Second Order Elliptic Equations with Discontinuous Coefficients in Unbounded domains (***)

SUMMARY. — This paper is concerned with the Dirichlet problem for linear second order elliptic partial differential equations with discontinuous coefficients in unbounded domains in $\mathbb{R}^n$. Some existence and uniqueness results have been proved.

Equazioni ellittiche del secondo ordine a coefficienti discontinui in aperti non limitati

SOMMARIO. — In questo lavoro ci occupiamo dello studio del problema di Dirichlet per le equazioni differenziali lineari ellittiche del secondo ordine a coefficienti discontinui in aperti non limitati di $\mathbb{R}^n$. Stabiliamo alcuni teoremi di esistenza ed unicità.

INTRODUCTION

Let $\Omega$ be an unbounded and sufficiently regular open subset in $\mathbb{R}^n$, $n > 2$ (see section 4).

In a recent paper M. Transirico and M. Troisi (see [TT]), have studied the Dirichlet problem

(1) \[ u \in W^2(\Omega) \cap W^1_0(\Omega), \quad Lu = f, \quad f \in L^2(\Omega), \]

where $L$ is the linear second order uniformly elliptic differential operator

\[ Lu = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial u}{\partial x_i} + au, \]

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with the following basic hypotheses on the coefficients:

\[(2) \quad a_{ij} = a_{ji} \in L^\infty(\Omega), \quad i, j = 1, \ldots, n,\]

\[(3) \quad (a_{ij})_{k} \in L^{\infty}_{\text{loc}}(\Omega), \quad \sup_{x \in \Omega} \| (a_{ij})_{k} \|_{L^{\infty}(\Omega \cap B(x, 1))} < +\infty, \quad i, j, k = 1, \ldots, n,\]

\[(4) \quad a_{i} \in L^{\infty}_{\text{loc}}(\Omega), \quad \sup_{x \in \Omega} \| a_{i} \|_{L^{\infty}(\Omega \cap B(x, 1))} < +\infty, \quad i = 1, \ldots, n,\]

\[(5) \quad a \in L^{1}_{\text{loc}}(\Omega), \quad \sup_{x \in \Omega} \| a \|_{L^{1}(\Omega \cap B(x, 1))} < +\infty,\]

where \(B(x, 1) = \{y \in \mathbb{R}^n : \|y - x\| < 1\}\) and \(t = 2\) if \(n = 3\), \(t > 2\) if \(n = 4\), \(t = n/2\) if \(n > 4\).

The aim of this paper is to extend the results in [TT], when the coefficients \(a_{ij}\) and \(a_{i}\) verify a weaker condition than (3) and (4).

We consider the Morrey type spaces \(M^{p,\alpha}(\Omega), \tilde{M}^{p,\alpha}(\Omega), M^{p,\alpha}_0(\Omega), 1 \leq p < +\infty\) and \(0 \leq \alpha < n\), introduced and studied by M. Transirico - M. Troisi - A. Vitolo [TTV] (see also F. Chiarenza - M. Franciosi [CF]).

We recall that \(M^{p,\alpha}(\Omega)\) is the space of functions \(g \in L^{1}_{\text{loc}}(\Omega)\) such that

\[(6) \quad \| g \|_{M^{p,\alpha}(\Omega)} = \sup_{\tau < 1} \tau^{-\alpha/p} \| g \|_{L^{p}(\Omega \cap B(\tau, 1))} < +\infty,\]

equipped with the norm defined in (6); \(\tilde{M}^{p,\alpha}(\Omega)\) is the closure of \(L^{\infty}(\Omega)\) in \(M^{p,\alpha}(\Omega)\); \(M^{p,\alpha}_0(\Omega)\) is the closure of \(C_{0}^{\infty}(\Omega)\) in \(M^{p,\alpha}(\Omega)\).

We study the problem when the coefficients of the operator \(L\) satisfy (2) and the conditions

\[(a_{ij})_{k}, \quad a_{i} \in \tilde{M}^{p,\alpha-\varepsilon}(\Omega), \quad a \in \tilde{M}^{p,\alpha-\varepsilon}(\Omega),\]

for some \(s \in [2, n]\) and for \(t\) defined above.

We prove that the problem

\[u \in W^{2}(\Omega) \cap W^{1}_{0}(\Omega), \quad Lu + \lambda u = f, \quad f \in L^{2}(\Omega)\]

is uniquely solvable for \(\lambda\) large enough.

