Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
114° (1996), Vol. XX, fasc. 1, pagg. 157-173

G. BRUNO - R. GRANDE (*)

Compact Embedding Theorems for Sobolev-Besicovitch Spaces of Almost Periodic Functions (**)

ABSTRACT. — Some compact embedding results are proved for Sobolev-Besicovitch spaces of type H and W. These results can be applied to obtain existence of almost periodic solutions of a non-linear differential equation.

Teoremi di immersione compatta negli spazi di Sobolev-Besicovitch di funzioni quasi periodiche

SUNTO. — In questo lavoro si provano dei risultati di immersione compatta per gli spazi di Sobolev-Besicovitch di tipo H e W. Questi risultati sono poi utilizzati per ottenere l'esistenza di soluzioni quasi periodiche di una equazione differenziale non lineare.

1. - INTRODUCTION

This paper is concerned with some compact embedding results for Sobolev-Besicovitch spaces analogous to the Rellich-Kondrachov theorem.

The study of continuous embeddings for these functional spaces in the spaces B^q_{ap} of Besicovitch almost periodic functions as well as in the space C^0_{ap} of uniformly almost periodic functions in the sense of Bohr, has been approached by Iannacci et al. [10] for $H^{m,q}_{ap}$, and $W^{m,q}_{ap}$ with $q > 1$, and by Dell'Acqua-Santucci [9] for $W^{m,1}_{ap}$. They consider subspaces with fixed spectrum of the wide spaces $H^{m,q}_{ap}$ and $W^{m,q}_{ap}$ that contain periodic functions. In this hypothesis, they obtain results similar to the classical Sobolev embeddings. Here, we also consider these subspaces.

In the proof of the Rellich-Kondrachov theorem for the usual Sobolev spaces, the classical Kolmogorov-Fréchet theorem plays a fundamental role [6, 15]. In the context

(*) Indirizzo degli Autori: Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università di Roma «La Sapienza», Via A. Scarpa 16, 00161 Roma.
(**) Memoria presentata il 18 giugno 1996 da Mario Troisi, uno dei XL.

of the Sobolev-Besicovitch spaces the same role is played by a compactness criterion in the B^q_{ap} spaces, given in [7] by the present authors.

This paper is organized as follows. In Section 2, we recall some results about the B^q_{ap} spaces (the Hausdorff-Young theorem and the compactness criterion), as well as the definitions relating to the Sobolev-Besicovitch spaces $W^{m,q}_{ap}$ and $H^{m,q}_{ap}$. An easy characterization of the elements in $H^{1,q}_{ap}$ is also given in the same section.

Section 3 is interely devoted to the proof of the compact embedding theorems and some corollaries.

Finally, in the last Section we apply the previous results to obtain the existence af almost periodic solution of the differential equation

$$-y''(x) + \gamma y(x) = f(x, y(x)),$$

by means the tool suggested by Avantaggiati in [3].

2. - Definitions and basic properties

Referring the reader to the monographs by Amerio-Prouse [1], Corduneanu [8], Levitan [12], Levitan-Zhikov [13] for the classical theory of the uniformly almost periodic functions and to the papers by Avantaggiati [2], Avantaggiati-Bruno-Iannacci [4], Bruno-Grande [7] for the main properties of the B^q_{ap} spaces, we wish to mention only the main results which we use in the following.

The first result we will need is a simple interpolation inequality for the $B^q_{ap}(\mathrm{R})$ spaces.

Proposition 2.1: *Let $f \in B^s_{ap}(\mathrm{R})$, $1 \leqslant s < +\infty$. Let $t \in \mathrm{R}$ such that $s \leqslant t < +\infty$. Then for any $r \in \mathrm{R}$ such that $s \leqslant r \leqslant t$ the following inequality holds*

$$\|f\|_r \leqslant \|f\|_s^{\alpha} \|f\|_t^{1-\alpha} \tag{2.1}$$

where $1/r = \alpha/s + (1-\alpha)/t$, $(0 \leqslant \alpha \leqslant 1)$. ■

Proposition 2.1 follows by Hölder inequality, in the same way as for the usual L^p spaces.

Let us remind the Hausdorff-Young theorem for $B^q_{ap}(\mathrm{R})$ spaces [3,5].

Theorem 2.2: *Let $f \in B^q_{ap}(\mathrm{R}^r)$ and $q' = q/(q-1)$; we have*

$$\Big(\sum_{\lambda \in \sigma(f)} |a(\lambda, f)|^{q'}\Big)^{1/q'} \leqslant \|f\|_q, \qquad \text{if } q \in]1, 2], \tag{2.2}$$

$$\|f\|_q \leqslant \Big(\sum_{\lambda \in \sigma(f)} |a(\lambda, f)|^{q'}\Big)^{1/q'}, \qquad \text{if } q \in [2, +\infty[, \tag{2.3}$$

and the series in the right-hand side of (2.3) may be divergent. ■

We would like to point out, that the Hausdorff-Young theorem plays a crucial role in our proofs.

Finally, we state a compactness criterion for $B^q_{ap}(\mathrm{R})$ spaces. We refer the reader to the paper [7] also for all the notations used here and in the following.

THEOREM 2.3: *Let $\mathcal{F}$ be a collection of elements of B^q_{ap}, $1 \leq q < +\infty$, closed and bounded. Then the following statements are equivalent:*

1) *$\mathcal{F}$ is compact (for the B^q_{ap}-norm).*
2) *$\mathcal{F}$ is B^q_{ap}-equicontinuous and B^q_{ap}-equi-almost periodic.* ■

In Iannacci et al. [10] the reader can find more informations about the basic properties of the Sobolev-Besicovitch $W^{m,q}_{ap}$ and $H^{m,q}_{ap}$ spaces. Here we only recall, for reader's convenience, the definitions.

