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A Homogenization Result for Unbounded Variational Functionals (**)

SUMMARY. — In the present paper we study the homogenization of classes of variational problems for integrals defined on functions subject to oscillating constraints on the gradient to describe some phenomena in elastic-plastic torsion theory.

Un risultato di omogeneizzazione per funzionali variazionali non limitati

SUNTO. — Nel presente lavoro si studia l'omogeneizzazione di una classe di problemi variazionali per integrali definiti su funzioni con vincolo oscillante sul gradiente che descrivono fenomeni nella teoria della torsione elastoplastica.

0. - INTRODUCTION

The homogenization of classes of variational problems for integrals defined on functions subject to oscillating constraints on the gradient has been treated in literature to describe some phenomena in elastic-plastic torsion theory (see [A], [BLP], [BS], [CR], [L1]-[L3], [T1]-[T3] for general reference on the subject).

In a general setting it is concerned with the asymptotic behaviour as $h (\in \mathbb{N})$ diverges of Dirichlet problems of the following type

$$(0.1) \quad m_h(\Omega, \beta) = \min \left\{ \int_\Omega \bar{f}(hx, Du)\,dx + \int_\Omega \beta u\,dx \colon u \text{ Lipschitz continuous}, \right.$$

$$\left. u = 0 \text{ on } \partial\Omega,\ |Du(x)| \le \varphi(hx) \text{ for a.e. } x \text{ in } \Omega \right\},$$

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where Ω is a bounded open subset of $\mathbf{R}^n$, β in $L^1(\Omega)$, and $\bar{f}$, φ are functions satisfying the following hypotheses:

$$(0.2)\quad \begin{cases} \bar{f}:(x,z)\in \mathbf{R}^n\times\mathbf{R}^n \to \bar{f}(x,z)\in[0,+\infty[\,, \\ \bar{f} \text{ measurable and }]0,1[^n \text{ periodic in the } x \text{ variable, convex in the } z \text{ one}, \\ \bar{f}(\cdot,z)\in L^1(]0,1[^n) \quad \text{for every } z \text{ in } \mathbf{R}^n, \\ \varphi: x\in\mathbf{R}^n\to\varphi(x)\in[0,+\infty[\,, \\ \varphi\ \ ]0,1[^n\text{-periodic}, \\ 0\le\varphi\le M<+\infty \quad \text{for a.e. } x \text{ in } \mathbf{R}^n. \end{cases}$$

This problem has been studied by many authors under different settings and in different hypotheses.

According to a conjecture of A. Bensoussan, J. L. Lions and G. Papanicolaou (see [BLP]), if, for every open bounded subset Ω of $\mathbf{R}^k$, β in $L^1(\Omega)$ and $h\in\mathbf{N}$, u_h is a solution of the problem (0.1), then the converging subsequences of $\{u_h\}_{h\in\mathbf{N}}$ converge to solutions of the problem

$$(0.3)\quad m_\infty(\Omega,\beta) = \min\left\{\int_\Omega f_\infty(Du)\,dx + \int_\Omega \beta u\,dx,\ u \text{ Lipschitz continuous}, u=0 \text{ on } \partial\Omega\right\}$$

where f_∞ is a convex lower semicontinuous function on $\mathbf{R}^n$ given by

$$(0.4)\quad f_\infty(z) = \min\left\{\int_{]0,1[^n} \bar{f}(y,z+Dv)\,dy : v \text{ Lipschitz continuous},\ v\ \ ]0,1[^n\text{-periodic},\ |z+Dv(y)|\le\varphi(y) \text{ a.e. } y \text{ in }]0,1[^n\right\}$$

(in (0.4) it is assumed that $\min\varnothing=+\infty$).

In some papers (cf. [C1]-[C4], [CS1], [CS2]), the above conjecture has been verified by assuming each time less restrictive assumptions on the constraints φ (see also [CS3], [DAV], [DAGP] for the study of some cases in which φ is unbounded), and in [CEDA] by assusming $\bar{f}$ and φ as in (0.2).

In this paper we study the asymptotic behaviour, for every bounded open subset Ω of $\mathbf{R}^n$, β in $L^1(\Omega)$ of solutions of problems

$$(0.5)\quad m_h(\Omega,\beta) = \min\left\{\int_\Omega f(hx,u,Du)\,dx + \int_\Omega \beta u\,dx : u \text{ Lipschitz continuous},\ u=0 \text{ on } \partial\Omega,\ |Du(x)|\le\varphi(hx) \text{ for a.e. } x \text{ in } \Omega\right\},\quad h\in\mathbf{N}$$

where f and φ are functions verifying

$$(0.6)\quad \begin{cases} f\colon (x,s,z)\in \mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n \to f(x,s,z)\in[0,+\infty[, \\ f \text{ measurable and }]0,1[^n\text{-periodic in the } x \text{ variable, continuous in the } s\text{, convex in the } z \text{ one,} \\ f(\cdot,s,z)\in L^1(]0,1[^n) \text{ for every } (s,z) \text{ in } \mathbf{R}\times\mathbf{R}^n, \\ |f(x,s_1,z)-f(x,s_2,z)| \leq \omega(x,|s_1-s_2|)\cdot\varrho(|z|) \\ \qquad\qquad \text{for a.e. } x \text{ in } \mathbf{R}^n, \text{ for every } s_1,s_2 \text{ in } \mathbf{R} \text{ and for every } z \text{ in } \mathbf{R}^n, \\ \text{where } \omega\colon \mathbf{R}^n\times[0,+\infty[\to[0,+\infty[, \\ ]0,1[^n\text{-periodic, } \omega(\cdot,s)\in L^1(]0,1[^n) \text{ for every } s \text{ in } \mathbf{R}, \\ \omega(x,\cdot) \text{ is increasing, } \omega(x,0)=0 \text{ and continuous in } 0 \text{ for a.e. } x \text{ in } \mathbf{R}^n, \\ \varrho\colon \mathbf{R}^+\to\mathbf{R}^+ \text{ increasing,} \\ \varphi\colon x\in\mathbf{R}^n\to\varphi(x)\in[0,+\infty[, \text{ measurable} \\ \varphi \ ]0,1[^n\text{-periodic and } 0\leq\varphi\leq M<+\infty \text{ for a.e. } x \text{ in } \mathbf{R}^n. \end{cases}$$

More precisely we prove that for every bounded open subset Ω of $\mathbf{R}^n$, β in $L^1(\Omega)$, the sequence $\{u_h\}_{h\in N}$ of solutions of (0.5) is compact in $C_0^0(\Omega)$ and that the converging subsequences of $\{u_h\}_{h\in N}$ converge to solutions of the problem

$$(0.7)\quad m_\infty(\Omega,\beta) = \\ = \min\left\{\int_\Omega f_\infty(u,Du)\,dx + \int_\Omega \beta u\,dx\colon u \text{ Lipschitz continuous, } u=0 \text{ on } \partial\Omega\right\}$$

where f_∞ is given by

$$(0.8)\quad f_\infty(s,z) = \min\left\{\int_{]0,1[^n} f(y,s,z+Dv)\,dy\colon\right.$$

$$\left. v \text{ Lipschitz continuous, } v\ ]0,1[^n\text{-periodic, } |z+Dv(y)|\leq\varphi(y) \text{ a.e. } y \text{ in }]0,1[^n\right\}.$$

A similar result has been proved in [F] by assuming the following condition on φ: there exist $\vartheta\in[0,1/2[$ and $m>0$ such that $0<m<\varphi(y)$ for a.e. y in $]0,1[^n\setminus]1/2-\vartheta, 1/2+\vartheta[^n$.

We will achieve such results by using Γ-convergence theory (see [DGF], [DG]) and some results and techniques from [CS2], [CEDA] and [F].

1. - Notations and preliminaries

Let us recall the definition and the main properties of Γ^--convergence (see [DGF]). Let (U, τ) be a topological space satisfying the first countability axiom.

Definition 1.1: *Let F_h, $(h \in \mathbb{N})$, F' and F'' be functional from U to $\bar{\mathbb{R}}$. We say that F' is the $\Gamma^-(\tau)$-lower limit of sequence $\{F_h\}_{h\in\mathbb{N}}$ and we write*

$$F'(u) = \Gamma^-(\tau) \liminf_{\substack{h\to+\infty \\ v\to u}} F_h(v) \quad \forall u, v \in U, \tag{1.1}$$

if for every $u \in U$ and for every sequence $\{u_h\}_{h\in\mathbb{N}} \subseteq U$ such that $u_h \xrightarrow{\tau} u$ it results

$$F'(u) \leqslant \liminf_{h\to+\infty} F_h(u_h) \tag{1.2}$$

and for every $u \in U$ there exists a sequence $\{v_h\}_{h\in\mathbb{N}} \subseteq U$ such that $v_h \xrightarrow{\tau} u$ and

$$F'(u) \geqslant \liminf_{h\to+\infty} F_h(v_h). \tag{1.3}$$

We say that F'' is the $\Gamma^-(\tau)$-upper limit of sequence $\{F_h\}_{h\in\mathbb{N}}$ and we write

$$F''(u) = \Gamma^-(\tau) \limsup_{\substack{h\to+\infty \\ v\to u}} F_h(v) \quad \forall u, v \in U, \tag{1.4}$$

if (1.2) and (1.3) hold with the operator «liminf» replaced by «limsup».

When $F' = F''$ we say that the sequence $\{F_h\}_{h\in\mathbb{N}}$ $\Gamma^-(\tau)$-converges on U and we write

$$F'(u) = F''(u) = \Gamma^-(\tau) \lim_{\substack{h\to+\infty \\ v\to u}} F_h(v) \quad \forall u \in U. \tag{1.5}$$

Remark 1.2: The limits in (1.1) and (1.4) exist and are given by

$$F'(u) = \min\left\{\liminf_{h\to+\infty} F_h(u_h): \{u_h\}_{h\in\mathbb{N}} \subseteq U \text{ and } u_h \xrightarrow{\tau} u\right\}, \tag{1.6}$$

$$F''(u) = \min\left\{\limsup_{h\to+\infty} F_h(u_h): \{u_h\}_{h\in\mathbb{N}} \subseteq U \text{ and } u_h \xrightarrow{\tau} u\right\}. \tag{1.7}$$

We recall the following properties of Γ^--convergence proved in [DGF].