Furthermore, if we have

\[(a_{ij})_{k}, \quad a_{i} \in M^{p,\alpha-\varepsilon}_{0}(\Omega),\]

\[a = a' + a'', \quad a' \in M^{p,0}_{0}(\Omega), \quad \text{ess inf}_{\Omega} a'' > 0,\]

then (1) is a zero index problem and it is uniquely solvable for \(a' = 0\).
1. Notations

Let $E$ be a Lebesgue measurable subset in $\mathbb{R}^n$. We denote by $\Sigma(E)$ the $\sigma$-algebra of Lebesgue measurable subsets of $E$.

For any $A \in \Sigma(E)$, $|A|$ is the Lebesgue measure of $A$, $\mathcal{O}(A)$ is the class of restrictions to $A$ of functions $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp } \zeta \subset A$, $L^p_w(A)$ is the class of functions $g : A \to \mathbb{R}$ such that $\zeta g \in L^p(\mathbb{R}^n)$ for all $\zeta \in \mathcal{O}(A)$, $\chi_A$ is the characteristic function of $A$.

Furthermore, we set

$$u_\zeta = \left( \sum_{i=1}^n u_{\zeta}^2 \right)^{1/2}, \quad u_m = \left( \sum_{i,j=1}^n u_{\zeta}^2 \right)^{1/2},$$

$$|u|_{p, E} = \|u\|_{L^p(E)}, \quad B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}, \quad B_r = B(0, r).$$

In the following we will use the lemma (similar to Lemma 2 of M. Transirico - M. Troisi [TT]$_2$):

**Lemma 1.1:** Let $p \in [1, +\infty[$, $r \in \mathbb{R}_+$, and $E \in \Sigma(\mathbb{R}^n)$. We have $f \in L^p(E)$ if and only if $f \in L^p_w(E)$ and the application $(x \in \mathbb{R}^n \to |f|_{p, E \cap B(x, r)}) \in L^p(\mathbb{R}^n)$.

Furthermore, the equality

$$(1.1) \quad \int_{E} |f|^p \, dx = \frac{1}{|B_r|} \int_{R^*} \int_{E \cap \overline{B(x, r)}} |f|^p \, dx \, dy.$$

holds.

**Proof:** If we consider the function

$$\varphi_r : (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \to \begin{cases} 1 & \text{if } |x - y| < r, \\ 0 & \text{if } |x - y| \geq r, \end{cases}$$

the result follows by observing that

$$\int_{\mathbb{R}^n} |f|_{p, E \cap \overline{B(x, r)}}^p \, dx = \int_{\mathbb{R}^n} \int_{E \cap \overline{B(x, r)}} \varphi_r(x, y) |f(y)|^p \, dy = \int_{E} |f(y)|^p \, dy \int_{\mathbb{R}^n} \varphi_r(x, y) \, dx. \quad \blacksquare$$

Let $\Omega$ be an unbounded open subset in $\mathbb{R}^n$.

For any $x \in \Omega$ and $t \in \mathbb{R}_+$, we set:

$$\Omega(x, t) = \Omega \cap B(x, t).$$

We denote by $W^k(\Omega)$, $k \in \mathbb{N}$, the space of real distributions $u$ on $\Omega$ such that $D^k u \in L^2(\Omega)$, with $|x| \leq k$, equipped with the norm

$$\|u\|_{W^k(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha u|^2 \, dx \right)^{1/2}.$$
and by $\mathcal{W}^k_0(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $\mathcal{W}^k(\Omega)$. In order to unify the notations often, in the following, we will denote by the symbol $U^k(\Omega)$ the space $\mathcal{W}^k(\Omega)$ in the case $\Omega$ has the cone property and the space $\mathcal{W}^k_0(\Omega)$ in the case $\Omega$ doesn’t have.

2. The spaces $M_p^{\alpha}(\Omega)$, $\overline{M}_p^{\alpha}(\Omega)$, $M_0^{\alpha}(\Omega)$.

If $1 \leq p < +\infty$, $0 \leq \alpha < n$ and $t \in \mathbb{R}_+$, we call $M_p^{\alpha}(\Omega, t)$ the space of the functions $g \in L^p_{\infty}(\tilde{\Omega})$ such that:

$$
\|g\|_{M_p^{\alpha}(\Omega, t)} = \sup_{\substack{\xi \in \tilde{\Omega} \\ 0 < \gamma < t}} \xi^{-\alpha}\|g|_{p, \Omega(\xi, \gamma)} < +\infty,
$$

equipped with the norm defined in (2.1).