For any $q \in [1 + \infty]$ and $m \in \mathrm{N}_0$, we set $\forall P \in \mathcal{P}(\mathrm{R})$

$$\|P\|_{W^{m,q}} := \left(\sum_{k=0}^{m} \|P^{(k)}\|_q^q\right)^{1/q}, \qquad \forall q \in [1 + \infty[,$$

(2.4)

$$\|P\|_{W^{m,\infty}} := \sum_{k=0}^{m} \|P^{(k)}\|_\infty .$$

Let us observe that, for any fixed q, (2.4) defines a norm on $\mathcal{P}(\mathrm{R})$ and we have $\|P\|_{W^{0,q}} = \|P\|_q$.

DEFINITION 2.4: For any fixed $q \in [1 + \infty]$ we denote by $W^{m,q}_{ap}(\mathrm{R})$ the completion of $\mathcal{P}(\mathrm{R})$ with respect to the norm $\|\cdot\|_{W^{m,q}}$ defined by (2.4).

These spaces are called Sobolev-Besicovitch spaces of order m and of type W.

Hence we can define the norm on the space $W^{m,q}_{ap}$ in the following way:

$$\|f\|_{W^{m,q}} := \left(\sum_{k=0}^{m} \|f^{(k)}\|_q^q\right)^{1/q}.$$

Of course, $W^{m,q}_{ap}(\mathrm{R}) \subset B^q_{ap}(\mathrm{R})$, $\forall m \geq 0$, $\forall q \geq 1$.

In the same way, for any trigonometrical polynomial

$$P(x) = \sum_{\lambda \in \sigma(P)} a(\lambda; P)\, e^{i\lambda x},$$

and $\forall q \in]1, +\infty[$, $m \in \mathrm{R}$, let us consider the norm $\|\cdot\|_{H^{m,q}}$ defined as follows

$$\|P\|^{q'}_{H^{m,q}} := \sum_{\lambda \in \sigma(P)} (1 + |\lambda|^2)^{mq'/2} |a(\lambda; P)|^{q'}, \tag{2.5}$$

where $q' = q/(q-1)$.

DEFINITION 2.5: We denote by $H^{m,q}_{ap}(\mathrm{R})$ the completion of $\mathcal{P}(\mathrm{R})$ with respect to the norm $\|\cdot\|_{H^{m,q}}$ defined by (2.5).

These spaces are called Sobolev-Besicovitch spaces of type H.

For any $f \in H^{m,q}_{ap}(\mathrm{R})$ we introduce the norm

$$\|f\|_{H^{m,q}} := \Big(\sum_{\lambda \in \sigma(f)} (1 + |\lambda|^2)^{mq'/2} |a(\lambda; f)|^{q'}\Big)^{1/q'}.$$

Observe that, $\forall m \geq 0$, $\forall q \geq 2$ we have

$$H^{m,q}_{ap}(\mathrm{R}) \subset B^{q}_{ap}(\mathrm{R}),$$

with continuous embedding.

REMARK 2.6: If $f \in H^{m,q}_{ap}(\mathrm{R})$ then

$$\sum_{\lambda \in \sigma(f)} (1 + |\lambda|^2)^{mq'/2} |a(\lambda; f)|^{q'} < \infty, \tag{2.6}$$

so that the series (2.6) is summable with respect to any summation method and the sum is the same. Therefore, since $\sigma(f)$ is countable, we have

$$\sigma(f) = \{\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots\},$$

and

$$\sum_{\lambda \in \sigma(f)} (1 + |\lambda|^2)^{mq'/2} |a(\lambda; f)|^{q'} = \sum_{k=1}^{\infty} (1 + |\lambda_k|^2)^{mq'/2} |a(\lambda_k; f)|^{q'}.$$

The following result gives a simple but very useful characterization of the elements in $H^{1,q}_{ap}$.

LEMMA 2.7: *Let $f \in H^{0,q}_{ap}(\mathrm{R})$, $1 < q < +\infty$. The following statements are equivalent:*

i) $f \in H^{1,q}_{ap}(\mathrm{R})$,

ii) *there exists a positive constant C such that*

$$\|\tau_h f - f\|_{H^{0,q}} \leq C|h|,$$

for all real numbers h.

PROOF: ii) $\Rightarrow$ i) Now, let $f \in H^{0,q}_{ap}(\mathrm{R})$, $1 \leq q < 2$; then

$$\|f\|^{q'}_{H^{0,q}} = \sum_{k=1}^{\infty} |a(\lambda_k; f)|^{q'} < +\infty, \tag{2.7}$$

with $q' \geq 2$. Let $(P_n)_{n \in \mathbb{N}}$ be a sequence of trigonometric polynomials approximating f, we have for every $\lambda_k \in \sigma(f)$, $\lambda_k \neq 0$

$$(e^{i\lambda_k b} - 1) \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} P_n(x) e^{-i\lambda_k x} dx = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (P_n(x+b) - P_n(x)) e^{-i\lambda_k x} dx$$

and passing to the limit for $n \to +\infty$

$$(e^{i\lambda_k b} - 1) a(\lambda_k; f) = a(\lambda_k, \tau_b f - f), \tag{2.8}$$

according to the definition of Bohr-Fourier transform. Hence, we have

$$\sum_{k=1}^{r} |e^{i\lambda_k b} - 1|^{q'} |a(\lambda_k; f)|^{q'} = \sum_{k=1}^{r} |a(\lambda_k, \tau_b f - f)|^{q'} \leq (C|b|)^{q'}.$$

Then

$$\sum_{k=1}^{r} \left| \frac{e^{i\lambda_k b} - 1}{b} \right|^{q'} |b|^{q'} |a(\lambda_k; f)|^{q'} \leq (C|b|)^{q'}.$$

