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ANNA CANALE - MARIA LONGOBARDI - GIULIANA MANZO (*)

Second Order Elliptic Equations with Discontinuous Coefficients in Unbounded domains (**)

SUMMARY. — This paper is concerned with the Dirichlet problem for linear second order elliptic partial differential equations with discontinuous coefficients in unbounded domains in R^n. Some existence and uniqueness results have been proved.

Equazioni ellittiche del secondo ordine a coefficienti discontinui in aperti non limitati

SOMMARIO. — In questo lavoro ci occupiamo dello studio del problema di Dirichlet per le equazioni differenziali lineari ellittiche del secondo ordine a coefficienti discontinui in aperti non limitati di R^n. Stabiliamo alcuni teoremi di esistenza ed unicità.

INTRODUCTION

Let Ω be an unbounded and sufficiently regular open subset in R^n, $n > 2$ (see section 4).

In a recent paper M. Transirico and M. Troisi (see [TT]$_1$) have studied the Dirichlet problem

$$u \in W^2(\Omega) \cap W_0^1(\Omega), \quad Lu = f, \quad f \in L^2(\Omega), \tag{1}$$

where L is the linear second order uniformly elliptic differential operator

$$Lu = -\sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} + \sum_{i=1}^{n} a_i\, u_{x_i} + au ,$$

(*) Indirizzo degli Autori: DIIMA, Università di Salerno, Sede distaccata, Via S. Allende, 84081 Baronissi (Salerno).

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with the following *basic* hypotheses on the coefficients:

$$a_{ij} = a_{ji} \in L^\infty(\Omega), \quad i, j = 1, \ldots, n, \tag{2}$$

$$(a_{ij})_{x_k} \in L^n_{loc}(\bar{\Omega}), \quad \sup_{x \in \Omega} \|(a_{ij})_{x_k}\|_{L^n(\Omega \cap B(x,1))} < +\infty, \quad i, j, k = 1, \ldots, n, \tag{3}$$

$$a_i \in L^n_{loc}(\bar{\Omega}), \quad \sup_{x \in \Omega} \|a_i\|_{L^n(\Omega \cap B(x,1))} < +\infty, \quad i = 1, \ldots, n, \tag{4}$$

$$a \in L^t_{loc}(\bar{\Omega}), \quad \sup_{x \in \Omega} \|a\|_{L^t(\Omega \cap B(x,1))} < +\infty, \tag{5}$$

where $B(x, 1) = \{y \in R^n : |y - x| < 1\}$ and $t = 2$ if $n = 3$, $t > 2$ if $n = 4$, $t = n/2$ if $n > 4$.

The aim of this paper is to extend the results in [TT], when the coefficients a_{ij} and a_i verify a weaker condition than (3) and (4).

We consider the Morrey type spaces $M^{p,\alpha}(\Omega)$, $\tilde{M}^{p,\alpha}(\Omega)$, $M_0^{p,\alpha}(\Omega)$, $1 \leq p < +\infty$ and $0 \leq \alpha < n$, introduced and studied by M. Transirico - M. Troisi - A. Vitolo [TTV] (see also F. Chiarenza - M. Franciosi [CF]).

We recall that $M^{p,\alpha}(\Omega)$ is the space of functions $g \in L^p_{loc}(\bar{\Omega})$ such that

$$\|g\|_{M^{p,\alpha}(\Omega)} = \sup_{\substack{x \in \Omega \\ 0 < \tau \leq 1}} \tau^{-\alpha/p} \|g\|_{L^p(\Omega \cap B(x,\tau))} < +\infty, \tag{6}$$

equipped with the norm defined in (6); $\tilde{M}^{p,\alpha}(\Omega)$ is the closure of $L^\infty(\Omega)$ in $M^{p,\alpha}(\Omega)$; $M_0^{p,\alpha}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $M^{p,\alpha}(\Omega)$.

We study the problem when the coefficients of the operator L satisfy (2) and the conditions

$$(a_{ij})_{x_k},\ a_i \in \tilde{M}^{s,n-s}(\Omega), \qquad a \in \tilde{M}^{t,0}(\Omega),$$

for some $s \in]2, n]$ and for t defined above.

We prove that the problem

$$u \in W^2(\Omega) \cap \overset{\circ}{W}{}^1(\Omega), \quad Lu + \lambda u = f, \quad f \in L^2(\Omega) \tag{1}$$

is uniquely solvable for λ large enough.

Furthermore, if we have

$$(a_{ij})_{x_k},\ a_i \in M_0^{s,n-s}(\Omega),$$

$$a = a' + a'', \quad a' \in M_0^{t,0}(\Omega), \quad \operatorname*{ess\,inf}_{\Omega} a'' > 0,$$

then (1) is a zero index problem and it is uniquely solvable for $a' = 0$.

1. - Notations

Let E be a Lebesgue measurable subset in R^n. We denote by $\Sigma(E)$ the σ-algebra of Lebesgue measurable subsets of E.

For any $A \in \Sigma(E)$, $|A|$ is the Lebesgue measure of A, $\mathcal{D}(A)$ is the class of restrictions to A of functions $\zeta \in C_0^\infty(R^n)$ such that $\operatorname{supp} \zeta \subset A$, $L^p_{loc}(A)$ is the class of functions $g: A \to R$ such that $\zeta g \in L^p(A)$ for all $\zeta \in \mathcal{D}(A)$, χ_A is the characteristic function of A.