Let $d$ be a fixed positive number. We put

$$
M_p^{\alpha}(\Omega) = M_p^{\alpha}(\Omega, d).
$$

The spaces $M_p^{\alpha}(\Omega, t)$ and $M_p^{\alpha}(\Omega)$, when $\Omega$ is an unbounded set, have been introduced and studied in [TTV]. From the results in [TTV] it follows that the definition of $M_p^{\alpha}(\Omega)$ is independent of $d$ and that, fixed $d_1, d_2 \in \mathbb{R}_+$, the norms in $M_p^{\alpha}(\Omega, d_1)$ and in $M_p^{\alpha}(\Omega, d_2)$ are equivalent.

Furthermore we have the imbedding:

$$
M_p^{\alpha}(\Omega) \hookrightarrow M_p^{\alpha}(\Omega), \quad p \leq p_0, \quad \frac{\alpha - n}{p} \leq \frac{\alpha_0 - n}{p_0},
$$

which implies in particular that:

$$
L^\infty(\Omega) \hookrightarrow M_p^{\alpha}(\Omega).
$$

We denote

by $\overline{M}_p^{\alpha}(\Omega)$ the closure of $L^\infty(\Omega)$ in $M_p^{\alpha}(\Omega)$;

by $M_0^{\alpha}(\Omega)$ the closure of $C_c^{\infty}(\Omega)$ in $M_p^{\alpha}(\Omega)$.

From the results in [TTV] we have the following characterizations of the spaces $\overline{M}_p^{\alpha}(\Omega)$ and $M_0^{\alpha}(\Omega)$:

$\overline{M}_p^{\alpha}(\Omega)$ is the subspace of $M_p^{\alpha}(\Omega)$ of the functions $g \in M_p^{\alpha}(\Omega)$ such that:

$$
\forall \varepsilon \in \mathbb{R}_+, \exists \delta_\varepsilon \in \mathbb{R}_+, \exists E \in \Sigma(\Omega), \sup_{\xi \in \tilde{\Omega}} |E \cap B(x, \delta_\varepsilon)| \leq \delta_\varepsilon \Rightarrow \|g|_{M_p^{\alpha}(\Omega)} \leq \varepsilon,
$$
$M_{0}^{p,*}(\Omega)$ is the subspace of $M^{p,*}(\Omega)$ of the functions $g \in M^{p,*}(\Omega)$ such that:

$\forall \varepsilon \in R_{+}, \exists h_{\varepsilon}, k_{i} \in R_{+}, \exists \varepsilon' \in \Sigma(\Omega), \ |E \cap B(0, k_{i})| \leq h_{\varepsilon} \Rightarrow \|\chi_{E} \|_{M^{p,*}(\Omega)} \leq \varepsilon$.

3. Some known results

Let us set:

(3.1) $M^{p}(\Omega) = M^{p,0}(\Omega)$, $\tilde{M}^{p}(\Omega) = \tilde{M}^{p,0}(\Omega)$, $M_{0}^{p}(\Omega) = M_{0}^{p,0}(\Omega)$.

It is known that (see, e.g., [TTV]):

**Lemma 3.1:** We have:

(3.2) $M_{0}^{p,n}(\Omega) \subset \tilde{M}^{p,n}(\Omega)$, $p < p_{0}$, $\frac{\alpha-n}{p} < \frac{\alpha_{0}-n}{p_{0}}$;

(3.3) $M_{0}^{p,\ast}(\Omega) = \tilde{M}^{p,\ast}(\Omega) \cap M_{0}^{p}(\Omega)$.

Furthermore $g \in \tilde{M}^{p,\ast}(\Omega)$ if and only if $g \in \tilde{M}^{p}(\Omega)$ and

(3.4) $\lim_{t \to 0} \|g\|_{M^{p,\ast}(\Omega, t)} = 0$.

We assign, for any $r \in R_{+}$, a function $\zeta_{r} \in C_{0}^{\infty}(\Omega)$ such that:

$0 \leq \zeta_{r} \leq 1$, $\zeta_{r}(x) = 1$ $\forall x \in B(0, r)$, $\text{supp} \zeta_{r} \subset B(0, 2r)$.