Proposition 1.3: *Let $\{F_h\}_{h\in\mathbb{N}}$ be a sequence of functionals from U to $\bar{\mathbb{R}}$, then the functionals $\liminf\limits_{\substack{h\to+\infty \\ v\to u}} F_h(v)$ and $\limsup\limits_{\substack{h\to+\infty \\ v\to u}} F_h(v)$ are τ-lower semicontinuous on U; moreover if*

$\{u_h\}_{h\in\mathbb{N}}$ *is an increasing sequence of integer numbers, then*

$$\Gamma^-(\tau)\liminf_{\substack{h\to+\infty\\ v\to u}} F_h(v) \le \Gamma^-(\tau)\liminf_{\substack{k\to+\infty\\ v\to u}} F_{h_k}(v) \le$$

$$\le \Gamma^-(\tau)\limsup_{\substack{k\to+\infty\\ v\to u}} F_{h_k}(v) \le \Gamma^-(\tau)\limsup_{\substack{h\to+\infty\\ v\to u}} F_h(v).$$

Definition 1.4: *Let $\{F_h\}_{h\in\mathbb{N}}$ be a sequence of functionals from U to $\bar{\mathbb{R}}$. We say that the functionals F_h are equicoercive if for any c in $\mathbb{R}$ there exists a compact set K_c in U such that*

$$\{u\in U: F_h(u)\le c\}\subseteq K_c \quad \forall h\in\mathbb{N}.$$

Theorem 1.5: *Let F_h, $h\in\mathbb{N}$, be functionals from U to $\bar{\mathbb{R}}$ and assume that there exists the limit*

$$F(u) = \Gamma^-(\tau)\lim_{\substack{h\to+\infty\\ v\to u}} F_h(v) \quad \text{for every } u, v\in U. \tag{1.8}$$

Then, if G is a τ-continuous functional from U to $\bar{\mathbb{R}}$ and $F_h + G$ are equicoercive, $F+G$ attains its minimum on U and we have:

$$\min\{F(v)+G(v): v\in U\} = \lim_{h\to+\infty}(\inf\{F_h(v)+G(v): v\in U\}).$$

Moreover, if $\{u_h\}_{h\in\mathbb{N}}\in U$ is such that

$$\lim_{h\to+\infty}(F_h(u_h)+G(u_h) - \inf\{F_h(v)+G(v): v\in U\}) = 0 \quad \text{and} \quad u_h \xrightarrow{\tau} u,$$

then u is a solution of

$$\min\{F(v)+G(v): v\in U\}.$$

If F is a functional from U to $\bar{\mathbb{R}}$, we denote by $sc^-(\tau)F$ the greatest τ-lower semicontinuous functional on U less then or equal to F.

For two bounded open subset of $\mathbb{R}^n$ A, and B, we write $A\subset\subset B$ if $\bar{A}\subseteq B$.

Id G is a real extended functional defined for any bounded open subset Ω of $\mathbb{R}^n$, we say that G is increasing if

$$\Omega_1\subseteq\Omega_2 \Rightarrow G(\Omega_1)\le G(\Omega_2),$$

moreover we define the inner regular envelope G_- of G (see [DGL]) on the open subset Ω of $\mathbb{R}^n$ as

$$G_-(\Omega) = \sup_{A\subset\subset\Omega} G(A). \tag{1.9}$$

For every bounded open subset Ω of $\mathbb{R}^n$ we denote by $C^0(\Omega)$ and $C^0_0(\Omega)$ the topologies induced on $C^0(\mathbf{R}^n)$ respectively by the extended metrics

$$d(u,v)=\|u-v\|_{C^0(\Omega)}=\sup_{x\in\Omega}|u(x)-v(x)|,\qquad \delta(u,v)=\begin{cases} d(u,v) & \text{if } u=v \text{ on } \partial\Omega,\\ +\infty & \text{otherwise}.\end{cases}$$

For every open subset Ω of $\mathbf{R}^n$ we denote by $\mathrm{Lip}_0(\Omega)$ the set of functions u in $\mathrm{Lip}_{loc}(\mathbf{R}^n)$ such that $u=0$ in $\mathbf{R}^n/\Omega$.

Moreover for every subset P of $\mathbf{R}^n$ and $\varepsilon>0$ we denote by P_ε^+ and P_ε^- the open sets defined by

$$P_\varepsilon^+=\{x\in\mathbf{R}^n: \mathrm{dist}(x,P)<\varepsilon\},\qquad P_\varepsilon^-=\{x\in P: \mathrm{dist}(x,\partial P)>\varepsilon\}.$$

Let $\bar f$ and φ be the functionals defined in (0.2) and set, for any b in $\mathbf{N}$,

$$\varphi_b(x)=\varphi(bx)\qquad \forall x\in\mathbf{R}.$$

For every bounded open subset Ω of $\mathbf{R}^n$, we define the following functional on $C^0(\mathbf{R}^n)$

$$\bar F_b(u,\Omega)=\begin{cases}\int_\Omega \bar f(bx,Du)\,dx & \text{if } u\in \mathrm{Lip}_{loc}(\mathbf{R}^n)\ |Du|\le\varphi_b \text{ a.e. } x \text{ in } \Omega,\\ +\infty & \text{otherwise on } C^0(\mathbf{R}^n).\end{cases}\tag{1.10}$$

We set

$$\begin{cases}\bar F'(u,\Omega)=\Gamma^-(C^0(\Omega))\liminf\limits_{\substack{b\to+\infty\\ v\to u}}\bar F_b(v,\Omega),\\ \bar F''(u,\Omega)=\Gamma^-(C^0(\Omega))\limsup\limits_{\substack{b\to+\infty\\ v\to u}}\bar F_b(v,\Omega),\end{cases}\tag{1.11}$$

and if

$$\bar F'(u,\Omega)=\bar F''(u,\Omega),$$

we set

$$\bar F(u,\Omega)=\Gamma^-(C^0(\Omega))\lim_{\substack{b\to+\infty\\ v\to u}}\bar F_b(v,\Omega).\tag{1.12}$$

Moreover, we set

$$\begin{cases}\bar F_0'(u,\Omega)=\Gamma^-(C_0^0(\Omega))\liminf\limits_{\substack{b\to+\infty\\ v\to u}}\bar F_b(v,\Omega),\\ \bar F_0''(u,\Omega)=\Gamma^-(C_0^0(\Omega))\limsup\limits_{\substack{b\to+\infty\\ v\to u}}\bar F_b(v,\Omega)\end{cases}\tag{1.13}$$

and if

$$\bar{F}_0'(u,\Omega) = \bar{F}_0''(u,\Omega),$$

we set

$$\bar{F}_0(u,\Omega) = \Gamma^-(C_0^0(\Omega)) \lim_{\substack{h \to +\infty \\ v \to u}} \bar{F}_h(v,\Omega). \tag{1.14}$$

We define, for every bounded open subset Ω of R^n and for every u in $C^0(R^n)$, $\bar{F}'_-(u,\Omega)$ and $\bar{F}''_-(u,\Omega)$ by (1.9) written with $G = \bar{F}'(u,\cdot)$ and $G = \bar{F}''(u,\cdot)$ respectively. Moreover it is easy to prove that

(1.15) $\bar{F}'(u,\Omega) \leq \bar{F}''(u,\Omega)$ for every bounded open subset Ω of R^n, and u in $C^0(R^n)$,

(1.16) $\bar{F}'(u+c,\Omega) = \bar{F}'(u,\Omega)$ and $\bar{F}''(u+c,\Omega) = \bar{F}''(u,\Omega)$

for every bounded open subset Ω of R^n, u in $C^0(R^n)$ and c in R,

and that the following locality property holds

(1.17) Ω bounded open subset of R^n and $u_1, u_2 \in C^0(R^n)$ with $u_1 = u_2$ on $\Omega \Rightarrow$

$$\Rightarrow \bar{F}'(u_1,\Omega) = \bar{F}'(u_2,\Omega) \quad \text{and} \quad \bar{F}''(u_1,\Omega) = \bar{F}''(u_2,\Omega).$$

REMARK 1.6: *Let φ be functions satisfying* (0.2), *$\bar{f}_\infty$ be given by* (0.2), *f be satisfying* (0.6) *and f_∞ be given by* (0.8). *Then we set*

$K_\varphi = \{z \in R^n : \exists v$ *Lipschitz continuous and* $]0,1[^n$*-periodic such that*

$$|z + Dv(y)| \leq \varphi(y) \text{ for a.e. } y \text{ in }]0,1[^n\},$$

$\operatorname{dom} f_\infty = \{(s,z) \in R \times R^n : f_\infty(s,z) < +\infty\} =$

$= \{(s,z) \in R \times R^n : \exists v$ *Lipschitz continuous and*

$]0,1[^n$*-periodic such that* $|z + Dv(y)| \leq \varphi(y)$ *for a.e.* y *in* $]0,1[^n\}$,

$$f_\infty^s : z \in R^n \to f_\infty(s,z) \in \bar{R} \quad s \in R,$$

$\operatorname{dom} f_\infty^s = \{z \in R^n : f_\infty^s(z) < +\infty\} = \{z \in R^n : \exists v$ *Lipschitz continuous and*

$]0,1[^n$*-periodic such that* $|z + Dv(y)| \leq \varphi(y)$ *for a.e.* y *in* $]0,1[^n\}$.

It is evident that

$$\operatorname{dom} f_\infty = R \times K_\varphi,$$

$$\operatorname{dom} f_\infty^s = K_\varphi \quad \forall s \in R.$$

Consequently

$$(\mathrm{dom}\, f_\infty)^0 \neq \emptyset \Rightarrow (K_\varphi)^0 \neq \emptyset\,.$$

REMARK 1.7: *Let us observe that the minimum in* (0.8) *is attained, (cf. for example* [Bu]).

DEFINITION 1.8: *Let* $\bar{f}$ *and* φ *be defined by* (0.2) *and let* $\bar{f}_\infty$ *be given by* (0.4). *Then for every bounded open subset* Ω *of* R^n *we define the functional* $\bar{F}_\infty(\cdot, \Omega)$ *on* $C^0(\mathrm{R}^n)$ *by*

$$\bar{F}_\infty(u, \Omega) = \begin{cases} \displaystyle\int_\Omega \bar{f}_\infty(Du)\,dx & \text{if } u \in C^0(\mathrm{R}^n) \cap W^{1,\infty}(\Omega)\,, \\ +\infty & \text{if } u \in C^0(\mathrm{R}^n) \setminus W^{1,\infty}(\Omega)\,. \end{cases} \tag{1.18}$$

The following results are proved in [CEDA] Proposition (2.4) and Theorem (4.10) respectively.