Furthermore we set

$$u_x = \left(\sum_{i=1}^n u_{x_i}^2 \right)^{1/2}, \qquad u_{xx} = \left(\sum_{i,j=1}^n u_{x_i x_j}^2 \right)^{1/2},$$

$$|u|_{p,E} = \|u\|_{L^p(E)}, \quad B(x,r) = \{y \in R^n : |y-x| < r\}, \qquad B_r = B(0,r).$$

In the following we will use the lemma (similar to Lemma 2 of M. Transirico - M. Troisi [TT]$_2$):

Lemma 1.1: *Let $p \in [1, +\infty[$, $r \in R_+$ and $E \in \Sigma(R^n)$. We have $f \in L^p(E)$ if and only if $f \in L^p_{loc}(\bar{E})$ and the application $(x \in R^n \to |f|_{p, E \cap B(x,r)}) \in L^p(R^n)$.*

Furthermore the equality

$$\int_E |f|^p \, dx = \frac{1}{|B_r|} \int_{R^n} |f|^p_{p, E \cap B(x,r)} \, dx. \tag{1.1}$$

holds.

Proof: If we consider the function

$$\psi_r : (x,y) \in R^n \times R^n \to \begin{cases} 1 & \text{if } |x-y| < r, \\ 0 & \text{if } |x-y| \geq r, \end{cases}$$

the result follows by observing that

$$\int_{R^n} |f|^p_{p, E \cap B(x,r)} \, dx = \int_{R^n} dx \int_E \psi_r(x,y) \, |f(y)|^p \, dy = \int_E |f(y)|^p \, dy \int_{R^n} \psi_r(x,y) \, dx. \qquad \blacksquare$$

Let Ω be an unbounded open subset in R^n.

For any $x \in \Omega$ and $t \in R_+$ we set:

$$\Omega(x,t) = \Omega \cap B(x,t).$$

We denote by $W^k(\Omega)$, $k \in N$, the space of real distributions u on Ω such that $D^\alpha u \in L^2(\Omega)$, with $|\alpha| \leq k$, equipped with the norm

$$\|u\|_{W^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^2 \, dx \right)^{1/2}$$

and by $W_0^k(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^k(\Omega)$. In order to unify the notations often, in the following, we will denote by the symbol $U^k(\Omega)$ the space $W^k(\Omega)$ in the case Ω has the cone property and the space $W_0^k(\Omega)$ in the case Ω doesn't have.

2. - The spaces $M^{p,\alpha}(\Omega)$, $\tilde{M}^{p,\alpha}(\Omega)$, $M_0^{p,\alpha}(\Omega)$.

If $1 \leq p < +\infty$, $0 \leq \alpha < n$ and $t \in R_+$, we call $M^{p,\alpha}(\Omega, t)$ the space of the functions $g \in L^p_{loc}(\bar{\Omega})$ such that:

$$\|g\|_{M^{p,\alpha}(\Omega,t)} = \sup_{\substack{x \in \Omega \\ 0 < \tau \leq t}} \tau^{-\alpha/p} |g|_{p,\Omega(x,\tau)} < +\infty, \tag{2.1}$$

equipped with the norm defined in (2.1).

Let d be a fixed positive number. We put

$$M^{p,\alpha}(\Omega) = M^{p,\alpha}(\Omega, d).$$

The spaces $M^{p,\alpha}(\Omega, t)$ and $M^{p,\alpha}(\Omega)$, when Ω is an unbounded set, have been introduced and studied in [TTV]. From the results in [TTV] it follows that the definition of $M^{p,\alpha}(\Omega)$ is independent of d and that, fixed $d_1, d_2 \in R_+$, the norms in $M^{p,\alpha}(\Omega, d_1)$ and in $M^{p,\alpha}(\Omega, d_2)$ are equivalent.

Furthermore we have the imbedding:

$$M^{p_0,\alpha_0}(\Omega) \hookrightarrow M^{p,\alpha}(\Omega), \quad p \leq p_0, \quad \frac{\alpha - n}{p} \leq \frac{\alpha_0 - n}{p_0}, \tag{2.2}$$

which implies in particular that:

$$L^\infty(\Omega) \hookrightarrow M^{p,\alpha}(\Omega). \tag{2.3}$$

We denote

by $\tilde{M}^{p,\alpha}(\Omega)$ the closure of $L^\infty(\Omega)$ in $M^{p,\alpha}(\Omega)$;

by $M_0^{p,\alpha}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $M^{p,\alpha}(\Omega)$.

From the results in [TTV] we have the following characterizations of the spaces $\tilde{M}^{p,\alpha}(\Omega)$ and $M_0^{p,\alpha}(\Omega)$:

$\tilde{M}^{p,\alpha}(\Omega)$ is the subspace of $M^{p,\alpha}(\Omega)$ of the functions $g \in M^{p,\alpha}(\Omega)$ such that:

$$\forall \varepsilon \in R_+ \ \exists \delta_\varepsilon \in R_+ \ni' \Big(E \in \Sigma(\Omega), \sup_{x \in \Omega} |E \cap B(x, d)| \leq \delta_\varepsilon \Rightarrow \|g\chi_E\|_{M^{p,\alpha}(\Omega)} \leq \varepsilon\Big), \tag{2.4}$$

$M_0^{p,\alpha}(\Omega)$ is the subspace of $M^{p,\alpha}(\Omega)$ of the functions $g \in M^{p,\alpha}(\Omega)$ such that:

(2.5) $\quad \forall \varepsilon \in R_+ \ \exists h_\varepsilon, k_\varepsilon \in R_+ \ni' (E \in \Sigma(\Omega),\ |E \cap B(0, k_\varepsilon)| \leq h_\varepsilon \Rightarrow \|g\chi_E\|_{M^{p,\alpha}(\Omega)} \leq \varepsilon)$.