From section 2 in [TTV] we deduce

**Lemma 3.2:** The following propositions are equivalent:

1) $g \in M_{0}^{p,\ast}(\Omega)$;

2) $g \in M^{p,\ast}(\Omega)$ and satisfies the relations

(3.5) $\lim_{r \to \ast} \|1 - \zeta_{r}\|_{M^{p,\ast}(\Omega)} = 0$, $\lim_{t \to 0} \|\psi g\|_{M^{p,\ast}(\Omega, t)} = 0$ $\forall \psi \in C_{0}^{\infty}(\Omega)$;

3) $g \in M^{p,\ast}(\Omega)$ and satisfies (3.4) together with

(3.6) $\lim_{r \to \ast} \|1 - \zeta_{r}\|_{M^{p,\ast}(\Omega)} = 0$;

4) $g \in M^{p,\ast}(\Omega)$ and satisfies (3.4) together with

(3.7) $\lim_{|x| \to +} |g|_{p, \phi(x)} = 0$.

From section 3 in [TTV] we have the following
**Lemma 3.3:** If \( n > 2 \) and \( g \in M^{1,\infty}_{s-1}(\Omega) \), with \( s \in [2, n] \), then for any \( u \in U^1(\Omega) \) we have \( gu \in L^2(\Omega) \).

Furthermore, the operator

\[ u \in U^1(\Omega) \rightarrow gu \in L^2(\Omega) \]

is bounded.

If \( g \in \widetilde{M}^{1,\infty}_{s-1}(\Omega) \), then there exists \( c(\varepsilon) \in R_+ \) such that:

\[ |gu|_{2,0} \leq \varepsilon |u|_{s,0} + c(\varepsilon) |u|_{2,0} \quad \forall u \in U^1(\Omega) \]

If \( g \in M^0_{s-1}(\Omega) \), then there exist \( c(\varepsilon) \in R_+ \), and a bounded open subset \( \Omega(\varepsilon) \) of \( \Omega \) such that:

\[ |gu|_{2,0} \leq \varepsilon \|u\|_{w^1(\Omega)} + c(\varepsilon) |u|_{2,0} \quad \forall u \in U^1(\Omega) \]

Furthermore, the operator (3.8) is compact.

As a consequence of the results of M. Transirico - M. Troisi [TT], and A. V. Glushak - M. Transirico - M. Troisi [GTT] we have

**Lemma 3.4:** Let \( k \in N \) and \( p \in [2, + \infty] \) be such that:

\[ p = 2 \quad \text{if} \quad n < 2k, \quad p > 2 \quad \text{if} \quad n = 2k, \quad p = \frac{n}{k} \quad \text{if} \quad n > 2k. \]

If \( g \in M^p(\Omega) \), then for any \( u \in U^k(\Omega) \) we have \( gu \in L^2(\Omega) \).

Furthermore, the operator:

\[ u \in U^k(\Omega) \rightarrow gu \in L^2(\Omega) \]

is bounded.

If \( g \in \widetilde{M}^p(\Omega) \), then there exists \( c(\varepsilon) \in R_+ \) such that:

\[ |gu|_{2,0} \leq \varepsilon \sum_{|\alpha| = k} |D^\alpha u|_{2,0} + c(\varepsilon) |u|_{2,0} \quad \forall u \in U^k(\Omega) \]

If \( g \in M^p_0(\Omega) \), then there exist \( c(\varepsilon) \in R_+ \), and a bounded open subset \( \Omega(\varepsilon) \) of \( \Omega \) such that:

\[ |gu|_{2,0} \leq \varepsilon \|u\|_{w^1(\Omega)} + c(\varepsilon) |u|_{2,0} \quad \forall u \in U^k(\Omega) \]

Furthermore, the operator (3.12) is compact.