PROPOSITION 1.9: *Let* $\bar{f}$ *and* φ *be defined in* (0.2), *let* $\bar{f}_\infty$ *be given by* (0.4), *let* $\bar{F}'$, $\bar{F}''$ *be defined in* (1.11), *and let* $\bar{F}_0'$, $\bar{F}_0''$ *be given by* (1.13).
Assume $(\mathrm{dom}\, \bar{f}_\infty)^0 \neq \emptyset$. *Then*

$$\bar{F}'(u, \Omega) = \bar{F}'_-(u, \Omega) = \bar{F}_0'(u, \Omega)\,, \qquad \bar{F}''(u, \Omega) = \bar{F}''_-(u, \Omega) = \bar{F}_0''(u, \Omega)\,,$$

for every bounded open subset Ω *of* R^n *and for every* u *in* $C^0(\mathrm{R}^n)$ *such that* $u = 0$ *on* $\partial\Omega$.

THEOREM 1.10: *Let* $\bar{f}$ *and* φ *be defined in* (0.2), *let* $\bar{F}'$, $\bar{F}''$ *be defined in* (1.11) *and let* $\bar{F}_\infty$ *be given by* (1.8).
Assume $(\mathrm{dom}\, \bar{f}_\infty)^0 \neq \emptyset$. *Then*

$$\bar{F}'_-(u, \Omega) = \bar{F}''_-(u, \Omega) = \bar{F}_\infty(u, \Omega)\,,$$

for every bounded open subset Ω *of* R^n *and for every* u *in* $C^0(\mathrm{R}^n)$.

From Proposition 1.9 and Theorem 1.10 we obtain the following result:

COROLLARY 1.11: *Let* $\bar{f}$ *be defined in* (0.2), *let* $\bar{F}_h$, $h \in \mathrm{N}$, *be given by* (1.10) *and let* $\bar{F}_\infty$ *be given by* (1.18).
Assume $(\mathrm{dom}\, \bar{f}_\infty)^0 \neq \emptyset$. *Then*

$$\Gamma^-(C^0(\Omega)) \lim_{\substack{h \to +\infty \\ v \to u}} \bar{F}_h(v, \Omega) = \Gamma^-(C_0^0(\Omega)) \lim_{\substack{h \to +\infty \\ v \to u}} \bar{F}_h(v, \Omega) = \bar{F}_\infty(u, \Omega)\,,$$

for every bounded open subset Ω *of* R^n *and for every* u *in* $C^0(\mathrm{R}^n)$ *such that* $u = 0$ *on* $\partial\Omega$.

2. · Some estimates for Γ-limits

Let f and φ be given by (0.6). We set, for every h in N,

$$f_h(x,s,z) = f(hx,s,z), \qquad \varphi_h(x) = \varphi(hx) \text{ for a.e. } x \text{ in } \mathbf{R}^n, \ \forall (s,z) \in \mathbf{R} \times \mathbf{R}^n .$$

Moreover for every h in N, for every u in $L^\infty_{loc}(\mathbf{R}^n)$, for every v in $C^0(\mathbf{R}^n)$ and for every bounded open subset Ω of $\mathbf{R}^n$, we define

$$\text{(2.1)} \quad \Phi_h(u,v,\Omega) = \begin{cases} \displaystyle\int_\Omega f_h(x,u,Dv)\,dx & \text{if } v \in \mathrm{Lip}_{loc}(\mathbf{R}^n) \ |Dv| \le \varphi_h \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise}, \end{cases}$$

$$\text{(2.2)} \quad \begin{cases} \Phi'(u,v,\Omega) = \Gamma^-(C^0(\Omega)) \liminf\limits_{\substack{h \to +\infty \\ w \to v}} \Phi_h(u,w,\Omega), \\ \Phi''(u,v,\Omega) = \Gamma^-(C^0(\Omega)) \limsup\limits_{\substack{h \to +\infty \\ w \to v}} \Phi_h(u,w,\Omega), \end{cases}$$

and if

$$\Phi'(u,v,\Omega) = \Phi''(u,v,\Omega),$$

we set

$$\text{(2.3)} \quad \Phi(u,v,\Omega) = \Gamma^-(C^0(\Omega)) \lim_{\substack{h \to +\infty \\ w \to v}} \Phi_h(u,w,\Omega).$$

Moreover we define

$$\text{(2.4)} \quad \begin{cases} \Phi'_0(u,v,\Omega) = \Gamma^-(C^0_0(\Omega)) \liminf\limits_{\substack{h \to +\infty \\ w \to v}} \Phi_h(u,w,\Omega), \\ \Phi''_0(u,v,\Omega) = \Gamma^-(C^0_0(\Omega)) \limsup\limits_{\substack{h \to +\infty \\ w \to v}} \Phi_h(u,w,\Omega), \end{cases}$$

and if

$$\Phi'_0(u,v,\Omega) = \Phi''_0(u,v,\Omega),$$

we set

$$\text{(2.5)} \quad \Phi_0(u,v,\Omega) = \Gamma^-(C^0_0(\Omega)) \lim_{\substack{h \to +\infty \\ w \to v}} \Phi_h(u,w,\Omega).$$

Furthermore, for every h in N, for every v in $C^0(\mathbf{R}^n)$ and for every bounded open subset Ω of $\mathbf{R}^n$, we define

$$\text{(2.6)} \quad F_h(v,\Omega) = \Phi_h(v,v,\Omega),$$

$$(2.7)\qquad \begin{cases} F'(v,\Omega) = \Gamma^-(C^0(\Omega)) \liminf\limits_{\substack{h\to+\infty \\ w\to v}} F_h(w,\Omega), \\ F''(v,\Omega) = \Gamma^-(C^0(\Omega)) \limsup\limits_{\substack{h\to+\infty \\ w\to v}} F_h(w,\Omega), \end{cases}$$

and if

$$F'(v,\Omega) = F''(v,\Omega),$$

we set

$$(2.8)\qquad F(v,\Omega) = \Gamma^-(C^0(\Omega)) \lim_{\substack{h\to+\infty \\ w\to v}} F_h(w,\Omega).$$

Moreover we define

$$(2.9)\qquad \begin{cases} F'_0(v,\Omega) = \Gamma^-(C^0_0(\Omega)) \liminf\limits_{\substack{h\to+\infty \\ w\to v}} F_h(w,\Omega), \\ F''_0(v,\Omega) = \Gamma^-(C^0_0(\Omega)) \limsup\limits_{\substack{h\to+\infty \\ w\to v}} F_h(w,\Omega), \end{cases}$$

and if

$$F'_0(v,\Omega) = F''_0(v,\Omega),$$

we set

$$(2.10)\qquad F_0(v,\Omega) = \Gamma^-(C^0_0(\Omega)) \lim_{\substack{h\to+\infty \\ w\to v}} F_h(w,\Omega).$$

We begin by proving the following lemma which will be used in Section 4 (see also [F]).

LEMMA 2.1: *Let f and φ be defined in* (0.6), *let f_∞ be given by* (0.8) *and let Φ', Φ'' be defined in* (2.2).

Assume $(K_\varphi)^0 \neq \emptyset$. Then, if Ω is a bounded open subset of $\mathbb{R}^n$, u is in $L^\infty_{loc}(\mathbb{R}^n)$ and v is in $C^0(\mathbb{R}^n)$ with $v = 0$ on $\partial\Omega$, it results

$$(2.11)\quad \Phi'(u,v,\Omega) < +\infty,\quad \Phi''(u,v,\Omega) < +\infty \Leftrightarrow v \in W^{1,\infty}(\Omega) \text{ and } Dv \in K_\varphi \text{ a.e. in } \Omega.$$

Moreover the functionals

$$(2.12)\qquad \Phi'(\cdot, v, \Omega) \quad \text{and} \quad \Phi''(\cdot, v, \Omega)$$

are continuous on $L^\infty_{loc}(\mathbb{R}^n)$ with respect to the uniform convergence in Ω, for every v in $C^0(\mathbb{R}^n)$ such that $v = 0$ on $\partial\Omega$ and Dv in K_φ a.e. in Ω.

Proof: We prove Lemma 2.1 for the functional Φ', the proof for Φ'' being analogous. Denote by

$$T = \{v \in C^0(\mathbf{R}^n) \cap W^{1,\infty}(\Omega) \text{ and } Dv \in K_\varphi \text{ a.e. in } \Omega\}.$$

Fix a bounded open subset Ω of $\mathbf{R}^n$, u in $L^\infty_{loc}(\mathbf{R}^n)$. First let us prove that

$$(2.13) \qquad \Phi'(u, v, \Omega) < +\infty \qquad \forall v \in T \text{ with } v = 0 \text{ on } \partial\Omega.$$

Let

$$(2.14) \qquad M: (x, z) \in \mathbf{R}^n \times \mathbf{R}^n \to \sup_{|s| \leq \|u\|_{L^\infty(\Omega)}} f(x, s, z),$$

and let M_∞ be defined by (0.4) with f replaced by M.

