close upper bounds of $\mu_1^{(l)}$ can be computed by applying the Method of Orthogonal Invariants. To this end, given an integer $\nu \geq 1$, let $\tau_1^{(l,\nu)} \geq \tau_2^{(l,\nu)} \geq \ldots \geq \tau_\nu^{(l,\nu)}$ be the roots of the determinant equation:

$$\det\{(S_l^* TS_l \omega_h, \omega_k)_\Omega - \tau(\omega_h, \omega_k)\}_{h,k=1,\ldots,\nu} = 0.$$

We have $\tau^{(l,\nu)} \leq \mu_1^{(l)}$; $\lim_{\nu \to \infty} \tau^{(l,\nu)} = \mu_1^{(l)}$. For $r > (m-1)/2$, denote by $\mathcal{S}_r^{(l)}(x, \xi)$ the $r$-th iterated kernel of $\mathcal{S}_1^{(l)}(x, \xi) = \mathcal{S}^{(l)}(x, \xi)$. Setting

$$\sigma^{(l,\nu)} = \left[ \int_\Sigma d\sigma_x \int_\Sigma |\mathcal{S}_r^{(l)}(x, \xi)|^2 d\sigma_\xi - \sum_{h=2}^{\nu} [\tau_h^{(l,\nu)}]^{2r} \right]^{1/2r}, \tag{4.12}$$

we have: $\mu_1^{(l)} \leq \sigma^{(l,\nu)}$; $\lim_{\nu \to \infty} \sigma^{(l,\nu)} = \mu_1^{(l)}$.

Assume $\nu = \nu(l)$ such that $\sigma^{(l,\nu(l))} - \tau^{(l,\nu(l))} < 2^{-l}$. Then $0 \leq \sigma^{(l,\nu(l))} - \mu_1^{(l)} < 2^{-l}$ and $\lim_{l \to \infty} \sigma^{(l,\nu(l))} = \lim_{l \to \infty} \mu_1^{(l)} = \mu_1$.

Set

$$c_l^* = \left[ \int_\Sigma d\sigma_x \int_\Sigma |\mathcal{S}_r^{(l)}(x, \xi)|^2 d\sigma_\xi - \sum_{h=2}^{\nu(l)} [\tau_h^{(l,\nu(l))}]^{2r} \right]^{1/2r} + \tag{4.13}$$

$$+ \frac{\mathfrak{M}}{2^l} \left[ 1 + 2(1 + \sqrt{2}) \sqrt{N} \frac{n}{\sqrt{\bar{\lambda}_n}} + 4\sqrt{2} N \frac{n^2}{\bar{\lambda}_n} \right].$$

$$\cdot \left[ 4 + \frac{1}{2^{l+1}} + 4\sqrt{N}\left( \frac{\sqrt{2}+1}{2^{l+1}} + 2\sqrt{2} \right) \frac{n}{\sqrt{\bar{\lambda}_n}} + 8\sqrt{2}N \frac{n^2}{\bar{\lambda}_n} \right].$$

All the quantities in the right-hand side of (4.13) can be explicitly computed. From (4.9), (4.10), (4.12), (3.19), (4.6) and (4.11) we have

$$c_\Omega \leq c_l^n ; \qquad \lim_{l \to \infty} c_l^n = c_\Omega .$$

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*Direttore responsabile*: Prof. A. Ballio - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
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