3. - Some known results

Let us set:

(3.1) $\quad M^p(\Omega) = M^{p,0}(\Omega), \qquad \tilde{M}^p(\Omega) = \tilde{M}^{p,0}(\Omega), \qquad M_0^p(\Omega) = M_0^{p,0}(\Omega)$.

It is known that (see, *e.g.*, [TTV]):

Lemma 3.1: *We have:*

(3.2) $$M^{p_0,\alpha_0}(\Omega) \subset \tilde{M}^{p,\alpha}(\Omega), \qquad p < p_0, \qquad \frac{\alpha - n}{p} < \frac{\alpha_0 - n}{p_0};$$

(3.3) $$M_0^{p,\alpha}(\Omega) = \tilde{M}^{p,\alpha}(\Omega) \cap M_0^p(\Omega).$$

Furthermore $g \in \tilde{M}^{p,\alpha}(\Omega)$ *if and only if* $g \in \tilde{M}^p(\Omega)$ *and*

(3.4) $$\lim_{t \to 0} \|g\|_{M^{p,\alpha}(\Omega, t)} = 0.$$

We assigne, for any $r \in R_+$, a function $\zeta_r \in C_0^\infty(\Omega)$ such that:

$$0 \leq \zeta_r \leq 1, \qquad \zeta_r(x) = 1 \quad \forall x \in B(0, r), \qquad \operatorname{supp} \zeta_r \subset B(0, 2r).$$

From section 2 in [TTV] we deduce

Lemma 3.2: *The following propositions are equivalent:*

1) $g \in M_0^{p,\alpha}(\Omega)$;

2) $g \in M^{p,\alpha}(\Omega)$ *and satisfies the relations*

(3.5) $$\lim_{r \to +\infty} \|(1 - \zeta_r) g\|_{M^{p,\alpha}(\Omega)} = 0, \qquad \lim_{t \to 0} \|\psi g\|_{M^{p,\alpha}(\Omega, t)} = 0 \qquad \forall \psi \in C_0^\infty(\bar{\Omega});$$

3) $g \in M^{p,\alpha}(\Omega)$ *and satisfies* (3.4) *together with*

(3.6) $$\lim_{r \to +\infty} \|(1 - \zeta_r) g\|_{M^{p,\alpha}(\Omega)} = 0;$$

4) $g \in M^{p,\alpha}(\Omega)$ *and satisfies* (3.4) *together with*

(3.7) $$\lim_{|x| \to +\infty} |g|_{p, \Omega(x)} = 0.$$

From section 3 in [TTV] we have the following

LEMMA 3.3: *If $n > 2$ and $g \in M^{s,n-s}(\Omega)$, with $s \in]2, n]$, then for any $u \in U^1(\Omega)$ we have $gu \in L^2(\Omega)$.*

Furthermore the operator

$$u \in U^1(\Omega) \to gu \in L^2(\Omega) \tag{3.8}$$

is bounded.

If $g \in \tilde{M}^{s,n-s}(\Omega)$, then there exists $c(\varepsilon) \in R_+$ such that:

$$|gu|_{2,\Omega} \leq \varepsilon |u_x|_{2,\Omega} + c(\varepsilon)\,|u|_{2,\Omega} \qquad \forall u \in U^1(\Omega). \tag{3.9}$$

If $g \in M_0^{s,n-s}(\Omega)$, then there exist $c(\varepsilon) \in R_+$ and a bounded open subset $\Omega(\varepsilon)$ of Ω such that:

$$|gu|_{2,\Omega} \leq \varepsilon \|u\|_{W^1(\Omega)} + c(\varepsilon)\,|u|_{2,\Omega(\varepsilon)} \qquad \forall u \in U^1(\Omega). \tag{3.10}$$

Furthermore the operator (3.8) is compact.

As a consequence of the results of M. Transirico - M. Troisi [TT]$_1$ and A. V. Glushak - M. Transirico - M. Troisi [GTT] we have

LEMMA 3.4: *Let $k \in N$ and $p \in [2, +\infty[$ be such that:*

$$p = 2 \quad \text{if} \quad n < 2k\,, \qquad p > 2 \quad \text{if} \quad n = 2k\,, \qquad p = \frac{n}{k} \quad \text{if} \quad n > 2k\,. \tag{3.11}$$

If $g \in M^p(\Omega)$, then for any $u \in U^k(\Omega)$ we have $gu \in L^2(\Omega)$.

Furthermore the operator:

$$u \in U^k(\Omega) \to gu \in L^2(\Omega) \tag{3.12}$$

is bounded.