Let us define, for every b in $\mathbf{N}$ and for every v in $C^0(\mathbf{R}^n)$

$$(2.15) \quad \Phi_b(v, \Omega) = \begin{cases} \displaystyle\int_\Omega M(bx, Dv)\,dx & \text{if } v \in \mathrm{Lip}_{loc}(\mathbf{R}^n) \text{ and } |Dv| \leq \varphi_b \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise}, \end{cases}$$

$$(2.16) \quad \Psi(v, \Omega) = \begin{cases} \displaystyle\int_\Omega M_\infty(Dv) & \text{if } v \in C^0(\mathbf{R}^n) \cap W^{1,\infty}(\Omega), \\ +\infty & \text{if } v \in C^0(\mathbf{R}^n) \setminus W^{1,\infty}(\Omega). \end{cases}$$

Then by Corollary 1.11 we deduce that

$$(2.17) \quad \Psi(v, \Omega) = \Gamma^-(C^0(\Omega)) \lim_{\substack{b \to +\infty \\ w \to v}} \Psi_b(w, \Omega) \quad \forall v \in C^0(\mathbf{R}^n) \text{ with } v = 0 \text{ on } \partial\Omega.$$

Consequently, being

$$(2.18) \qquad \Phi_b(u, v, \Omega) \leq \Psi_b(v, \Omega) \qquad \forall v \in C^0(\mathbf{R}^n)\ \forall b \in \mathbf{N},$$

it results

$$(2.19) \qquad \Phi'(u, v, \Omega) \leq \Psi(v, \Omega) < +\infty \qquad \forall v \in T \text{ with } v = 0 \text{ on } \partial\Omega.$$

Let us prove, now, that

$$(2.20) \qquad \Phi'(u, v, \Omega) = +\infty \qquad \forall v \in C^0(\mathbf{R}^n) - T \text{ with } v = 0 \text{ on } \partial\Omega.$$

Let, in fact, for every b in $\mathbf{N}$ and for every v in $C^0(\mathbf{R}^n)$

$$(2.21) \qquad \Theta_b(v, \Omega) = \begin{cases} 0 & \text{if } v \in \mathrm{Lip}_{loc}(\mathbf{R}^n) \text{ and } |Dv| \leq \varphi_b \text{ a.e. x in } \Omega, \\ +\infty & \text{otherwise}, \end{cases}$$

and

$$\Theta(v,\Omega)=\begin{cases}0 & \text{if } v\in T,\\ +\infty & \text{otherwise}.\end{cases}\tag{2.22}$$

Hence from Corollary 1.11 it follows

$$\Theta(v,\Omega)=\Gamma^-(C^0(\Omega))\lim_{\substack{h\to+\infty\\ w\to v}}\Theta_h(w,\Omega)\quad \forall v\in C^0(\mathbf{R}^n): v=0 \text{ on } \partial\Omega.\tag{2.23}$$

Consequently, being

$$\Phi_h(u,v,\Omega)\geq\Theta_h(v,\Omega)\quad \forall v\in C^0(\mathbf{R}^n)\ \forall h\in\mathbf{N},\tag{2.24}$$

it results

$$\Phi'(u,v,\Omega)\geq\theta(v,\Omega)=+\infty\quad \forall v\in C^0(\mathbf{R}^n)-T:\ v=0 \text{ on } \partial\Omega.\tag{2.25}$$

Let us prove, now, that $\Phi'(\cdot,v,\Omega)$ is continuous with respect the uniform convergence in Ω. Fix $v\in T$. Moreover fix $\varepsilon>0$ and let u_1, u_2 be in $L^\infty_{loc}(\mathbf{R}^n)$, such that $\sup\limits_{\Omega}\|u_2-u_1\|<\varepsilon$.

By (2.11) there exists a sequence $\{v_h\}_{h\in\mathbf{N}}$ in $C^0(\mathbf{R}^n)$ and a sequence $\{h_k\}_{k\in\mathbf{N}}$ of integer numbers such that

$$|Dv_{h_k}|\leq\varphi_{h_k}\quad \text{a.e. in } \Omega\quad \forall k\in\mathbf{N},\tag{2.26}$$

$$v_h\to v \text{ in } C^0(\Omega),\tag{2.27}$$

$$+\infty>\Phi'(u_1,v,\Omega)=\liminf_{k\to+\infty}\int_\Omega f_{h_k}(x,u_1,Dv_{h_k})\,dx.\tag{2.28}$$

Consequently

$$\begin{aligned}\Phi'(u_2,v,\Omega)&\leq\liminf_{k\to+\infty}\int_\Omega f_{h_k}(x,u_2,Dv_{h_k})\,dx\leq\\ &\leq\liminf_{k\to+\infty}\int_\Omega\big(f_{h_k}(x,u_2,Dv_{h_k})+f_{h_k}(x,u_1,Dv_{h_k})-f_{h_k}(x,u_1,Dv_{h_k})\big)\,dx\leq\\ &\leq\liminf_{k\to+\infty}\left(\int_\Omega f_{h_k}(x,u_1,Dv_{h_k})\,dx+\varrho(M)\int_\Omega\omega(h_kx,|u_2-u_1|)\,dx\right)\leq\\ &\leq\Phi'(u_1,v,\Omega)+\varrho(M)\operatorname{meas}(\Omega)\int_{]0,1[^n}\Omega(y,\varepsilon)\,dy.\end{aligned}\tag{2.29}$$

Hence, it results

$$(2.30)\qquad \Phi'(u_2, v, \Omega) - \Phi'(u_1, v, \Omega) \le \varrho(M)\,\mathrm{meas}(\Omega) \int_{]0,1[^n} \omega(y, \varepsilon)\,dy\,.$$

Similarly we have

$$(2.31)\qquad \Phi'(u_1, v, \Omega) - \Phi'(u_2, v, \Omega) \le \varrho(M)\,\mathrm{meas}(\Omega) \int_{]0,1[^n} \omega(y, \varepsilon)\,dy\,.$$

Finally from the assumption on ω the desired result follows. ■

As in [F] we prove the following result:

Proposition 2.2: *Let f and φ be defined in* (0.6). *Let Φ', Φ'' be defined in* (2.2), *let Φ_0', Φ_0'' be given by* (2.4), *Let F', F'' be defined in* (2.7) *and let F_0', F_0'' be defined in* (2.9). *Then*

$$(2.32)\qquad F'(u, \Omega) = \Phi'(u, u, \Omega),$$

$$(2.33)\qquad F''(u, \Omega) = \Phi''(u, u, \Omega),$$

$$(2.34)\qquad F_0'(u, \Omega) = \Phi_0'(u, u, \Omega),$$

$$(2.35)\qquad F_0''(u, \Omega) = \Phi_0''(u, u, \Omega),$$

for every bounded open subset Ω of $\mathbf{R}^n$ and for every u in $C^0(\mathbf{R}^n)$.

Proof: We prove only (2.32), the proof of (2.33), (2.34) and (2.35) being analogous. Let u be in $C^0(\mathbf{R}^n)$, and assume $F'(u, \Omega) < +\infty$. Then there exists a sequence $\{v_k\}_{k\in\mathbf{N}}$ in $C^0(\mathbf{R}^n)$ and a sequence $\{h_k\}_{k\in\mathbf{N}}$ of integer numbers such that

$$(2.36)\qquad |Dv_{h_k}| \le \varphi_{h_k} \quad \text{a.e. in } \Omega \ \forall k\in\mathbf{N},$$

$$(2.37)\qquad v_k \to u \text{ in } C^0(\Omega),$$

$$(2.38)\qquad F'(u, \Omega) = \liminf_{k\to+\infty} F_{h_k}(v_{h_k}, \Omega) = \liminf_{k\to+\infty} \int_\Omega f_{h_k}(x, v_{h_k}, Dv_{h_k})\,dx\,.$$

Consequently

$$(2.39)\qquad \Phi'(u, u, \Omega) \le \liminf_{k\to+\infty} \Phi_{h_k}(u, v_{h_k}, \Omega) = \liminf_{k\to+\infty} \int_\Omega f_{h_k}(x, u, Dv_{h_k})\,dx\,.$$

Let, for every k in N, be $\varepsilon_k = \|u - v_{h_k}\|_{L^\infty(\Omega)}$. Then, for a fixed m in N, it results

$$(2.40)\qquad \int_\Omega f_{h_k}(x, u, Dv_{h_k})\,dx =$$

$$= \int_\Omega f_{h_k}(x, v_{h_k}, Dv_{h_k})\,dx + \int_\Omega (f_{h_k}(x, u, Dv_{h_k}) - f_{h_k}(x, v_{h_k}, Dv_{h_k}))\,dx \leq$$

$$\leq \int_\Omega f_{h_k}(x, v_{h_k}, Dv_{h_k})\,dx + \varrho(M)\int_\Omega \omega(h_k x, \varepsilon_k)\,dx \qquad \forall k \geq m\,.$$

Consequently, passing to the liminf as $k \to \infty$, $\forall k \geq m$ we obtain

$$(2.41)\qquad \Phi'(u, u, \Omega) \leq \liminf_{k \to +\infty}\left(\int_\Omega f_{h_k}(x, v_{h_k}, Dv_{h_k})\,dx + \varrho(M)\int_\Omega \omega(hx, \varepsilon_k)\,dx\right) =$$

$$= F'(u, \Omega) + \varrho(M)\,\mathrm{meas}\,(\Omega)\int_{]0,1[^n} \omega(y, \varepsilon_m)\,dy\,.$$

Finally, as $m \to \infty$, we obtain

$$(2.42)\qquad \Phi'(u, u, \Omega) \leq F'(u, \Omega)\,.$$

On the other hand if we assume $\Phi'(u, u, \Omega) < +\infty$, it follows similarly

$$(2.43)\qquad F'(u, \Omega) \leq \Phi'(u, u, \Omega)\,. \qquad ■$$

In the following we shall prove the continuity of $F_\infty(\cdot, v, \Omega)$ with respect to the uniform convergence in Ω. To obtain this, we need the following lemma, which is a slight variation of a result in [F].

LEMMA 2.3: *Let f and φ be defined in* (0.6) *and f_∞ in* (0.8). *Then*

$$(2.44)\quad |f_\infty(s_1, z) - f_\infty(s_2, z)| \leq \varrho(M)\int_{]0,1[^n} \omega(y, |s_1 - s_2|)\,dx \qquad \forall s_1, s_2 \in \mathrm{R},\ \forall z \in K_\varphi\,.$$

PROOF: Fix s_1, s_2 in R and z in K_φ. There exists a w Lipschitz continuous function, $]0,1[^n$-periodic with $|z + Dw| \leq \varphi$ a.e. in $]0,1[^n$ and such that

$$(2.45)\qquad f_\infty(s_2, z) = \int_{]0,1[^n} f(y, s_2, z + Dw)\,dy\,.$$

Then it results

$$(2.46)\qquad f_\infty(s_1, z) - f_\infty(s_2, z) = f_\infty(s_1, z) - \int_{]0,1[^n} f(x, s_2, z + Dw)\,dx \leq$$

$$\leq \int_{]0,1[^n} (f(y, s_1, z + Dw) - f(y, s_2, z + Dw))\,dy \leq \varrho(M)\int_{]0,1[^n} \omega(y, |s_1 - s_2|)\,dy\,.$$

Analogously it results

$$f_\infty(s_2,z)-f_\infty(s_1,z)\le\varrho(M)\int_{]0,1[^n}\omega(y,|s_1-s_2|)\,dy\,. \qquad \blacksquare \tag{2.47}$$

Let f and φ be defined in (0.6) and f_∞ be given by (0.8). For every u in $L^\infty_{loc}(\mathbb{R}^n)$ we set

$$F_\infty(u,v,\Omega)=\begin{cases}\displaystyle\int_\Omega f_\infty(u,Dv)\,dx & \text{if } v\in C^0(\mathbb{R}^n)\cap W^{1,\infty}(\Omega),\\ +\infty & \text{if } v\in C^0(\mathbb{R}^n)\setminus W^{1,\infty}(\Omega).\end{cases} \tag{2.48}$$

Proposition 2.4: *Let F_∞ be defined in (2.48). For every v in $C^0(\Omega)$ and $Dv\in K_\varphi$ a.e. in Ω the functional $u\in L^\infty_{loc}(\mathbb{R}^n)\to F_\infty(u,v,\Omega)$ is continuous with respect to the uniform convergence in Ω.*

Proof: The result is an immediate consequence of Lemma 2.3. $\blacksquare$

We conclude this Section with an estimate for Φ^u_-.