4. - Hypotheses

We set:

\[ B = B(0, 1), \quad B_+ = \{x \in B \colon x_n > 0\}, \quad B_0 = \{x \in B \colon x_n = 0\} \]
and, for any \( a \in \mathbb{R}_+ \),
\[
\Omega_a = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < a \}.
\]

Let us suppose \( n \geq 3 \) and that the following condition holds:
\( i_1 \) there exist a number \( \delta \in \mathbb{R}_+ \), an open covering \( \{ U_i \}_{i \in I} \) of \( \partial \Omega \) and, for any \( i \in I \),
a diffeomorphism of class \( C^2 \), \( \psi_i : \overline{U_i} \to \overline{B} \), such that:
1) \( \psi_i(U_i \cap \Omega) = B_+ \), \( \psi_i(U_i \cap \partial \Omega) = B_0 \);
2) the components of \( \psi_i \) and \( \psi_i^{-1} \) and their weak derivatives are bounded by a constant independent of \( i \);
3) \( \forall x \in \Omega \); there exists \( i \in I \) such that \( B(x, \delta) \subset U_i \) and \( B(x, \delta) \subset \Omega \) for any \( x \in \Omega \setminus \Omega_a \).

**Remark 4.1:** It's easy to prove that \( i_1 \) holds when \( \Omega \) has the uniform \( C^2 \)-regularity property defined in section 4.6 of R. A. Adams [A].

**Remark 4.2:** The condition \( i_1 \) implies that there exists a number \( \rho \in \mathbb{R}_+ \) such that, for any \( x \in \mathbb{R}^n \), \( B(x, \rho) \cap \partial \Omega = \emptyset \) or \( B(x, \rho) \cap \partial \Omega \neq \emptyset \) and \( B(x, \rho) \subset U_i \) for some \( i \in I \).

Let us give in \( \Omega \) the linear second order differential operator:
\[
Lu = -\sum_{i,j=1}^n a_{ij} u_{x_i x_j} + \sum_{i=1}^n a_i u_{x_i} + au,
\]
and suppose that
\( i_2 \) the real coefficients satisfy the following conditions:
\[
\sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geq \nu |\xi|^2 \quad \text{a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^n, \quad \nu \in \mathbb{R}_+;
\]
\[
a_{ij} = a_{ij} \in L^\infty(\Omega), \quad i, j = 1, \ldots, n;
\]
\[
(a_{ij})_{i,j \in [2,n]} \in \widetilde{M}^{n-2} (\Omega) \quad \text{for some } s \in [2,n], \quad i, j, k = 1, \ldots, n;
\]
\[
a \in \widetilde{M}^t (\Omega),
\]
where
\[
t = 2 \quad \text{if } \ n = 3, \quad t > 2 \quad \text{if } \ n = 4, \quad t = \frac{n}{2} \quad \text{if } \ n > 4.
\]

Let us set
\[
b = \sum_{i,j=1}^n (a_{ij})_x + \sum_{i=1}^n |a_i|.
\]
We observe that if (4.4), (4.5) hold and $u \in \mathcal{W}^2(\Omega)$, from Lemma 3.3 and Lemma 3.4 it follows that:

\begin{align}
(4.7) \quad |b u_x|_{2,\Omega} & \leq \varepsilon |u_{xx}|_{2,\Omega} + c_1(\varepsilon) |u|_{2,\Omega}, \\
(4.8) \quad |a u|_{2,\Omega} & \leq \varepsilon |u_{xx}|_{2,\Omega} + c_2(\varepsilon) |u|_{2,\Omega},
\end{align}

where the constants $c_1(\varepsilon)$ and $c_2(\varepsilon)$ are independent of $u$.

5. - A Priori Bounds

Let $\beta : \Omega \to R_+$ be a function such that:

\begin{align}
(5.1) \quad \beta \in \bar{M}^\prime(\Omega), \quad \exists \gamma \in \bar{M}^\prime(\Omega) \Rightarrow \beta \gamma \in \bar{M}^\prime(\Omega),
\end{align}

For example:

\begin{align}
\beta = 1 \quad \text{or} \quad \beta(x) = \frac{1}{(1 + |x|^2)\tau}, \quad \tau \in R_+.
\end{align}

We denote by $\mathcal{W}^2_{bc}(\bar{\Omega})$ (resp. $\mathcal{W}^2_{bc}(\bar{\Omega})$) the space of the functions $u : \Omega \to R$ such that $\zeta u \in \mathcal{W}^2(\Omega)$ (resp. $\zeta u \in \mathcal{W}^2(\Omega)$) for any $\zeta \in \mathcal{O}(\bar{\Omega})$.