Passing to the limit for $b \to 0$ and then for $r \to \infty$, we obtain,

$$\sum_{k=1}^{\infty} |\lambda_k|^{q'} |a(\lambda_k; f)|^{q'} \leq C^{q'}. \tag{2.9}$$

Hence, from (2.7) and (2.9) it follows

$$\sum_{k=1}^{\infty} (1 + |\lambda_k|^{q'}) |a(\lambda_k; f)|^{q'} < +\infty.$$

On the other hand, from the inequality $a^r + b^r \leq 2^{1-r}(a+b)^r$, $\forall a, b \in \mathrm{R}^+$, $\forall r \in [0, 1]$ choosing $r = 2/q'$, $a = 1$, $b = |\lambda_k|^{q'}$, it follows that $(1 + |\lambda_k|^2)^{q'/2} \leq 2^{(1-2/q')q'/2}(1 + |\lambda_k|^{q'})$ holds, and therefore we have

$$\|f\|^{q'}_{H^{1,q}} = \sum_{k=1}^{\infty} (1 + |\lambda_k|^2)^{q'/2} |a(\lambda_k; f)|^{q'} < +\infty.$$

Now, let $q \geq 2$, then $q' \in (1, 2]$, but

$$\sum_{k=1}^{\infty} (1 + |\lambda_k|^{q'}) |a(\lambda_k; f)|^{q'} < +\infty,$$

still holds. From the inequality $a^r + b^r \leq (a+b)^r$, $\forall a, b \in \mathrm{R}^+ \cup \{0\}$, $\forall r > 1$ choosing $r = 2/q' > 1$, $a = 1$, $b = |\lambda_k|^{q'}$, it follows $(1 + |\lambda_k|^2)^{q'/2} \leq (1 + |\lambda_k|^{q'})$. Finally we have

$$\|f\|^{q'}_{H^{1,q}} = \sum_{k=1}^{\infty} (1 + |\lambda_k|^2)^{q'/2} |a(\lambda_k; f)|^{q'} < +\infty,$$

that is $f \in H^{1,q}_{ap}(\mathrm{R})$.

i) ⇒ ii) Now let $f \in H^{1,q}_{\varphi}(\mathbf{R})$. Then

$$\|f\|^{q'}_{H^{1,q}} = \sum_{k=1}^{\infty} (1 + |\lambda_k|^2)^{q'/2} |a(\lambda_k; f)|^{q'} < +\infty .$$

Since for every index k

$$|\lambda_k|^{q'} |a(\lambda_k; f)|^{q'} \leq (1 + |\lambda_k|^2)^{q'/2} |a(\lambda_k; f)|^{q'}$$

we have

$$\sum_{k=1}^{\infty} |\lambda_k|^{q'} |a(\lambda_k; f)|^{q'} < +\infty .$$

Furthermore, from the inequality

$$|a(\lambda_k; \tau_b f - f)| = \left| \frac{e^{i\lambda_k b} - 1}{b} \right| |b| |a(\lambda_k; f)| \leq |\lambda_k| |b| |a(\lambda_k; f)|$$

for $\lambda_k \in \sigma(f)$, $\lambda_k \neq 0$, it follows that

$$\sum_{k=1}^{\infty} |a(\lambda_k; \tau_b f - f)|^{q'} \leq |b|^{q'} \sum_{k=1}^{\infty} |\lambda_k|^{q'} |a(\lambda_k; f)|^{q'},$$

that is $\|\tau_b f - f\|_{H^{0,q}} \leq C|b|$, for a suitable positive constant C. ■

3. - Compact embedding theorems for $H^{1,q}_{\varphi}$ spaces

Let us fix a set $\Lambda \subset \mathbf{R} \setminus \{0\}$, satisfying the following properties:

a) $\operatorname{card} \Lambda = \operatorname{card} \mathbf{N}$;

b) Λ has an unique limit point and this is the point at infinity;

c) Λ is ordered with respect to the absolute value:

$$\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}, \qquad \text{with } |\lambda_1| \leq |\lambda_2| \leq \ldots;$$

d) there exists $\beta \geq 0$ such that

$$\sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^{\gamma}} < +\infty, \qquad \forall \gamma > \beta . \tag{3.1}$$

We shall consider the almost periodic functions such that $\sigma(f) \subset \Lambda$.

Let us observe that $\Lambda \subset \mathbf{R} \setminus \{0\}$ means that we consider almost periodic functions with asymptotic mean equal to zero.

Furthermore, observe that these classes of almost periodic functions include the periodic functions.

In this cases we have $\beta = 1$, since the series (3.1) has the same behaviour of the generalized harmonic series $\sum_{k=1}^{\infty} 1/k^{\gamma}$.

Let us set

$$B^q_{ap}(\Lambda) = \{f \in B^q_{ap}(\mathbb{R}) \colon \sigma(f) \subseteq \Lambda\},$$

and, analogously, $W^{1,q}_{ap}(\Lambda)$, $H^{1,q}_{ap}(\Lambda)$ and $C^0_{ap}(\Lambda)$. Observe that these spaces are separable.

Then the following result holds.

THEOREM 3.1: *Suppose that* $\Lambda \subset \mathbb{R}\backslash\{0\}$ *satisfies the previous assumptions. Then*

i) *if* $q < \beta$ *and* $q \geq 2$,

$$H^{1,q}_{ap}(\Lambda) \hookrightarrow\hookrightarrow B^r_{ap}(\Lambda) \quad \forall r \in \left[q, \frac{\beta q}{\beta - q}\right[;$$

ii) *if* $q = \beta$ *and* $q \geq 2$,

$$H^{1,q}_{ap}(\Lambda) \hookrightarrow\hookrightarrow B^r_{ap}(\Lambda) \quad \forall r \in [q, +\infty[;$$

iii) *if* $q > \beta$

$$H^{1,q}_{ap}(\Lambda) \hookrightarrow\hookrightarrow C^0_{ap}(\Lambda).$$

PROOF.: i) Let $\mathcal{B} = \{f \in H^{1,q}_{ap}(\Lambda), \|f\|_{H^{1,q}} < 1\}$, the open unit ball of $H^{1,q}_{ap}(\Lambda)$. Then, by Sobolev embedding Theorem [10, theorem 5.1], $\mathcal{B}$ is a bounded subset of $B^r_{ap}(\Lambda)$, for any $r \in [q, \beta q/(\beta - q)[$. We shall show that $\bar{\mathcal{B}}$ is a compact subset of $B^r_{ap}(\Lambda)$. Observe first that $\bar{\mathcal{B}}$ is closed and bounded. Thus, by the compactness criterion 2.3, we need to show that $\mathcal{B}$ is B^r_{ap}-equicontinuous and B^r_{ap}-equi-almost periodic.