Proposition 2.5: *Let f and φ be defined in* (0.6) *and let Φ^u be defined in* (2.2). *Then*

$$|\Phi^-_-(u_1,v,\Omega)-\Phi^-_-(u_2,v,\Omega)|\le\varrho(M)\int_\Omega\omega(x,|u_1-u_2|)\,dx \tag{2.49}$$

for every u_1,u_2, in $L^\infty_{loc}(\mathbb{R}^n)$ and for every v in $C^0(\mathbb{R}^n)$ such that $\Phi^u(u_1,v,\Omega)<+\infty$, $\Phi^u(u_2,v,\Omega)<+\infty$.

Proof: Fix u_1, u_2, in $L^\infty_{loc}(\mathbb{R}^n)$, v in $C^0(\mathbb{R}^n)$ and an open set $A\subset\subset\Omega$.

Then there exist a sequence $\{v_k\}_{k\in\mathbb{N}}$ in $C^0(\mathbb{R}^n)$ and a sequence $\{h_k\}_{k\in\mathbb{N}}$ of integer numbers such that

$$|Dv_{h_k}|\le\varphi_{h_k}\quad\text{a.e. in } A\ \ \forall k\in\mathbb{N}, \tag{2.50}$$

$$v_k\to v\ \text{in } C^0(A), \tag{2.51}$$

$$\Phi^u(u_1,v,A)=\limsup_{h\to+\infty}\int_A f(hx,u_1,Dv_h)\,dx. \tag{2.52}$$

Consequently

$$\Phi''(u_2, v, A) \leq \limsup_{h\to+\infty} \int_A f(hx, u_2, Dv_{h_k})\,dx \leq \tag{2.53}$$

$$\leq \limsup_{h\to+\infty} \int_A f(hx, u_1, Dv_{h_k})\,dx + \varrho(M) \int_\Omega \omega(x, |u_1 - u_2|)\,dx =$$

$$= \Phi''(u_1, v, A) + \varrho(M) \int_\Omega \omega(x, |u_1 - u_2|)\,dx .$$

This implies that

$$\Phi''_-(u_2, v, \Omega) \leq \Phi''_-(u_1, v, \Omega) + \varrho(M) \int_\Omega \omega(x, |u_1 - u_2|)\,dx . \tag{2.54}$$

Analogously it results

$$\Phi''_-(u_1, v, \Omega) \leq \Phi''_-(u_2, v, \Omega) + \varrho(M) \int_\Omega \omega(x, |u_1 - u_2|)\,dx \tag{2.55}$$

and the desired result follows. ■

3. - Some technical results

In this section we adapt to our situation some result on the sub-additivity of Γ-limits proved in [CEDA].

Denote with $\bar{z}_1, \bar{z}_2, \dots, \bar{z}_{2^n}$ the vertices of $]0, 1[^n$.

Lemma 3.1: *Let f be defined in (0.6), E a bounded measurable open subset of $\mathbb{R}^n$ and let $\{m_h\}_{h\in\mathbb{N}} \subseteq (L^\infty_{loc}(\mathbb{R}^n))^n$, $r > 0$ be such that $\|m_h(x)\|_{(L^\infty(E))^n} \leq r$. Then*

$$\limsup_{h\to+\infty} \int_E f(hx, u, m_h)\,dx \leq |E| \sum_{j=1}^{2^n} \int_Y \sup_{|s| \leq \|u\|_{L^\infty(E)}} f(y, s, r\bar{z}_j)\,dy \qquad \forall u \in L^\infty_{loc}(\mathbb{R}^n). \tag{3.1}$$

Moreover if $m_h \to 0$ in $(L^\infty(E))^n$ then it results

$$\limsup_{h\to+\infty} \int_E f(hx, u, m_h)\,dx \leq 2^n |E| \int_Y \sup_{|s| \leq \|u\|_{L^\infty(E)}} f(y, s, 0)\,dy \qquad \forall u \in L^\infty_{loc}(\mathbb{R}^n). \tag{3.2}$$

Proof: Fix u in $L^\infty_{loc}(\mathbb{R}^n)$ and let $M(x, z) = \sup_{|s| \leq \|u\|_{L^\infty(E)}} f(x, s, z)$.
Hence

$$\limsup_{h\to+\infty} \int_E f(hx, u, m_h)\,dx \leq \limsup_{h\to+\infty} \int_E M(hx, m_h)\,dx . \tag{3.3}$$

Since $M(x, z)$ verifies the hypotheses of Lemma 1.3 in [CEDA], it results

$$\limsup_{h \to +\infty} \int_E M(hx, m_h)\, dx \leq |E| \sum_{i=1}^{r} \int_Y M(y, r\bar{z}_i)\, dy, \tag{3.4}$$

hence

$$\limsup_{h \to +\infty} \int_E f(hx, u, m_h)\, dx \leq |E| \sum_{i=1}^{r} \int_Y \sup_{|s| \leq \|u\|_{L^\infty(E)}} f(y, s, r\bar{z}_i)\, dy. \tag{3.5}$$

The second part of Lemma can be achieved by using a proof similar to the one in Lemma 1.3 of [CEDA]. ∎

Lemma 3.2: *Let φ be defined in* (0.6) *and f_∞ be given by* (0.8). *Assume that $(K_\varphi)^0 \neq \emptyset$ and $\delta = (1/2)\,\mathrm{dist}(0, \mathbb{R}^n / k_\varphi)$. Then an explicitly computed constant L depending only on n exists such that for every bounded open subset Ω of $\mathbb{R}^n$ and for every compact subset B with $B \subsetneq \Omega$ there exists a sequence $\{\psi_h\}_{h \in \mathbb{N}} \subsetneq \mathrm{Lip}_0(\Omega)$ and ψ in $\mathrm{Lip}_0(\Omega)$ with*

$$0 \leq \psi_h \leq 1 \qquad \text{in } \Omega\ \forall h \in \mathbb{N}, \tag{3.6}$$

$$\psi_h(x) = 1 \qquad \forall x \in B,\ \forall h \in \mathbb{N}, \tag{3.7}$$

$$\psi_h(x) \to \psi \qquad \text{uniformly on } \Omega \text{ as } h \text{ diverges}, \tag{3.8}$$

$$|D\psi_h| \leq \frac{L}{\delta\, \mathrm{dist}(B, \partial\Omega)} \varphi_h(x) \qquad \text{for a.e. } x\ \Omega \text{ and } \forall h \in \mathbb{N}. \tag{3.9}$$

Proof: The proof of this Lemma follows, with some modifications, the same outlines of Lemma 2.3 of [CEDA]. ∎

Lemma 3.1 and 3.2 allow us to prove the following result:

Proposition 3.3: *Let f and φ be defined in* (0.6). *Let Φ', Φ'' be defined in* (2.2), *let Φ'_0, Φ''_0 be given by* (2.4) *and let f_∞ be defined in* (0.8).
Assume $(K_\varphi)^0 \neq \emptyset$. Then

$$\begin{cases} \Phi'(u, v, \Omega) = \Phi'_-(u, v, \Omega) = \Phi'_0(u, v, \Omega), \\ \Phi''(u, v, \Omega) = \Phi''_-(u, v, \Omega) = \Phi''_0(u, v, \Omega), \end{cases} \tag{3.10}$$

for every bounded open subset Ω of $\mathbb{R}^n$, for every u in $L^\infty_{loc}(\mathbb{R}^n)$, and for every v in $C^0(\mathbb{R}^n)$ such that $v = 0$ on $\partial\Omega$.

PROOF: Let Ω be a bounded open subset of $\mathbb{R}^n$ and u in $L^\infty_{loc}(\mathbb{R}^n)$ any $v = 0$ on $\partial\Omega$.

Let $\{\varepsilon_k\}_k$ be a sequence of positive numbers such that $\varepsilon_k \to 0^+$ as $k \to +\infty$ and let, for every $k \in \mathbb{N}$, χ_k be defined by

$$\chi_k(t) = \begin{cases} t + \varepsilon_k & \text{if } t < -\varepsilon_k, \\ 0 & \text{if } -\varepsilon_k \leq t \leq \varepsilon_k, \\ t - \varepsilon_k & \text{if } t > \varepsilon_k. \end{cases} \tag{3.11}$$

For every $k \in \mathbb{N}$ let $\Omega_k \subset\subset \Omega$ be such that

$$\sup_{x \in \Omega \setminus \Omega_k} |v(x)| < \frac{\varepsilon_k}{2}, \qquad |\Omega \setminus \Omega_k| < \varepsilon_k. \tag{3.12}$$

Let us prove that

$$\Phi^*_0(u, v, \Omega) \leq \Phi^*_-(u, v, \Omega). \tag{3.13}$$

To this aim we can assume that $\Phi^*_-(u, v, \Omega) < +\infty$, then for every $k \in \mathbb{N}$ there exists a sequence $\{v_b\}_{b \in \mathbb{N}}$ in $\mathrm{Lip}_{loc}(\mathbb{R}^n)$, such that $v^k_b \to v$ in $C^0(\Omega_k)$, and there exists a $r_k \in \mathbb{N}$, such that $r_k \geq k$,

$$|Dv^k_b| \leq \varphi_k \quad \text{a.e. in } \Omega_k \text{ for every } b \geq r_k, \tag{3.14}$$

and

$$\Phi^*(u, v, \Omega_k) \geq \limsup_{b \to +\infty} \int_{\Omega_k} f(bx, u, Dv^k_b)\, dx. \tag{3.15}$$

For every $k \in \mathbb{N}$, let $s_k \in \mathbb{N}$ be such that $s_k \geq r_k$

$$\Phi^*(u, v, \Omega_k) + \frac{1}{k} \geq \int_{\Omega_k} f(bx, u, Dv^k_b) \quad \text{for every } b > s_k, \tag{3.16}$$

and

$$\|v - v^k_b\|_{C^0(\Omega_k)} \leq \frac{\varepsilon_k}{2} \quad \forall b > s_k. \tag{3.17}$$

For every b sufficiently large let $k_b \in \mathbb{N}$ be defined by $k_b = \max\{k : s_k \leq b\}$ and let us define the functions v_b and $\bar{v}_b$ by

$$v_b(x) = v^{k_b}_b(x) \quad \bar{v}_b(x) = \chi_k(v_b(x)) \quad \forall x \in \mathbb{R}^n. \tag{3.18}$$

Being $h \geq r_{k_h}$, by (3.14) we have

$$|Dv_h| \leq \varphi_{k_h} \quad \text{in } \Omega_{k_h} \text{ definitively in } h, \tag{3.19}$$

moreover by (3.17) and (3.12) it follows that

$$\bar{v}_h = 0 \quad \text{on } \partial\Omega_{k_h}. \tag{3.20}$$

Let us extend $\bar{v}_h$ in $\mathbb{R}^n$ by defining $\bar{v}_h = 0$ in $\mathbb{R}^n \setminus \Omega_{k_h}$ and denote again by $\bar{v}_h$ such extension.