If $g \in \tilde{M}^p(\Omega)$, then there exists $c(\varepsilon) \in R_+$ such that:

$$|gu|_{2,\Omega} \leq \varepsilon \sum_{|\alpha|=k} |D^\alpha u|_{2,\Omega} + c(\varepsilon)\,|u|_{2,\Omega} \qquad \forall u \in U^k(\Omega). \tag{3.13}$$

If $g \in M_0^p(\Omega)$, then there exist $c(\varepsilon) \in R_+$ and a bounded open subset $\Omega(\varepsilon)$ of Ω such that:

$$|gu|_{2,\Omega} \leq \varepsilon \|u\|_{W^k(\Omega)} + c(\varepsilon)\,|u|_{2,\Omega(\varepsilon)} \qquad \forall u \in U^k(\Omega). \tag{3.14}$$

Furthermore the operator (3.12) is compact.

4. - HYPOTHESES

We set:

$$B = B(0, 1), \qquad B_+ = \{x \in B : x_n > 0\}, \qquad B_0 = \{x \in B : x_n = 0\}$$

and, for any $a \in R_+$,

$$\Omega_a = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < a\}.$$

Let us suppose $n \geq 3$ and that the following condition holds:

i_1) there exist a number $\delta \in R_+$, an open covering $\{U_i\}_{i \in I}$ of $\partial\Omega$ and, for any $i \in I$, a diffeomorphism of class C^2, $\psi_i : \bar{U}_i \to \bar{B}$, such that:

1) $\psi_i(U_i \cap \Omega) = B_+$, $\psi_i(U_i \cap \partial\Omega) = B_0$;

2) the components of ψ_i and ψ_i^{-1} and their weak derivatives are bounded by a constant independent of i;

3) $\forall x \in \Omega$, there exists $i \in I$ such that $B(x, \delta) \subset U_i$ and $B(x, \delta) \subset \Omega$ for any $x \in \Omega \setminus \Omega_\delta$.

REMARK 4.1: It's easy to prove that i_1) holds when Ω has the uniform C^2-regularity property defined in section 4.6 of R. A. Adams [A].

REMARK 4.2: The condition i_1) implies that there exists a number $\rho \in R_+$ such that, for any $x \in R^n$, $B(x, \rho) \cap \partial\Omega = \emptyset$ or $B(x, \rho) \cap \partial\Omega \neq \emptyset$ and $B(x, \rho) \subset U_i$ for some $i \in I$.

Let us give in Ω the linear second order differential operator:

$$Lu = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + au, \tag{4.1}$$

and suppose that

i_2) the real coefficients satisfy the following conditions:

$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq \nu |\xi|^2 \quad \text{a.e. in } \Omega, \quad \forall \xi \in R^n, \quad \nu \in R_+; \tag{4.2}$$

$$a_{ij} = a_{ji} \in L^\infty(\Omega), \quad i, j = 1, \ldots, n; \tag{4.3}$$

$$(a_{ij})_{x_k}, \; a_i \in \tilde{M}^{s, n-s}(\Omega) \quad \text{for some } s \in]2, n], \quad i, j, k = 1, \ldots, n; \tag{4.4}$$

$$a \in \tilde{M}^t(\Omega), \tag{4.5}$$

where

$$t = 2 \;\text{ if }\; n = 3, \qquad t > 2 \;\text{ if }\; n = 4, \qquad t = \frac{n}{2} \;\text{ if }\; n > 4.$$

Let us set

$$b = \sum_{i,j=1}^{n} (a_{ij})_x + \sum_{i=1}^{n} |a_i|. \tag{4.6}$$

We observe that if (4.4), (4.5) hold and $u \in W^2(\Omega)$, from Lemma 3.3 and Lemma 3.4 it follows that:

$$|bu_x|_{2,\Omega} \leq \varepsilon |u_{xx}|_{2,\Omega} + c_1(\varepsilon)\, |u|_{2,\Omega}, \tag{4.7}$$

$$|au|_{2,\Omega} \leq \varepsilon |u_{xx}|_{2,\Omega} + c_2(\varepsilon)\, |u|_{2,\Omega}, \tag{4.8}$$

where the constants $c_1(\varepsilon)$ and $c_2(\varepsilon)$ are independent of u.

5. - Apriori bounds

Let $\beta \colon \Omega \to R_+$ be a function such that:

$$\beta \in \tilde{M}^{t}(\Omega), \qquad \exists \gamma \in \tilde{M}^{t,n-t}(\Omega) \ni' \beta_x \leq \beta\gamma. \tag{5.1}$$

For example:

$$\beta = 1 \quad \text{or} \quad \beta(x) = \frac{1}{(1+|x|^2)^{\tau}}, \qquad \tau \in R_+.$$

We denote by $W^2_{loc}(\bar{\Omega})$ (resp. $\mathring{W}^1_{loc}(\bar{\Omega})$) the space of the functions $u \colon \Omega \to R$ such that $\zeta u \in W^2(\Omega)$ (resp. $\zeta u \in W^1_0(\Omega)$) for any $\zeta \in \mathcal{D}(\bar{\Omega})$.

Theorem 5.1: *Let us suppose that i_1) and i_2) are satisfied. Therefore for any function $u \colon \Omega \to R$ such that*

$$u \in W^2_{loc}(\bar{\Omega}) \cap \mathring{W}^1_{loc}(\bar{\Omega}) \cap L^2(\Omega), \qquad Lu \in L^2(\Omega), \tag{5.2}$$

we have $u \in W^2(\Omega)$. Moreover, for any $\lambda_1 \in R$, it holds

$$|u_{xx}|_{2,\Omega} \leq c\,(|Lu + \lambda\beta u|_{2,\Omega} + |u|_{2,\Omega}) \qquad \forall \lambda \in [\lambda_1, +\infty[, \tag{5.3}$$

where c is a constant independent of u and λ.