Let $q < r < q^* < \beta q/(\beta - q)$, and choose $\alpha \in (0, 1)$ so that

$$\frac{1}{r} = \frac{\alpha}{q} + \frac{1-\alpha}{q^*}.$$

Therefore, by Proposition 2.2, it follows that

$$\|\tau_h u - u\|_r \leq \|\tau_h u - u\|^{\alpha}_q \|\tau_h u - u\|^{1-\alpha}_{q^*}, \qquad h \in \mathbb{R}.$$

Also, by the Hausdorff-Young theorem for the B^q_{ap} spaces,

$$\|\tau_h u - u\|_q \leq \left(\sum_{k=1}^{\infty} |a(\lambda_k; \tau_h u - u)|^{q'}\right)^{1/q'} = \|\tau_h u - u\|_{H^{0,q}}$$

holds with $1/q + 1/q' = 1$.

Then by applying Lemma 2.7, we get

$$\|\tau_h u - u\|_{H^{0,q}} \leq C|h|, \qquad h \in \mathbb{R}\backslash\{0\},$$

where C is a positive constant.

Hence

$$\|\tau_b u - u\|_r \leqslant (C|b|)^{\alpha} \|\tau_b u - u\|_{q^*}^{1-\alpha} \leqslant (C|b|)^{\alpha} (2\|u\|_{q^*})^{1-\alpha} \leqslant K|b|^{\alpha}.$$

This shows that $\mathcal{B}$ is B^r_{ap}-equicontinuous.

Now we verify that $\mathcal{B}$ is also a B^r_{ap}-equi-almost periodic set. To this end, we fix $\varepsilon > 0$ and observe that, by Hausdorff-Young theorem,

$$\|\Delta^{\zeta} u\|_r = \lim_{T \to \infty} \left(\frac{1}{2T} \int_{-T}^{T} |u(x+\zeta) - u(x)|^r dx \right)^{1/r} \leqslant$$

$$\leqslant \left(\sum_{k=1}^{\infty} |a(\lambda_k; \Delta^{\zeta} u)|^{r'} \right)^{1/r'}, \qquad \forall u \in \mathcal{B}.$$

From (2.8) it follows, by Hölder inequality

$$\sum_{k=1}^{\infty} |a(\lambda_k; \Delta^{\zeta} u)|^{r'} = \sum_{k=1}^{\infty} |a(\lambda_k; \Delta^{\zeta} u)|^{r'} |\lambda_k|^{r'} \frac{1}{|\lambda_k|^{r'}} =$$

$$= \sum_{k=1}^{\infty} |e^{i\lambda_k \zeta} - 1|^{r'} |a(\lambda_k; u)|^{r'} |\lambda_k|^{r'} \frac{1}{|\lambda_k|^{r'}} \leqslant$$

$$\leqslant \left(\sum_{k=1}^{\infty} |a(\lambda_k; u)|^{q'} |\lambda_k|^{q'} \right)^{r'/q'} \left(\sum_{k=1}^{\infty} \frac{|e^{i\lambda_k \zeta} - 1|^{r'(q'/r')'}}{|\lambda_k|^{r'(q'/r')'}} \right)^{1/(q'/r')'} \leqslant$$

$$\leqslant \left(\sum_{k=1}^{\infty} (1 + |\lambda_k|^2)^{q'/2} |a(\lambda_k; u)|^{q'} \right)^{r'/q'} \left(\sum_{k=1}^{\infty} \frac{|e^{i\lambda_k \zeta} - 1|^{r'(q'/r')'}}{|\lambda_k|^{r'(q'/r')'}} \right)^{1/(q'/r')'} \leqslant$$

$$\leqslant \left(\sum_{k=1}^{\infty} \frac{|e^{i\lambda_k \zeta} - 1|^{r'(q'/r')'}}{|\lambda_k|^{r'(q'/r')'}} \right)^{1/(q'/r')'}.$$

Observe that $r'(q'/r')' = rq/(r-q) > \beta$, hence from hypotesis (3.1)

$$\sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^{rq/(r-q)}} < +\infty$$

holds. Then there exists a natural number $N = N_{\varepsilon}$, such that

$$\sum_{k=N+1}^{\infty} \frac{1}{|\lambda_k|^{r'(q'/r')'}} < \frac{\varepsilon^{r'(q'/r')'}}{2^{r'(q'/r')'+1}}, \tag{3.2}$$