By (3.17) and (3.18) it results that

$$\begin{aligned}|\bar{v}_h(x) - v(x)| &\leq |\bar{v}_h(x) - v_h(x)| + |v_h(x) - v(x)| \leq \\ &\leq \varepsilon_{k_h} + \frac{1}{2}\varepsilon_{k_h} = \frac{3}{2}\varepsilon_{k_h} \quad \text{in } \Omega_{k_h},\end{aligned} \tag{3.21}$$

and by (3.12) that

$$|\bar{v}_h(x) - v(x)| = |v(x)| \leq \frac{1}{2}\varepsilon_{k_h} \quad \text{in } \Omega \setminus \Omega_{k_h}, \tag{3.22}$$

therefore by (3.21) and (3.22) we deduce that

$$\bar{v}_h \to v \quad \text{in } C_0^0(\Omega), \tag{3.23}$$

and by (3.19) that

$$|D\bar{v}_h| \leq |\chi'_{k_h}(v_h)|\,|Dv_h| \leq \varphi_h \quad \text{a.e. in } \Omega. \tag{3.24}$$

Now let B_1 and B_2 be open sets such that $B_1 \subset\subset B_2 \subset\subset \Omega_{k_h}$ for h large enough, let be $\delta = (1/2)\,\mathrm{dist}(0, \mathbb{R}^n / K_\varphi)$, let $\{\psi_h\}_h \subseteq \mathrm{Lip}_0(B_2)$ be the functions given by Lemma 3.2 with $B = \bar{B}_1$ and set

$$w_h = \psi_h v_h + (1 - \psi_h)\bar{v}_h. \tag{3.25}$$

Obviously $w_h \to v$ in $C_0^0(\Omega)$, for every $t \in [0, 1]$ and by (3.19), (3.24), Lemma 3.1, and (3.23) it results

$$\begin{aligned} t|Dw_h| &\leq t|\psi_h Dv_h + (1 - \psi_h)D\bar{v}_h + D\psi_h(v_h - \bar{v}_h)| \leq \\ &\leq t(\psi_h\varphi_h + (1 - \psi_h)\varphi_h + |D\psi_h|\,|v_h - \bar{v}_h|) \leq \\ &\leq t\left(\varphi_h + \frac{L}{\delta\,\mathrm{dist}(B_1, \partial B_2)}\varphi_h \|v_h - \bar{v}_h\|_{C_0^0(\Omega)}\right) \leq \varphi_h \end{aligned} \tag{3.26}$$

for h sufficiently large.

By using the convexity of $f(x, y, \cdot)$ we have:

$$(3.27)\quad \int_\Omega f(hx, u, tDw_h)\,dx \le t\left(\int_\Omega \psi_h f(hx, u, Dv_h)\,dx + \int_\Omega (1-\psi_h) f(hx, u, D\bar{v}_h)\,dx\right) +$$

$$+(1-t)\int_\Omega f\left(hx, u, \frac{t}{1-t}(v_h - \bar{v}_h) D\psi_h\right)dx \le$$

$$\le t\left(\int_{\Omega_{k_0}} \psi_h f(hx, u, Dv_h)\,dx + \int_{\Omega_{k_0}} (1-\psi_h) f(hx, u, D\bar{v}_h)\,dx + \int_{\Omega\setminus\Omega_{k_0}} \psi_h f(hx, u, Dv_h)\,dx + \right.$$

$$\left. + \int_{\Omega\setminus\Omega_{k_0}} (1-\psi_h) f(hx, u, D\bar{v}_h)\,dx\right) + (1-t)\int_\Omega f\left(hx, u, \frac{t}{1-t}(v_h - \bar{v}_h) D\psi_h\right)dx .$$

Hence being

$$\begin{cases} \psi_h = 0 & \text{in } \Omega\setminus B_2 \Rightarrow \psi_h = 0 \text{ in } \Omega\setminus\Omega_{k_0}, \\ \bar{v}_h = 0 & \text{in } \Omega\setminus\Omega_{k_0}, \end{cases}$$

we have

$$\int_\Omega f(hx, u, tDw_h)\,dx \le$$

$$\le t\left(\int_{\Omega_{k_0}} \psi_h f(hx, u, Dv_h)\,dx + \int_{\Omega_{k_0}} (1-\psi_h) f(hx, u, D\bar{v}_h)\,dx + \int_{\Omega\setminus\Omega_{k_0}} f(hx, u, 0)\,dx\right) +$$

$$+(1-t)\int_\Omega f\left(hx, u, \frac{t}{1-t}(v_h - \bar{v}_h) D\psi_h\right)dx \le$$

$$\le t\left(\int_{\Omega_{k_0}} \psi_h f(hx, u, Dv_h)\,dx + \int_{\Omega_{k_0}} (1-\psi_h)\chi^i_{k_0} f(hx, u, D\bar{v}_h)\,dx + \right.$$

$$\left. + \int_{\Omega_{k_0}} (1-\psi_h)(1-\chi^i_{k_0}) f(hx, u, 0)\,dx + \int_{\Omega\setminus\Omega_{k_0}} f(hx, u, 0)\,dx\right) +$$

$$+(1-t)\int_\Omega f\left(hx, u, \frac{t}{1-t}(v_h - \bar{v}_h) D\psi_h\right)dx .$$

On the other hand $B_1 \subseteq \Omega_{k_b}$, $\psi_b = 1$ on B_1 and $\Omega \backslash \Omega_{k_b} \subseteq \Omega \backslash B_1$ therefore:

$$\int_\Omega f(bx, u, tDw_b)\,dx \le t\left(\int_{\Omega_{k_b}} \psi_b f(bx, u, Dv_b)\,dx + 2\int_{\Omega\backslash B_1} f(bx, u, 0)\,dx\right) +$$

$$+(1-t)\int_\Omega f\left(bx, u, \frac{t}{1-t}(v_b - \bar{v}_b)D\psi_b\right)dx,$$

hence we have

$$\int_\Omega f(bx, u, tDw_b)\,dx \le t\left(\int_{\Omega_{k_b}} f(bx, u, Dv_b)\,dx + 2\int_{\Omega\backslash B_1} f(bx, u, 0)\,dx\right) + \tag{3.28}$$

$$+(1-t)\int_\Omega f\left(bx, u, \frac{t}{1-t}(v_b - \bar{v}_b)D\psi_b\right)dx.$$

On the other hand by (3.17), (3.23) and Lemma 3.1 applied with $m_b = 1/(1-t)\cdot$ $\cdot(v_b - \bar{v}_b)D\psi_b$ we obtain:

$$\limsup_{b\to+\infty}\int_\Omega f\left(bx, u, \frac{t}{1-t}(v_b - \bar{v}_b)\right)D\psi_b \le 2^n|\Omega|\int_Y \sup_{|s|\le\|u\|_{L^\infty(\Omega)}} f(y, s, 0)\,dy, \tag{3.29}$$

hence by (3.28) and (3.29) we get as $b \to +\infty$

$$\Phi_0''(u, tv, \Omega) \le t\Phi''_-(u, v, \Omega) + 2t|\Omega\backslash B_1|\int_Y \sup_{|s|\le\|u\|_{L^\infty(\Omega)}} f(y, s, 0)\,dy + \tag{3.30}$$

$$+(1-t)2^n|\Omega|\int_Y \sup_{|s|\le\|u\|_{L^\infty(\Omega)}} f(y, s, 0)\,dy.$$

Since $v = 0$ on $\partial\Omega$ it follows that $tv \to v$ in $C^0(\Omega)$, therefore by (3.30) as $t \to 1^-$ we deduce that

$$\Phi_0''(u, v, \Omega) \le \Phi''_-(u, v, \Omega). \tag{3.31}$$

On the other side, being always true that:

$$\Phi''_-(\Omega, u, v) \le \Phi''(\Omega, u, v) \le \Phi_0''(\Omega, u, v), \tag{3.32}$$

by (3.31) and (3.32) the thesis follows. ∎

Proposition 3.4: *Let f and φ be defined in (0.6), Φ', Φ'' in (2.2), f_∞ in (0.8) and let Ω, Ω_1, Ω_2 be bounded open subset of $\mathbb{R}^n$.*

Then if $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1 \cup \Omega_2 \subseteq \Omega$

$$\Phi'_-(\Omega, u, v) \ge \Phi'_-(\Omega_1, u, v) + \Phi'_-(\Omega_2, u, v) \tag{3.33}$$

for every u in $L^\infty_{loc}(\mathbb{R}^n)$ and for every v in $C^0(\mathbb{R}^n)$;

if $\Omega \subseteq \Omega_1 \cup \Omega_2$ and $(K_\varphi)^0 \neq \emptyset$

$$\Phi''_-(\Omega, u, v) \ge \Phi''_-(\Omega_1, u, v) + \Phi''_-(\Omega_2, u, v) \quad \textit{for every } v \textit{ in } C^0(\mathbb{R}^n); \tag{3.34}$$

Proof: Let u in $L^\infty_{loc}(\mathbb{R}^n)$ and let be v in $C^0(\mathbb{R}^n)$.

Inequality (3.33) follows directly by the definition of Φ'_{-}.

Let us now prove (3.34).