Proof: We consider $r, r' \in R_+$, with $r < r' < 1$, and a function $\varphi \in C^\infty_0(R^n)$ such that (see, e.g., P. Di Gironimo - M. Transirico [DT]):

$$\varphi|_{B_r} = 1, \qquad \operatorname{supp} \varphi \subset B_{r'}, \qquad \sup_{R^n} |\partial^\alpha \varphi| \leq c_\alpha (r' - r)^{-|\alpha|} \qquad \forall \alpha \in N^n_0,$$

where c_α is a constant depending only on α.

Moreover let us fix $x \in \Omega$ and set

$$\psi = \psi^x \colon y \in R^n \to \varphi\left(\frac{x-y}{\rho}\right),$$

where ρ is the number defined in the Remark 4.2.

We observe that, if $u \colon \Omega \to R$ satisfies (5.2), the function $v = \psi u$ verifies the hypotheses of theorem 7.1 where $U = B(x, \rho)$ (see section 7); therefore, for any $\lambda_1 \in R$, we have

(5.4) $$\|\psi u\|_{W^2(\Omega)} \leq c_1(|L\psi u + \lambda\beta\psi u|_{2,\Omega} + |\psi u|_{2,\Omega}) \quad \forall\lambda \in [\lambda_1, +\infty[.$$

In this proof the constants $c_1, \ldots, c_8$ are independent of u, x, r, r', λ.
Let us set:

$$\chi = \left(\sum_{i=1}^{n} \psi_{y_i}^2\right)^{1/2}, \quad \Omega_t(x) = \Omega(x,t) \quad \text{for } t \in R_+ .$$

From the relation

$$L(\psi u) = \psi Lu - \sum_{i,j=1}^{n} a_{ij}(\psi_{y_j} u_{y_i} + \psi_{y_i} u_{y_j} + \psi_{y_i y_j} u) + u \sum_{i=1}^{n} a_i \psi_{y_i}$$

it follows:

$$|L(\psi u) + \lambda\beta\psi u|_{2,\Omega} \leq c_2(|Lu + \lambda\beta u|_{2,\Omega_{r'}(x)} + (r'-r)^{-2}|u|_{2,\Omega_{r'}(x)} + \\ + \sum_{i=1}^{n} |a_i \chi u|_{2,\Omega} + |(\chi u)_y|_{2,\Omega}).$$

Lemma 3.3 gives

$$|a_i \chi u|_{2,\Omega} \leq c_3 \|\chi u\|_{W^1(\Omega)}$$

and, from known results, it follows that:

$$\|\chi u\|_{W^1(\Omega)} \leq c_4(|(\chi u)_{yy}|_{2,\Omega}^{1/2} |\chi u|_{2,\Omega}^{1/2} + |\chi u|_{2,\Omega}) \leq \\ \leq c_5(r'-r)^{-2}(\|u\|_{W^2(\Omega_{r'}(x))}^{1/2} |u|_{2,\Omega_{r'}(x)}^{1/2} + |u|_{2,\Omega_{r'}(x)}).$$

From previous inequalities we get

(5.5) $$(r'-r)^2\|u\|_{W^2(\Omega_r(x))} \leq \\ \leq c_6(|Lu + \lambda\beta u|_{2,\Omega_{r'}(x)} + |u|_{2,\Omega_{r'}(x)} + \|u\|_{W^2(\Omega_{r'}(x))}^{1/2} |u|_{2,\Omega_{r'}(x)}^{1/2}).$$

By a lemma of monotonicity of C. Miranda (see Lemma 3.1 in $[M]_3$) and from (5.5) we deduce the following result:

(5.6) $$\|u\|_{W^2(\Omega(x,\rho/2))} \leq c_7(|Lu + \lambda\beta u|_{2,\Omega(x,\rho)} + |u|_{2,\Omega(x,\rho)}) \quad \forall\lambda \in [\lambda_1, +\infty[.$$

By using inequality (5.6) with $\lambda = 0$ and Lemma 1.1 we have $u \in W^2(\Omega)$. It follows that $\beta u \in L^2(\Omega)$ and, then, by (5.5) and Lemma 1.1 we deduce (5.3). ■

Corollary 5.1: *If i_1), i_2) and the condition* $\operatorname{ess\,inf}_{\Omega} \beta > 0$ *hold, then there exist* λ_0, $c \in R_+$ *such that:*

(5.7) $$\|u\|_{W^2(\Omega)} \leq c|Lu + \lambda\beta u|_{2,\Omega} \quad \forall u \in W^2(\Omega) \cap W_0^1(\Omega), \quad \forall\lambda \in [\lambda_0, +\infty[.$$

Proof: The result can be deduced from Theorem 5.1 using arguments similar to those used by G. Viola [V] in order to get Theorem 6 from Proposition 5. ■