Now set

(3.3) $$\delta = \delta_\varepsilon = \left(\frac{\varepsilon^{r'(q'/r')'}}{2C} \right)^{1/(r'(q'/r')')},$$

where

$$C = \sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^{r'(q'/r')'}}.$$

If $\zeta \in \mathrm{R}$ is a solution of the following system of inequalities

(3.4) $$|\lambda_k \zeta| < \delta (\mathrm{mod}\, 2\pi), \qquad k = 1, 2, \ldots, N,$$

then

$$|e^{i\lambda_k \zeta} - 1| = ((1 - \cos \lambda_k \zeta)^2 + \sin^2 \lambda_k \zeta)^{1/2} = 2 \left| \sin \frac{\lambda_k \zeta}{2} \right| < \delta,$$

so that ζ is a solution of the following system

$$|e^{i\lambda_k \zeta} - 1| < \delta, \qquad k = 1, 2, \ldots, N.$$

But, from (3.2) and (3.3), we get

$$\sum_{k=1}^{\infty} \frac{|e^{i\lambda_k \zeta} - 1|^{r'(q'/r')'}}{|\lambda_k|^{r'(q'/r')'}} < \delta^{r'(q'/r')'} \sum_{k=1}^{N} \frac{1}{|\lambda_k|^{r'(q'/r')'}} +$$

$$+ 2^{r'(q'/r')'} \sum_{k=N+1}^{\infty} \frac{1}{|\lambda_k|^{r'(q'/r')'}} = \frac{\varepsilon^{r'(q'/r')'}}{2} + \frac{\varepsilon^{r'(q'/r')'}}{2} = \varepsilon^{r'(q'/r')'}.$$

Therefore

(3.5) $$\|\Delta^\zeta u\|_r = \left(\sum_{k=1}^{\infty} |a(\lambda_k; \Delta^\zeta u)|^{r'} \right)^{1/r'} < \varepsilon, \qquad \forall u \in \mathcal{B}.$$

As it is well known [13], by the Kronecker's theorem, the system (3.4) has a relatively dense set in R of solutions. Hence, corrisponding to $\varepsilon > 0$ it follows from (3.5), that there exists a relatively dense set in R of common ε-almost periods, in the sense of B^r_{ap}, for all $u \in \mathcal{B}$. This shows that $\mathcal{B}$ is B^r_{ap}-equi-almost periodic.

ii) The proof follows in similar way as in i).

iii) Let $\mathcal{B}$ the open unit ball of $H^{1,q}_{ap}(\Lambda)$, with $q > \beta$. Then, by Sobolev embedding Theorem, $\mathcal{B}$ is a bounded subset of $C^0_{ap}(\Lambda)$. Moreover, we observe that

(3.6) $$|u(x+h) - u(x)| \leq C \|u\|_{H^{1,q}} |h|^\alpha \leq C |h|^\alpha, \qquad u \in \mathcal{B}.$$

Indeed

$$\frac{|u(x+h)-u(x)|}{|h|^{\alpha}} \leqslant 2^{1-\alpha} \sum_{k=1}^{\infty} |\lambda_k|^{\alpha} |a(\lambda_k;u)|, \qquad u \in \mathcal{B},$$

and by Hölder inequality, it follows

$$\frac{|u(x+h)-u(x)|}{|h|^{\alpha}} \leqslant 2^{1-\alpha} \left(\sum_{k=1}^{\infty} |\lambda_k|^{q'} |a(\lambda_k;u)|^{q'} \right)^{1/q'} \left(\sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^{(1-\alpha)q}} \right)^{1/q},$$

for every $u \in \mathcal{B}$. Therefore (3.6) holds whenever

$$\alpha < 1 - \frac{\beta}{q}.$$

Hence

$$\|\tau_h u - u\|_{\infty} \leqslant \varepsilon,$$

if $|h|$ is sufficiently small. This shows that $\mathcal{B}$ is equicontinuous. On the other hand, if $\zeta \in \mathrm{R}$, the following inequality

$$|u(x+\zeta)-u(x)| \leqslant \sum_{k=1}^{\infty} |e^{i\lambda_k \zeta}-1|\,|a(\lambda_k;u)| \leqslant$$

$$\leqslant \left(\sum_{k=1}^{\infty} |\lambda_k|^{q'} |a(\lambda_k;u)|^{q'} \right)^{1/q'} \left(\sum_{k=1}^{\infty} \frac{|e^{i\lambda_k \zeta}-1|^{q}}{|\lambda_k|^{q}} \right)^{1/q} \leqslant$$

$$\leqslant \left(\sum_{k=1}^{\infty} \frac{|e^{i\lambda_k \zeta}-1|^{q}}{|\lambda_k|^{q}} \right)^{1/q}, \qquad u \in \mathcal{B},$$

holds. Thus, it should be clear that $\mathcal{B}$ is also equi-almost periodic in the sense of $C^0_{ap}(\Lambda)$. Hence, $\mathcal{B}$ satisfies the assumptions of the Lyusternik's theorem [13, p. 7], and so it is relatively compact in $C^0_{ap}(\Lambda)$. ■

Note that it is possible to obtain other results of classical type. For example [15, Lemma 1.37, p. 61].

Corollary 3.2: *Let Λ be as in Theorem 3.1 and take $\varepsilon > 0$, $h, k \in \mathrm{N}$ with $h < k$. Then there exists a constant C such that*

$$\|u^{(h)}\|_q \leqslant \varepsilon \|u^{(k)}\|_q + C\|u\|_q,$$

whenever $u \in H^{k,q}_{ap}(\Lambda)$, $q \geqslant 2$.

Proof: Observe, first of all, that if $q \geqslant 2$ then

$$H^{k,q}_{ap}(\Lambda) = H^{k,q}_{ap}(\Lambda) \cap W^{k,q}_{ap}(\Lambda),$$

by Theorem 4.1 in [10].

Suppose now that there exixts $\varepsilon > 0$ and $u_n \in H^{k,q}_{ap}(\Lambda)$, that is $u_n \in W^{k,q}_{ap}(\Lambda)$, $n \in \mathbb{N}$, with $\|u_n\|_{W^{k,q}} = 1$ and

$$\|u^{(h)}\|_q > \varepsilon \|u^{(k)}\|_q + n\|u\|_q \,. \tag{3.7}$$

By Theorem 3.1 we may assume that $\{u_n^\gamma\}$ converges in $B^q_{ap}(\Lambda)$ whatever the index γ, $\gamma < k - 1$, hence that $u_n \to u$ in $W^{k-1,q}_{ap}(\Lambda)$.