To this aim it suffices to consider the case in which $\Omega \subset\subset \Omega_1 \cup \Omega_2$ and prove that:

$$(3.35) \qquad \Phi''(\Omega, u, v) \geq$$
$$\geq \Phi''(\Omega_1, u, v) + \Phi''(\Omega_2, u, v) \qquad \forall u \in L^\infty_{loc}(\mathrm{R}^n) \text{ and } \forall v \in C^0(\mathrm{R}^n).$$

We can assume that the right hand side of (3.35) is finite.

For $i = 1, 2$, let $\{v_h\}_h \subsetneq \mathrm{Lip}_{loc}$ be such that $v^i_h \to v$ in $C^0(\Omega_i)$ as $h \to \infty$, $|Dv^i_h| \leq \leq \varphi_h$ a.e. in Ω_i definitively in h in N and

$$(3.36) \qquad \Phi''(\Omega_i, u, v) \geq \limsup_{h \to +\infty} \int_{\Omega_i} f(hx, u, Dv^i_h)\, dx\,.$$

Being $\Omega \subset\subset \Omega_1 \cup \Omega_2$, it results for every ε small enough that $\Omega \subset\subset \Omega^-_{1,\varepsilon} \cup \Omega_2$.

Let $\{\psi_h\}_h \subsetneq \mathrm{Lip}_0(\Omega_1)$ be given by Lemma 3.2 applied to $B = \mathrm{cl}(\Omega^-_{1,\varepsilon})$ and set:

$$(3.37) \qquad w_h = \psi_h v^1_h + (1 - \psi_h) v^2_h\,.$$

We now observe that $w_h \to v$ in $C^0(\Omega)$. Moreover, as in (3.26), for every $t \in [0, 1[$ we have $t|Dw_h| \leq \varphi_h$ a.e. in Ω definitively in h in N. By using the convexity of $f(x, y, \cdot)$ we have:

$$(3.38) \qquad \Phi''(\Omega, u, tv) \leq \limsup_{h \to +\infty} \int_{\Omega} f(hx, u, t\,Dw_h)\, dx \leq$$
$$\leq t \limsup_{h \to +\infty} \int_{\Omega} \psi_h f(hx, u, Dv^1_h)\, dx + t \limsup_{h \to +\infty} \int_{\Omega} \psi_h f(hx, u, Dv^2_h)\, dx +$$
$$+ (1 - t) \limsup_{h \to +\infty} \int_{\Omega} \psi_h f\left(hx, u, \frac{t}{1-t}(v^1_h - v^2_h) D\psi_h\right) dx \leq$$
$$\leq t \limsup_{h \to +\infty} \int_{\Omega_1} \psi_h f(hx, u, Dv^1_h)\, dx + t \limsup_{h \to +\infty} \int_{\Omega \setminus \Omega^-_{1,\varepsilon}} \psi_h f(hx, u, Dv^2_h)\, dx +$$
$$+ (1 - t) \limsup_{h \to +\infty} \int_{\Omega \cap (\Omega_1 \setminus \Omega^-_{1,\varepsilon})} \psi_h f\left(hx, u, \frac{t}{1-t}(v^1_h - v^2_h) D\psi_h\right) dx +$$
$$+ (1 - t) \limsup_{h \to +\infty} \int_{\Omega \setminus (\Omega_1 \setminus \Omega^-_{1,\varepsilon})} \psi_h f\left(hx, u, \frac{t}{1-t}(v^1_h - v^2_h) D\psi_h\right) dx\,,$$

we obtain

(3.39) $\Phi''(\Omega, u, tv) \leq$

$$\leq t \limsup_{b \to +\infty} \int_{\Omega_1} f(bx, u, Dv_b^1)\,dx + t \limsup_{b \to +\infty} \int_{\Omega \setminus \Omega_{1,\varepsilon}^-} f(bx, u, Dv_b^2)\,dx +$$

$$+ (1-t) \limsup_{b \to +\infty} \int_{\Omega \cap (\Omega_1 \setminus \Omega_{1,\varepsilon}^-)} f\left(bx, u, \frac{t}{1-t}(v_b^1 - v_b^2)\,D\psi_b\right) dx +$$

$$+ (1-t) \limsup_{b \to +\infty} \int_{\Omega \setminus (\Omega_1 \setminus \Omega_{1,\varepsilon}^-)} f(bx, u, 0)\,dx)\,.$$

We now observe that $\Omega \cap (\Omega_1 \setminus \Omega_{1,\varepsilon}^-) \subseteq \Omega_1 \cap \Omega_2$, hence $1/(1-t)(v_b - v_b^2)\cdot$ $\cdot D\psi_b \to 0$ in $(C^0(\Omega \cap (\Omega_1 \setminus \Omega_{1,\varepsilon}^-)))^n$ as $b \to +\infty$ therefore by Lemma 3.1 applied with $m_b = 1/(1-t)(v_b - v_b^2)\,D\psi_b$ we deduce

(3.40) $$\limsup_{b \to +\infty} \int_{\Omega \cap (\Omega_1 \setminus \Omega_{1,\varepsilon}^-)} f\left(bx, u, \frac{t}{1-t}(v_b^1 - v_b^2)\,D\psi_b\right) dx \leq$$

$$\leq 2^n |\Omega| \int_Y \sup_{|s| \leq \|u\|_{L^\infty(\Omega)}} f(y, s, 0)\,dy\,.$$

By (3.36), (3.38) and (3.40) we have

(3.41) $$\Phi''(\Omega, u, tv) \leq t\Phi''(\Omega_1, u, v) + t\Phi \notin \{\Omega_2, u, v) +$$

$$+ (1-t)\,2^n |\Omega| \int_Y \sup_{|s| \leq \|u\|_{L^\infty(\Omega)}} f(y, s, 0)\,dy + (1-t)\,|\Omega| \int_Y \sup_{|s| \leq \|u\|_{L^\infty(\Omega)}} f(y, s, 0)\,dy\,.$$

By (2.40) and by the $C^0(\Omega)$-lower semicontinuity of $\Phi''(\Omega, u, \cdot)$ we obtain as $t \to 1^-$ that

(3.42) $$\Phi''(\Omega, u, v) \leq \liminf_{t \to 1^-} \Phi''(\Omega, u, tv) \leq \Phi''(\Omega_1, u, v) + \Phi''(\Omega_2, u, v)$$

therefore (3.35) and the thesis follow. ■

4. - Integral representation results

In this Section we prove a representation theorem for $\Gamma^-(C^0(\Omega)) \lim_{b \to +\infty} F_b(w, \Omega)$ where F_b is defined in (2.6).

Proposition 4.1: *Let f and φ be defined in (0.6) and f_∞ be given by (0.8). Let Φ be given in* (2.3), *Φ_0 in* (2.5), *and let F_∞ be given by* (2.48).

Assume $(K_\varphi)^0 \neq \emptyset$. *Then*

$$\Phi''_-(u,v,\Omega) = F_\infty(u,v,\Omega) \tag{4.1}$$

for every bounded open subset Ω *of* $\mathbb{R}^n$, *for every constant function* u *and for every* v *in* $C^0(\mathbb{R}^n)$.

PROOF: Let, for every h in $\mathbb{N}$ and w in $C^0(\mathbb{R}^n)$,

$$\Phi_h(u,w,\Omega) = \tag{4.2}$$

$$= \begin{cases} \int_\Omega f_h(x,u,Dw)\,dx & \text{if } w \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n) \text{ and } |Dw| \leq \varphi \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise} \end{cases}$$

and

$$F_\infty(u,w,\Omega) = \begin{cases} \int_\Omega f_\infty(u,Dw)\,dx & \text{if } w \in C^0(\mathbb{R}^n) \cap W^{1,\infty}(\Omega), \\ +\infty & \text{if } w \in C^0(\mathbb{R}^n) \backslash W^{1,\infty}(\Omega). \end{cases} \tag{4.3}$$

Then by Theorem 1.10 it follows that:

$$F_\infty(u,w,\Omega) = (\Gamma^-(C^0(\Omega)) \lim_{\substack{h \to +\infty \\ w \to v}} \Phi_h(u,w,\Omega))_- \qquad \forall v \in C^0(\mathbb{R}^n). \tag{4.4}$$

Hence

$$\Phi''_-(u,v,\Omega) = F_\infty(u,v,\Omega) \quad \forall v \in C^0(\mathbb{R}^n). \quad \blacksquare \tag{4.5}$$

By making use of Proposition 2.5, Proposition 3.3, Proposition 3.4, and Proposition 4.1, we prove the following results.

THEOREM 4.2: *Let* f *and* φ *be defined in* (0.6) *and* f_∞ *be given by* (0.8). *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$, *let* Φ', Φ'' *be defined in* (2.2) *and* F_∞ *be given by* (2.48).

Assume $(K_\varphi)^0 \neq \emptyset$. *If* v *is in* $C^0(\mathbb{R}^n)$ *with* $v = 0$ *on* $\partial\Omega$ *and* u *in* $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ *and if there exist* $\Omega_1, \Omega_2, \ldots, \Omega_r$ *bounded open subsets of* $\mathbb{R}^n$ *such that* $\Omega_i \cap \Omega_j = \emptyset$ *if* $i \neq j$, $\Omega \supseteq \bigcup_{i=1}^{r} \Omega_i$, $\left|\Omega - \bigcup_{i=1}^{r} \Omega_i\right| = 0$, $|\partial\Omega_i| = 0$ *and there exist* $c_1, c_2, \ldots, c_r$ *real numbers such that* $u = c_i$ *on* Ω_i, *then*

$$\Phi'_-(u,v,\Omega) \geq F_\infty(u,v,\Omega), \tag{4.6}$$

$$\Phi''_-(u,v,\Omega) \leq F_\infty(u,v,\Omega). \tag{4.7}$$

PROOF: First let us prove (4.6). Fix u in $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ as above and v in $C^0(\mathbb{R}^n)$ such that $v = 0$ on $\partial\Omega$. Assume $\Phi'(u,v,\Omega) < +\infty$. Then there exists a sequence $\{v_h\}_{h \in \mathbb{N}}$

in $C^0(\mathrm{R}^n)$ and a sequence $\{h_k\}_{k\in\mathrm{N}}$ of integer numbers such that

$$|Dv_{h_k}| \leq \varphi_{h_k} \quad \text{a.e. in } \Omega \ \forall k \in \mathrm{N}, \tag{4.8}$$

$$v_k \to v \quad \text{in } C^0(\Omega), \tag{4.9}$$

$$\Phi'(u, v, \Omega) \geq \liminf_{k\to+\infty} \int_\Omega f_{h_k}(x, u, Dv_{h_k})\,dx . \tag{4.10}$$

By Lemma 2.1 v is in $C^0(\mathrm{R}^n) \cap W^{1,\infty}(\Omega)$. Consequently, by Proposition 4.1, it follows

$$\Phi'(u, v, \Omega) \geq \liminf_{k\to+\infty} \int_\Omega f_{h_k}(x, u, Dv_{h_k})\,dx \geq \liminf_{k\to+\infty} \sum_{i=1}^{\nu} \int_{\Omega_i} f_{h_k}(x, c_i, Dv_{h_k})\,dx \geq \tag{4.11}$$

$$\geq \sum_{i=1}^{\nu} \int_{\Omega_i} f_\infty(c_i, Dv)\,dx = \int_\Omega f_\infty(u, Dv)\,dx = F_\infty(u, v, \Omega).$$

Let us, now, prove (4.7).