In the following corollary we suppose that there exists $\lambda_1 \in R$ satisfying

$$\operatorname*{ess\,inf}_{\Omega} (\lambda_1 \beta + a) > 0 . \tag{5.8}$$

Corollary 5.2: *If i_1), i_2), (5.8) and the condition*

$$a_i \,,\ (a_{ij})_x \in M_0^{t,n-t}(\Omega) , \tag{5.9}$$

hold, then there exists $c \in R_+$ such that

$$\|u\|_{W^2(\Omega)} \leq c|Lu + \lambda\beta u|_{2,\Omega} \quad \forall u \in W^2(\Omega) \cap W_0^1(\Omega), \quad \forall \lambda \in [\lambda_1, +\infty[. \tag{5.10}$$

Proof: Using Theorem 4.2 in [TTV], the result is a consequence of Theorem 5.1. ■

Now we consider the condition

$$a = a' + a'', \quad a' \in M_0^t(\Omega), \quad \operatorname*{ess\,inf}_{\Omega} a'' > 0 . \tag{5.11}$$

Corollary 5.3: *If i_1), i_2), (5.9), (5.11) hold and if λ_2 is a real number satisfying the condition*

$$\operatorname*{ess\,inf}_{\Omega} (\lambda_2 \beta + a'') > 0 ,$$

then there exist $c \in R_+$ and a bounded open subset Ω_0 in Ω such that:

$$\|u\|_{W^2(\Omega)} \leq c(|Lu + \lambda\beta u|_{2,\Omega} + |u|_{2,\Omega_0}) \qquad \forall u \in W^2(\Omega) \cap W_0^1(\Omega), \quad \forall \lambda \in [\lambda_2, +\infty[. \tag{5.12}$$

Proof: From corollary 5.2 it follows that

$$\|u\|_{W^2(\Omega)} \leq c_1 \left| \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + a'' u + \lambda\beta u \right|_{2,\Omega} \leq c_1 (|Lu + \lambda\beta u|_{2,\Omega} + |a' u|_{2,\Omega}),$$

$$c_1 \in R_+, \quad \forall u \in W^2(\Omega) \cap W_0^1(\Omega), \quad \forall \lambda \in [\lambda_2, +\infty[.$$

From the last relation and Lemma 3.4 we deduce (5.12). ■

Corollary 5.4: *If the hypotheses of Corollary 5.3 are satisfied and if $\beta^{-1} \in L^\infty_{loc}(\Omega)$, then there exist $c, \lambda_0 \in R_+$ such that*

$$\|u\|_{W^2(\Omega)} \leq c|Lu + \lambda\beta u|_{2,\Omega} \quad \forall u \in W^2(\Omega) \cap W_0^1(\Omega), \quad \forall \lambda \in [\lambda_0, +\infty[. \tag{5.13}$$

Proof: The result can be deduced from Corollary 5.3 in the same way Corollary 5.1 follows from Theorem 5.1. ■

6. - Existence theorems

Theorem 6.1: *Using either the hypotheses i_1), i_2) and the condition $\operatorname{ess\,inf}_{\Omega} \beta > 0$ or the hypotheses i_1), i_2), (5.9), (5.11) and the condition $\beta^{-1} \in L^\infty_{loc}(\bar{\Omega})$, the problem*

(6.1) $$u \in W^2(\Omega) \cap W^1_0(\Omega), \quad Lu + \lambda\beta u = f, \quad f \in L^2(\Omega)$$

is uniquely solvable for λ large enough.

Proof: We denote by A the operator $-\Delta$ if i_1), i_2) and $\operatorname{ess\,inf}_{\Omega} \beta > 0$ hold, the operator $-\Delta + a''$ in the other case.

It's known (see, e.g., Theorem 5.4 in [TT$_1$]) that, for any $\lambda \in R_+$, the problem

$$u \in W^2(\Omega) \cap W^1_0(\Omega), \quad Au + \lambda\beta u = f, \quad f \in L^2(\Omega)$$

is uniquely solvable.

For any $\tau \in [0, 1]$ we set

$$L_\tau = (1 - \tau)A + \tau(L).$$

Using the Corollary 5.1 when i_1), i_2) and the condition $\operatorname{ess\,inf}_{\Omega} \beta > 0$ are verified, the corollary 5.4 in the other case, we have that there exist c_0, $\lambda_0 \in R_+$ such that

$$\|u\|_{W^2(\Omega)} \leq c_0 \|L_\tau u + \lambda\beta u\|_{2,\Omega}$$

$$\forall u \in W^2(\Omega) \cap W^1_0(\Omega), \quad \forall \lambda \in [\lambda_0, +\infty[, \quad \forall \tau \in [0, 1].$$

Therefore observing that:

$$L_\tau u + \lambda\beta u = (1 - \tau)(Au + \lambda\beta u) + \tau(Lu + \lambda\beta u),$$

we obtain the result by means of the method of continuity. ■

Theorem 6.2: *If i_1), i_2), (5.9) and (5.11) hold, then the problem*

(6.2) $$u \in W^2(\Omega) \cap W^1_0(\Omega), \quad Lu = f, \quad f \in L^2(\Omega)$$

is a zero index problem.

Furthermore if the condition

(6.3) $$\operatorname*{ess\,inf}_{\Omega} a > 0,$$

holds, then the problem (6.2) is uniquely solvable.