Since all norms $\|u^{(h)}\|_q$ are uniformly bounded it follows from (3.7) that $u_n \to 0$ in $B^q_{ap}(\Lambda)$ hence $u = 0$.

But then $\|u^{(h)}\|_q \to 0$ and (3-7) can again be applied to yield

$$\|u^{(k)}\|_q \to 0$$

hence

$$\|u_n\|_{W^{k,q}} \to 0$$

a contradiction. ■

If $1 < q < 2$, because of the restrictions by the Hausdorff-Young theorem, to obtain compact embedding we need more «regularity», as the following result shows.

Theorem 3.3: *Suppose $\Lambda \subset \mathbb{R}\backslash\{0\}$ as in Theorem 3.1. Then*

i) *if* $(1+\eta)q < \beta < (2\eta q)/(2-q)$ *and* $1 < q < 2$ *with* $(2-q)/ /q < \eta < 1$,

$$H^{1+\eta,q}_{ap}(\Lambda) \hookrightarrow\hookrightarrow B^r_{ap}(\Lambda), \qquad \forall r \in \left[q, \frac{\beta q}{\beta - \eta q}\right[;$$

ii) *if* $(1+\eta)q = \beta$, $(2-q)/q < \eta$ *and* $1 < q < 2$,

$$H^{1+\eta,q}_{ap}(\Lambda) \hookrightarrow\hookrightarrow B^r_{ap}(\Lambda), \qquad \forall r \in [q, (1+\eta)q[.$$

Proof: i) If $1 < q < 2$ and

$$(1+\eta)q < \beta < \frac{2\eta q}{2-q} < \frac{2(1+\eta)q}{2-q},$$

then, by Sobolev embedding Theorem, we have

$$H^{1+\eta,q}_{ap}(\Lambda) \hookrightarrow\hookrightarrow B^r_{ap}(\Lambda), \qquad \forall r \in \left[q, \frac{\beta q}{\beta - (1+\eta)q}\right[.$$

On the other hand, if $u \in H_{ap}^{1+\eta,q}(\Lambda)$, we have

$$\left(\sum_{k=1}^{\infty} |a(\lambda_k;u)|^{r'}\right)^{1/r'} = \left(\sum_{k=1}^{\infty} |a(\lambda_k;u)|^{r'} |\lambda_k|^{(1+\eta)r'} \frac{1}{|\lambda_k|^{(1+\eta)r'}}\right)^{1/r'} \leqslant$$

$$\leqslant \left(\sum_{k=1}^{\infty} |a(\lambda_k;u)|^{q'} (1+|\lambda_k|^2)^{(1+\eta)q'/2}\right)^{r'/q'} \left(\sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^{(1+\eta)r'(q'/r')'}}\right)^{1/(q'/r')'},$$

and so

$$H_{ap}^{1+\eta,q}(\Lambda) \hookrightarrow H_{ap}^{0,r}(\Lambda), \qquad \forall r \in \left[2, \frac{\beta q}{\beta-(1+\eta)q}\right[.$$

From the equality (2.6) we obtain

$$|a(\lambda_k; \tau_b u - u)| \leqslant |\lambda_k|\,|b|\,|a(\lambda_k;u)|.$$

Therefore if

$$2 < r < \frac{\beta q}{\beta - \eta q} < \frac{\beta q}{\beta-(1+\eta)q},$$

we have

$$\left(\sum_{k=1}^{\infty} |a(\lambda_k;\tau_b u - u)|^{r'}\right)^{1/r'} \leqslant |b| \left(\sum_{k=1}^{\infty} |a(\lambda_k;u)|^{r'} |\lambda_k|^{r'}\right)^{1/r'} =$$

$$= |b| \left(\sum_{k=1}^{\infty} |a(\lambda_k;u)|^{r'} |\lambda_k|^{r'} |\lambda_k|^{(1+\eta)r'} \frac{1}{|\lambda_k|^{(1+\eta)r'}}\right)^{1/r'} \leqslant$$

$$\leqslant |b| \left(\sum_{k=1}^{\infty} (1+|\lambda_k|^2)^{(1+\eta)q'/2} |a(\lambda_k;u)|^{q'}\right)^{r'/q'} \left(\sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^{\eta r'(q'/r')'}}\right)^{1/(q'/r')'},$$

and by applying Lemma 2.7, it follows that $u \in H_{ap}^{1,r}(\Lambda)$, $\forall r \in [2, \beta q/(\beta-\eta q)[$.

Let $\mathcal{B}$, the open unit ball of $H_{ap}^{1+\eta,q}(\Lambda)$. We shall show that $\bar{\mathcal{B}}$ is a compact subset of $B_{ap}^{r}(\Lambda)$, $\forall r \in [q, \beta q/(\beta-\eta q)[$.

Let $q < 2 < r < q^* < \beta q/(\beta-\eta q)$, and choose $\alpha \in (0, 1]$, as in the proof of i) in Theorem 3.1, so that

$$\frac{1}{r} = \frac{\alpha}{q} + \frac{1-\alpha}{q^*}.$$

Therefore, by Proposition 2.1, it follows that

$$\|\tau_b u - u\|_r \leqslant \|\tau_b u - u\|_q^{\alpha} \|\tau_b u - u\|_{q^*}^{1-\alpha},$$

and, by Hausdorff-Young theorem

$$\|\tau_b u - u\|_{q^*} \leq \left(\sum_{k=1}^{\infty} |a(\lambda_k; \tau_b u - u)|^{q^{*\prime}}\right)^{1/q^{*\prime}} = \|\tau_b u - u\|_{H^{0,q^*}},$$

holds, with $1/q^* + 1/q^{*\prime} = 1$.