Assume v in $C^0(\mathrm{R}^n) \cap W^{1,\infty}(\Omega)$.

To verify (4.7) it is enough to prove that

$$\Phi''_-(u, v, \Omega') \leq F_\infty(u, v, \Omega') \quad \forall \Omega' \subset\subset \Omega . \tag{4.12}$$

Fix $\Omega' \subset\subset \Omega$ and assume $F_\infty(u, v, \Omega') < +\infty$ (this implies that Dv is in K_φ a.e. in Ω). Let A_i^m, for every m in N and for every $i = 1, \ldots, \nu$, be an open subset of R^n such that:

$$|\partial\Omega'| = 0, \quad \Omega' \subseteq \dot{\bigcup_{i=1}^{\nu}} A_i^m \subset \Omega, \quad \Omega' \cap \Omega_i \subseteq A_i^m \quad \text{and} \quad |A_i^m \setminus (\Omega' \cap \Omega_i)| < \frac{1}{m}.$$

By Lemma 2.3 it follows

$$\int_{A_i^m} f_\infty(c_i, Dv)\,dx = \int_{\Omega'\cap\Omega_i} f_\infty(u, Dv)\,dx + \int_{A_i^m\setminus(\Omega'\cap\Omega_i)} f_\infty(c_i, Dv)\,dx = \tag{4.13}$$

$$= \int_{\Omega'\cap\Omega_i} f_\infty(u, Dv)\,dx + \sum_{j=1}^{\nu} \int_{(A_i^m\setminus(\Omega'\cap\Omega_i))\cap\Omega_j} f_\infty(c_i, Dv)\,dx \leq \int_{\Omega'\cap\Omega_i} f_\infty(u, Dv)\,dx +$$

$$+ \sum_{j=1}^{\nu} \left(\int_{(A_i^m\setminus(\Omega'\cap\Omega_i))\cap\Omega_j} f_\infty(c_j, Dv)\,dx + |(A_i^m \setminus (\Omega'\cap\Omega_i)) \cap \Omega_j|\, \varrho(M) \int_{]0,1[^n} \omega(y, |c_i - c_j|)\,dy \right)$$

$$\forall m \in \mathrm{N}, \ \forall i = 1, \ldots, \nu .$$

Consequently for every $\varepsilon > 0$ there exists $\bar{m} > 0$ such that

$$\sum_{i=1}^{\nu} \int_{\Omega' \cap \Omega_i} f_\infty(u, Dv)\,dx \geqslant \sum_{i=1}^{\nu} \int_{A_i^m} f_\infty(c_i, Dv)\,dx - \varepsilon \quad \forall m > \bar{m} \quad \forall i = 1, \ldots, \nu . \tag{4.14}$$

By (4.14), Proposition 4.1, Proposition 3.3, Proposition 2.5 and Proposition 3.4 we obtain

$$\begin{aligned}\int_{\Omega'} f_\infty(u, Dv)\,dx &\geqslant \sum_{i=1}^{\nu} \Phi''_-(c_i, v, A_i^m) - \varepsilon = \\ &= \sum_{i=1}^{\nu} (\Phi''_-(c_i, v, A_i^m) - \phi''_-(u, v, A_i^m) + \Phi''_-(u, v, A_i^m)) - \varepsilon \geqslant \\ &\geqslant \sum_{i=1}^{\nu} \Phi''_-(u, v, A_i^m) - \sum_{i=1}^{\nu} \varrho(M) \int_{A_i^m} \omega(x, |c_i - u|)\,dx - \varepsilon \geqslant \\ &\geqslant \Phi''_-(u, v, \Omega') - \sum_{i=1}^{\nu} \varrho(M) \int_{A_i^m \setminus (\Omega' \cap \Omega_i)} \omega(x, |c_i - u|\,dx - \varepsilon \\ &\qquad \forall \varepsilon,\ \forall m > \bar{m},\ \forall i = 1, \ldots, \nu .\end{aligned}$$

Finally, passing to the limit first for $m \to +\infty$ and for $\varepsilon \to 0$ we obtain (4.12). ■

Corollary 4.3: *Let f and φ be defined in (0.6) and f_∞ be given by (0.8). Let Φ', Φ'' be defined in (2.2), and let F_∞ be given by (2.48).*
Assume $(K_\varphi)^0 \neq \emptyset$. Then

$$\Phi'_-(u, v, \Omega) \geqslant F_\infty(u, v, \Omega), \tag{4.15}$$

$$\Phi''_-(u, v, \Omega) \leqslant F_\infty(u, v, \Omega), \tag{4.15}$$

for every bounded open subset Ω of $\mathbb{R}^n$, for every u in $L^\infty(\mathbb{R}^n)$, and for every v in $C^0(\mathbb{R}^n)$ such that $v = 0$ on $\partial\Omega$.

Proof: The result follows immediately by Lemma 2.1, Proposition 2.4, Proposition 3.3 and Theorem 4.2. ■

Theorem 4.4: *Let f and φ be defined in (0.6) and f_∞ be given by (0.8). Let Φ be given by (2.3), let Φ_0 be given by (2.5) and let F_∞ be defined in (2.48).*
Assume $(K_\varphi)^0 \neq \emptyset$. Then

$$\Phi_0(u, v, \Omega) = \Phi(u, v, \Omega) = F_\infty(u, v, \Omega)$$

for every bounded open subset Ω of $\mathbb{R}^n$, for every u in $L^\infty_{loc}(\mathbb{R}^n)$ and for every v in $C^0(\mathbb{R}^n)$ with $v = 0$ on $\partial\Omega$.

PROOF: Let u in $L^\infty_{loc}(\mathbf{R}^n)$ and v in $C^0(\mathbf{R}^n)$ with $v = 0$ on $\partial\Omega$. By Corollary 4.3 and Proposition 3.3 it results

$$F_\infty(u, v, \Omega) \geq \Phi''_-(u, v, \Omega) =$$

$$= \Phi''(u, v, \Omega) \geq \Phi'(u, v, \Omega) = \Phi'_-(u, v, \Omega) \geq F_\infty(u, v, \Omega),$$

i.e.

$$F_\infty(u, v, \Omega) = \Phi(u, v, \Omega),$$

Moreover, being by Proposition 3.3

$$\Phi'_0(u, v, \Omega) = \Phi'(u, v, \Omega), \qquad \Phi''_0(u, v, \Omega) = \Phi''(u, v, \Omega),$$

it follows

$$F_\infty(u, v, \Omega) = \Phi_0(u, v, \Omega). \quad \blacksquare$$

THEOREM 4.5: *Let f and φ be defined in* (0.6) *and f_∞ be given by* (0.8). *Let F be given in* (2.8) *and let F_∞ be defined in* (2.48).
Assume $(K_\varphi)^0 \neq \emptyset$. Then

$$F(u, \Omega) = F_0(u, \Omega) = F_\infty(u, v, \Omega),$$

for every bounded open subset Ω of $\mathbf{R}^n$, for every v in $C^0(\mathbf{R}^n)$ with $v = 0$ on $\partial\Omega$.

PROOF: It is straightforward to obtain the theorem by Proposition 2.2 and Theorem 4.4. $\blacksquare$

5. - THE CONVERGENCE OF MINIMUM POINTS

THEOREM 5.1: *Let f and φ be defined in* (0.6) *and f_∞ be given by* (0.8). *Let u be in $L^\infty_{loc}(\mathbf{R}^n)$ and v in $C^0(\mathbf{R}^n)$ with $v = 0$ on $\partial\Omega$.*

Moreover for every bounded open subset Ω of $\mathbf{R}^n$, β in $L^1(\Omega)$ and for every h in $\mathbf{N}$ let u_h be a solution of the problem

$$m_h(\Omega, \beta) = \min\left\{ \int_\Omega f(hx, u, Du)\,dx + \int_\Omega \beta u\,dx: \right.$$

$$\left. u \text{ Lipschitz continuous, } u = 0 \text{ on } \partial\Omega,\ |Du(x)| \leq \varphi_h \text{ a.e. } x \text{ in } \Omega \right\}.$$

Then the following facts hold:

a) if $(K_\varphi)^0 \neq \emptyset$ the sequence $\{u_h\}$ is compact in $C^0_0(\Omega)$ and the converging subse-

quences $\{u_{h_k}\}$ *converge to solutions of*

$$m_\infty(\Omega,\beta) = \min\left\{\int_\Omega f_\infty(u, Du)\,dx + \int_\Omega \beta u\,dx\colon u \text{ Lipschitz continuous}, u = 0 \text{ on } \partial\Omega\right\},$$

moreover the sequence $\{m_h(\Omega,\beta)\}_{h\in\mathbb{N}}$ *converges to* $m_\infty(\Omega,\beta)$;

b) if $(K_\varphi)^0 \neq \emptyset$, $u_0 = 0$ *is the only solution of* $m_\infty(\Omega,\beta)$ *and the sequence* $\{u_h\}_{h\in\mathbb{N}}$ *converges in* $C_0^0(\Omega)$ *to* u_0, *moreover if in addition we assume hat*

$$f(y,0,0) = \min_{s,z} f(y,s,z) \quad \text{for a.e. } y \in Y$$

it turns out that the sequence $\{m_h(\Omega,\beta)\}_{h\in\mathbb{N}}$ *converges to* $m_\infty(\Omega,\beta)$ *and that*

$$m_\infty(\Omega,\beta) = \int_Y f(y,0,0)\,dy\,.$$

Proof: By making use of Theorem 4.5, Theorem 1.5 and by following the same outline of proof of Theorem 5.2 in [CEDA], we obtain the desired results. ■

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