Proof: We consider the function

$$\beta : x \in \Omega \to (1 + |x|^2)^{-1}.$$

We observe that β so defined verifies (5.1) and the condition $\beta^{-1} \in L^{\infty}_{loc}(\bar{\Omega})$. It follows, from Theorem 6.1, that the problem (6.1) is uniquely solvable for λ large enough. On the other hand, since $\beta \in M_0^r(\Omega)$, from Lemma 3.4 we have that the operator

$$u \in W^2(\Omega) \to \beta u \in L^2(\Omega)$$

is compact.

From known results we deduce that (6.2) is a zero index problem. The uniqueness of the solution follows from Corollary 5.2. ■

7. - Appendix

In this section we derive an apriori local estimate (see (7.1)) which implies (5.3).

Using notations given in section 4, let us fix an open set U in R^n such that $U \subset\subset \Omega$ or $U \cap \partial\Omega \neq \emptyset$ and $U \subset U_i$ for some $i \in I$.

If $U \cap \partial\Omega \neq \emptyset$, let us fix a bounded domain Ω^* of class C^2 in R^n, an open set V in R^n and a diffeomorphism $g: \bar{U} \to \bar{V}$ of class C^2 such that:

$$g(U \cap \Omega) = V \cap \Omega^*, \quad g(U \cap \partial\Omega) = V \cap \partial\Omega^*.$$

Let us set

$$L_0 u = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j}.$$

Theorem 7.1: *If the conditions i_1), i_2) are satisfied and λ_1 is a real number, then there exists a constant $c \in R_+$ such that*

$$|v_{xx}|_{2,\Omega} \leq c(|Lv + \lambda\beta v|_{2,\Omega} + |v|_{2,\Omega}) \tag{7.1}$$

for any $\lambda \in [\lambda_1, +\infty[$ and for any function $v: \Omega \to R$ with the conditions

$$v \in W^2(\Omega) \cap \overset{\circ}{W}{}^1_2(\Omega), \qquad \operatorname{supp} v \subset U. \tag{7.2}$$

We give the following lemmas before the proof of Theorem 7.1.

Lemma 7.1: *Any function v satisfying (7.2) is a W^2-limit of a sequence of functions v_n such that*

$$v_n \in C^2(\bar{\Omega}), \quad v_n|_{\partial\Omega} = 0, \quad \operatorname{supp} v_n \subset K, \tag{7.3}$$

where K is a compact set in R^n contained in $\bar{\Omega} \cap U$ and independent of n.

PROOF: Let $f \in W^2(\Omega^*) \cap \overset{\circ}{W}{}^1_2(\Omega^*)$ such that $f(z) = v(g^{-1}(z))$ if $z \in V \cap \Omega^*$, $f(z) = 0$ if $z \in \Omega^* \setminus V$. Let $(f_n)_{n \in N}$ be a sequence of functions in $C^\infty(\bar{\Omega}^*)$ convergent to f in $W^2(\Omega^*)$.

We know (see, *e.g.*, Theorem 6.14 of D. Gilbarg - N. S. Trudinger [GT]) that there exists a function $w_n \in C^2(\bar{\Omega}^*)$ satisfying the relations

$$\Delta w_n = 0, \qquad w_n|_{\partial\Omega^*} = f_n|_{\partial\Omega^*}.$$

Therefore, if $\varphi_n = f_n - w_n$, we have

$$\varphi_n - f \in W^2(\Omega^*) \cap \overset{\circ}{W}{}^1_2(\Omega^*), \quad \Delta(\varphi_n - f) = \Delta(f_n - f)$$

and, from Theorem 9.17 in [GT], it follows

$$\|\varphi_n - f\|_{W^2(\Omega^*)} \leq c\,|\Delta(f_n - f)|_{2,\Omega^*}, \qquad c \in R_+, \qquad \forall n \in N.$$

Let $\psi \in C_0^\infty(R^n)$ be a function such that $\operatorname{supp}\psi \subset U$ and $\psi|_{\operatorname{supp} v} = 1$. We have the assertion if we define v_n as follows:

$$v_n(x) = \psi\varphi_n(g(x)) \quad \text{if} \quad x \in \Omega \cap U, \qquad v_n(x) = 0 \quad \text{if} \quad x \in \Omega \setminus U. \quad \blacksquare$$

LEMMA 7.2: *If $v: \Omega \to R$ is a function which belongs to $C^2(\bar{\Omega})$ with $\operatorname{supp} v \subset \bar{\Omega} \cap U$, then there exist a compact set $K \subset \bar{\Omega} \cap U$ and a sequence of functions $v_n \in \mathcal{D}(\bar{\Omega})$, with $\operatorname{supp} v_n \subset K$, convergent to v in $C^2(\bar{\Omega})$.*

PROOF: It is known (see, *e.g.*, Theorem 6.37 in [GT]) that there exists a function $w \in C_0^2(R^n)$ such that $w|_\Omega = v$. Let $\psi \in C_0^\infty(R^n)$ be a function verifying the conditions:

$$\psi|_{\operatorname{supp} v} = 1, \qquad \operatorname{supp}\psi \subset U$$

and let $(J_n)_{n \in N}$ be a sequence of mollifiers.