Then by applying Lemma 2.7, there exists a positive constant C such that

$$\|\tau_b u - u\|_{H^{0,q^*}} \leq C|b|, \qquad b \in \mathrm{R}.$$

Hence

$$\|\tau_b u - u\|_r \leq (C|b|)^{1-\alpha} \|\tau_b u - u\|_q^{\alpha} \leq (C|b|)^{1-\alpha} (2\|u\|_q)^{\alpha} \leq K|b|^{1-\alpha},$$

holds for a suitable positive constant K. Since $B_{ap}^r \hookrightarrow B_{ap}^s$, if $r > s$, by a simple application of the Hölder inequality, we obtain

$$\|\tau_b u - u\|_s \leq \|\tau_b u - u\|_r \leq \varepsilon, \qquad \forall u \in \mathcal{B},\ \forall s \in [q, r[,$$

provided $|b|$ sufficiently small. This shows that $\mathcal{B}$ is B_{ap}^r-equicontinuous $\forall r \in$ $\in [q, \beta q/(\beta - \eta q)[$. Proceeding further as in the proof of i) in Theorem 3.1, it is easy to verify that $\mathcal{B}$ is also B_{ap}^r-equi-almost periodic.

ii) We observe that if $(1+\eta)q = \beta$, $1 < q < 2$ and $\eta > 0$, then

$$\frac{\eta q r}{r - q} > \beta,$$

with $r \geq 2$, implies $\beta > r$ and $(2-q)/q < \eta$. Now, repeating the argument in the proof of i), the claim follows. ∎

From Theorem 4.1 in [10] and Theorem 3.1 and 3.3 we obtain immediately

Corollary 3.4: *Suppose* $\Lambda \subset \mathrm{R}\backslash\{0\}$ *as in the previous theorem and* $1 < q < 2$. *Then*

i) *if* $(1+\eta)q < \beta < 2\eta q/(2-q)$ *with* $(2-q)/q < \eta < 1$,

$$W_{ap}^{1+\eta,q}(\Lambda) \hookrightarrow\hookrightarrow B_{ap}^r(\Lambda) \qquad \forall r \in \left[q, \frac{\beta q}{\beta - \eta q}\right[;$$

ii) *if* $(1+\eta)q = \beta$, $(2-q)/q < \eta$,

$$W_{ap}^{1+\eta,q}(\Lambda) \hookrightarrow\hookrightarrow B_{ap}^r(\Lambda) \qquad \forall r \in [q, (1+\eta)q[.$$

iii) *if* $q > \beta$,

$$W_{ap}^{1,q}(\Lambda) \hookrightarrow\hookrightarrow C_{ap}^0(\Lambda). \qquad ∎$$

4. - An application

In this section, following a suggestion given by Avantaggiati [3], we shall show how it is possible to solve the problem of finding solutions $y \in H^2_{\varphi}(\mathbb{R})$ of the equation

$$(4.1) \qquad -y''(x) + \gamma y(x) = f[x, y(x)],$$

with $x \in \mathbb{R}$ and $\gamma > 0$, by reducing it to the problem of finding solutions of a particular integral equation, which is similar to the so-called Hammerstein integral equations.

We assume that the following hypothesis hold.

A) The composition operator

$$u \to f[x, u(x)],$$

is continuous and bounded from $B^2_{\varphi}(\mathbb{R})$ in $B^2_{\varphi}(\mathbb{R})$.

B) There exists a discrete and countable semigroup $S \subset \mathbb{R}(+)$ such that

$$\sigma(u) \subset S \Rightarrow \sigma(f[\cdot, u(\cdot)]) \subset S.$$

We remark that condition *A)* can be generalized. However, it is verified if suitable growth conditions on f, like in Krasnoselski [11] and Pankov [14], are imposed.

The following definition can be given

Definition 4.4: Let $S \subset \mathbb{R}(+)$ be a fixed semigroup. A function $f(x, y)$ is said C^0_{φ}-admissible with respect to S, if for any $y \in C^0_{\varphi}(\mathbb{R})$

a) $f[\cdot, u(\cdot)] \in C^0_{\varphi}(\mathbb{R})$.

b) $\sigma(u) \subset S \Rightarrow \sigma(f[\cdot, u(\cdot)]) \subset S$.

On the other hand, the condition

$$(4.2) \qquad \sum_{\lambda \in S} \frac{1}{|\lambda|^{\gamma}} < +\infty, \qquad \forall \gamma > \beta,$$

which appears in the compact embedding theorems, is assumed true.

Observe that, if S is a group with two rationally indipendent generators, η_1 and η_2, we have $S = \{n\eta_1 + m\eta_2\}_{m,n \in \mathbb{Z}}$ and S is dense in $\mathbb{R}$. Therefore, conditions like (4.2) cannot be verified.

About condition *B)*, we note that it is suggested from the fact that (4.1) implies

$$(\gamma + \lambda^2) a(\lambda; y) = a(\lambda; f[\cdot, y(\cdot)]), \qquad \forall \lambda \in \mathbb{R},$$

for the elements in $H^2_{\varphi}(\mathbb{R})$.

The method, that we present, essentially depends on the construction of a fundamental solution G for the differential operator $-y'' + \gamma y$, in relation to the subspaces $B^2_{ap}(\mathcal{S})$.

It is not difficult to establish that the Bohr-Fourier series of G has the following expression [3]

$$G \sim \sum_{\lambda \in \mathcal{S}} \frac{e^{i\lambda x}}{\lambda^2 + \gamma}. \tag{4.3}$$

On the other hand, we observe that for the linear operator

$$b \to G * b, \tag{4.4}$$

defined by means of the convolution

$$(G * b)(x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} G(x - t) b(t)\, dt = \sum_{\lambda \in \mathcal{S}} \frac{a(\lambda; b)}{\lambda^2 + \gamma} e^{i\lambda x}, \tag{4.5}$$

the following result holds.

THEOREM 4.1: *Let $\mathcal{S}$ as above. Then (4.3) is the continuous inverse operator defined from $B^2_{ap}(\mathcal{S})$ in $H^2_{ap}(\mathcal{S})$ of the operator*

$$\mathfrak{A}(D) \colon y \to -y'' + \gamma y, \quad y \in H^2_{ap}(\mathcal{S}).$$

PROOF: It is sufficient to observe that from (4.5) easily follows

$$a(\lambda; \mathfrak{A}(D)(G * b)) = a(\lambda; b), \quad \forall b \in B^2_{ap}(\mathcal{S}),$$

and moreover

$$a(\lambda; G * \mathfrak{A}(D) b) = a(\lambda; b), \quad \forall b \in H^2_{ap}(\mathcal{S}).$$

From the Parseval equality we have

$$\|\mathfrak{A}(D)(G * b)\|_2^2 = \|b\|_2^2 = \sum_{\lambda \in \mathcal{S}} |a(\lambda; b)|^2, \tag{4.6}$$

and so

$$\|G * b\|^2_{H^2} = \sum_{\lambda \in \mathcal{S}} (1 + \lambda^2)^2 |a(\lambda; G * b)|^2 = \tag{4.7}$$

$$= \sum_{\lambda \in \mathcal{S}} \left(\frac{1 + \lambda^2}{\gamma + \lambda^2} \right)^2 |a(\lambda; b)|^2 \leq [C(\gamma)]^2 \|b\|_2^2,$$

where

$$C(\gamma) = \max_{\lambda \in \mathbb{R}} \frac{1+\lambda^2}{\gamma+\lambda^2} = \max\left\{1, \frac{1}{\gamma}\right\}. \quad ■$$

By virtue of Theorem 4.1, we obtain the following

COROLLARY 4.2: *The problem*

$$y \in H^2_{ap}(S) \quad \text{and} \quad -y'' + \gamma y = f[\cdot, y(\cdot)], \tag{4.8}$$

is equivalent to the problem

$$y \in H^2_{ap}(S) \quad \text{and} \quad y = G * f[\cdot, y(\cdot)]. \quad ■$$

Indeed, it is sufficient to consider the integral equation

$$y(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(x-t) f[t, y(t)]\, dt\,, \tag{4.9}$$

and to find $y \in B^2_{ap}(S)$.

Actually, when $y \in B^2_{ap}(S)$ exists, from hypothesis *A*) and from (4.7) then $y \in H^2_{ap}(S)$ directly follows.

A first result, which permits to apply Theorem 3.1, is the following

THEOREM 4.3: *Suppose the condition A) holds. If the embedding of $H^2_{ap}(S)$ in $B^2_{ap}(S)$ is compact, then there exists $\gamma_0 \in \mathbb{R}_+$ such that for any $\gamma > \gamma_0$ the problem (4.7) has at least a solution.*

PROOF: It is sufficient to observe that in these hypothesis the operator $y \to \to G * f[\cdot, y(\cdot)]$ is compact and therefore the Schauder's fixed point Theorem can be easily applied. ■

We mention that, if condition in Definition 4.4 is specialized, the previous result can be used to obtain existence theorems for periodic and quasi periodic solutions of the equation (4.1). However, to this problem, we shall devote a further note.

Acknowledgements. The authors thank Prof. Antonio Avantaggiati for his precious and helpful comments and suggestions.

REFERENCES

[1] L. AMERIO - G. PROUSE, *Almost Periodic Functions and Functional Equations*, Van Nostrand Reinhold Co., New York (1971).

[2] A. Avantaggiati, *Il teorema di Hausdorff-Young negli spazi $B^{\sharp}_{ap}$ di funzioni quasi periodiche secondo Besicovitch con applicazioni*, Conf. Sem. Mat. Un. Bari (1993).

[3] A. Avantaggiati, *Teoremi di immersione negli spazi di Sobolev-Besicovitch e applicazion*, preprint Dept. Me. Mo. Mat. Rome (1995).

[4] A. Avantaggiati - G. Bruno - R. Iannacci, *The Hausdorff-Young theorem for almost periodic functions and applications*, J. Nonlin. An. Appl., 25 (1995), 61-87.

[5] A. M. Bersani, *The Hausdorff-Young theorem for Besicovitch spaces of vector-valued almost periodic functions*, Rend. Mat., VII, 15 (1995), 5-24.

[6] H. Brezis, *Analyse Fonctionnelle. Théorie et applications*, Masson, Paris (1983).

[7] G. Bruno - R. Grande, *A compactness criterion in $B^{\sharp}_{ap}$ spaces*, preprint Dept. Me. Mo. Mat. Rome (1996).

[8] C. Corduneanu, *Almost Periodic Functions*, Interscience Publ., New York (1968).

[9] G. Dell'Acqua - P. Santucci, *Embedding theorems for Sobolev-Besicovitch spaces $W^{m,1}_{ap}$*, preprint Dept. Me. Mo. Mat. Rome (1996).

[10] R. Iannacci - A. M. Bersani - G. Dell'Acqua - P. Santucci, *Embedding theorems for Sobolev-Besicovitch spaces of almost periodic functions*, preprint Dept. Me. Mo. Mat. Rome (1996).

[11] M. A. Krasnoselski, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon, Oxford (1965).

[12] B. M. Levitan, *Almost Periodic Functions*, Gostekhizdat, Moscow (1953) (in Russian).

[13] B. M. Levitan - V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, London (1982).

[14] A. A. Pankov, *Bounded and Almost Periodic Solutions of Non Linear Operator Differential Equations*, Kluwer Academic Publisher, London (1990).

[15] G. M. Troianiello, *Elliptic Differential Equations and Obstacle Problems*, Plenum Press, New York (1987).