If we set

$$v_n = \psi(J_n * v)$$

the lemma follows. $\blacksquare$

PROOF OF THEOREM 7.1: By Lemma 7.1 it is enough to prove (7.1) when

$$v \in C^2(\bar{\Omega}), \qquad v|_{\partial\Omega} = 0, \qquad \operatorname{supp} v \in \bar{\Omega} \cap U. \tag{7.4}$$

We denote by $X_1, \dots, X_n$ the direction cosines of the exterior normal to Ω. We start proving that, for any v satisfying (7.4), it holds

$$\nu^2 |v_{xx}|^2_{2,\Omega} \leq |L_0 v|^2_{2,\Omega} + I_1(v) + I_2(v), \tag{7.5}$$

where

$$I_1(v) = \sum_{i,k,r,s}^{1\dots n} \int_{\partial\Omega} (a_{ir} a_{ks} - a_{ik} a_{rs}) X_k v_{x_i} v_{x_r x_s} \, d\sigma ,$$

$$I_2(v) = - \sum_{i,k,r,s}^{1\dots n} \int_{\Omega} (a_{ir} a_{ks} - a_{ik} a_{rs})_{x_k} v_{x_i} v_{x_r x_s} \, dx .$$

We remark that, by Lemma 7.2, it is enough to prove (7.5) when

$$v \in \mathcal{D}(\bar{\Omega}), \qquad \operatorname{supp} v \in \bar{\Omega} \cap U .$$

Therefore the inequality (7.5) is proved integrating on Ω both sides of the following well-known inequality (see, *e.g.*, C. Miranda $[M]_1$)

$$\nu^2 v_{xx}^2 \leq (L_0 v)^2 + \sum_{i,k,r,s}^{1\dots n} (a_{ir} a_{ks} - a_{ik} a_{rs}) \frac{\partial}{\partial x_k} (v_{x_i} v_{x_r x_s})$$

and, then, integrating by parts.

We will use well known methods (see, *e.g.*, C.Miranda $[M]_2$, O. A. Ladyženskaja - N. N. Ural'ceva [LU], A. A. Sharovskii [S], G. Viola [V]) to prove that, if v verifies (7.4), for any $\varepsilon \in R_+$ there exists $c_1(\varepsilon)$ such that

$$|I_1(v)| \leq \varepsilon |v_{xx}|^2_{2,\Omega} + c_1(\varepsilon) |v|^2_{2,\Omega} , \tag{7.6}$$

where $c_1(\varepsilon)$ and the other constants in this proof are independent of v and λ.

Moreover, if we set $g = \sum_{i,j=1}^{n} (a_{ij})_x$, it is easy to see that

$$|I_2(v)| \leq \varepsilon |v_{xx}|^2_{2,\Omega} + c_2(\varepsilon) |g v_x|^2_{2,\Omega} . \tag{7.7}$$

From previous relations and from (4.7) we can deduce the estimate

$$|v_{xx}|_{2,\Omega} \leq c_3 (|L_0 v|_{2,\Omega} + |v|_{2,\Omega}), \tag{7.8}$$

for any function v satisfying (7.2).

Now we prove that (7.8) implies (7.1) by means of known tecniques (see, *e.g.*, M. Chicco [C]).

We have for $\lambda \geq 0$:

$$\int_\Omega (L_0 v + \lambda \beta v)^2 \, dx = \int_\Omega (L_0 v)^2 \, dx + \lambda^2 \int_\Omega \beta^2 v^2 \, dx + \tag{7.9}$$

$$+ 2\lambda \sum_{i,j=1}^{n} \int_\Omega (\beta a_{ij} v_{x_i} v_{x_j} + (\beta (a_{ij})_{x_j} + a_{ij} \beta_{x_j}) v_{x_i} v) \, dx \geq$$

$$\geq \int_\Omega (L_0 v)^2 \, dx + \lambda^2 \int_\Omega \beta^2 v^2 \, dx + 2\lambda \nu \int_\Omega \beta v_x^2 \, dx - 2\lambda \int_\Omega \beta r_i |v| v_x \, dx ,$$

where

$$\eta = \sum_{i,j=1}^{n} ((a_{ij})_x + |a_{ij}|\gamma).$$

Since $\eta \in \tilde{M}^{t,n-t}(\Omega)$, by Lemma 3.3 we have

$$(7.10) \qquad \int_\Omega \beta\eta|v|\,v_x\,dx \leq \frac{\lambda}{2}\int_\Omega \beta^2 v^2\,dx + \frac{1}{2\lambda}\int_\Omega |\eta v_x|^2\,dx \leq$$

$$\leq \frac{\lambda}{2}\int_\Omega \beta^2 v^2\,dx + \frac{1}{2\lambda}\left(\varepsilon\int_\Omega v_{xx}^2\,dx + c_4(\varepsilon)\int_\Omega v^2\,dx\right).$$

From (7.8), (7.9), (7.10) we deduce the bound

$$|v_{xx}|_{2,\Omega} \leq c_5(|L_0 v + \lambda\beta v|_{2,\Omega} + |v|_{2,\Omega}).$$

From last relation and from (4.7), (4.8) the inequality (7.1) follows for $\lambda \geq 0$.

Starting from (7.1) with $\lambda = 0$, we deduce (7.1) for $\lambda \in [\lambda_1, 0[$ by means of the following inequality which is a consequence of the hypothesis $\beta \in \tilde{M}^t(\Omega)$ and of the Lemma 3.4:

$$|\lambda\beta v|_{2,\Omega} \leq |\lambda_1|(\varepsilon|v_{xx}|_{2,\Omega} + c_6(\varepsilon)|v|_{2,\Omega}). \qquad \blacksquare$$

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«Monograf» - Via Collamarini, 5 - Bologna