**Theorem 5.1:** Let us suppose that $i_1)$ and $i_2)$ are satisfied. Therefore for any function $u : \Omega \to R$ such that

\begin{align}
(5.2) \quad u \in \mathcal{W}^2_{bc}(\bar{\Omega}) \cap \mathcal{W}^1_{bc}(\bar{\Omega}) \cap L^2(\Omega), \quad Lu \in L^2(\Omega),
\end{align}

we have $u \in \mathcal{W}^2(\Omega)$. Moreover, for any $\lambda_1 \in R$, it holds

\begin{align}
(5.3) \quad |u_{\lambda_1}|_{2,\Omega} \leq c \left( |Lu + \lambda_2 u|_{2,\Omega} + |u|_{2,\Omega} \right) \forall \lambda \in [\lambda_1, + \infty[, \quad c \text{ is a constant independent of } u \text{ and } \lambda.
\end{align}

**Proof:** We consider $r, r' \in R_+$, with $r < r' < 1$, and a function $\varphi \in C_\sigma^n(R^n)$ such that (see, e.g., P. Di Gironimo - M. Transirico [DT]):

\begin{align}
\varphi|_{B_r} = 1, \quad \sup \varphi \subset B_{r'}, \quad \sup_{x \in R^n} |\partial^\alpha \varphi| \leq c_\alpha (r' - r)^{-|\alpha|} \forall \alpha \in N^n_0,
\end{align}

where $c_\alpha$ is a constant depending only on $\alpha$.

Moreover let us fix $x \in \Omega$ and set

\begin{align}
\varphi = \varphi^\ast : y \in R^n \to \varphi \left( \frac{x - y}{\rho} \right),
\end{align}

where $\rho$ is the number defined in the Remark 4.2.

We observe that, if $u : \Omega \to R$ satisfies (5.2), the function $v = \varphi u$ verifies the hypotheses of theorem 7.1 where $U = B(x, \rho)$ (see section 7); therefore, for any $\lambda_1 \in R$,
we have

$$\|\psi u\|_{W^2(\tilde{\Omega})} \leq c_1 (|L \psi u + \lambda \beta \psi u|_{2, \tilde{\Omega}} + |\psi u|_{2, \tilde{\Omega}}) \quad \forall \lambda \in [\lambda_1, +\infty[.$$  

In this proof the constants $c_1, \ldots, c_8$ are independent of $u$, $\lambda$, $r$, $r'$, $\psi$, $\tilde{\Omega}$.

Let us set:

$$\chi = \left( \sum_{i=1}^{n} \psi_i^2 \right)^{1/2}, \quad \Omega_i(x) = \Omega(x, t) \quad \text{for } t \in \mathbb{R}_+.$$

From the relation

$$L(\psi u) = \psi Lu - \sum_{i,j=1}^{n} \alpha_{ij} (\psi_i u_j + \psi_j u_i + \psi_{ij} u) + \psi \sum_{i=1}^{n} a_i \psi_i,$$

it follows:

$$|L(\psi u) + \lambda \beta \psi u|_{2, \tilde{\Omega}} \leq c_2 (|Lu + \lambda \beta u|_{2, \Omega_i(x)} + (r' - r)^{-2} |u|_{2, \tilde{\Omega}_i(x)} +$$

$$+ \sum_{i=1}^{n} |a_i \chi u|_{2, \tilde{\Omega}} + |\chi u|_{2, \tilde{\Omega}}).$$

Lemma 3.3 gives

$$|a_i \chi u|_{2, \tilde{\Omega}} \leq c_3 \|\chi u\|_{W^1(\tilde{\Omega})}$$

and, from known results, it follows that:

$$\|\chi u\|_{W^1(\tilde{\Omega})} \leq c_4 (|\chi u|_{2, \tilde{\Omega}}^{1/2} \|\chi u\|_{2, \tilde{\Omega}}^{1/2} + |\chi u|_{2, \tilde{\Omega}}) \leq$$

$$\leq c_5 (r' - r)^{-2} (\|u\|_{W^2(\Omega_i(x))}^{1/2} \|\chi u\|_{2, \tilde{\Omega}_i(x)} + |u|_{2, \tilde{\Omega}_i(x)} + |u|_{2, \tilde{\Omega}_i(x)}) \leq$$

From previous inequalities we get

$$|L(\psi u) + \lambda \beta \psi u|_{2, \tilde{\Omega}} \leq$$

$$\leq c_6 (|Lu + \lambda \beta u|_{2, \Omega_i(x)} + |u|_{2, \Omega_i(x)} + \|u\|_{W^2(\Omega_i(x))} + |u|_{2, \tilde{\Omega}_i(x)}).$$

By a lemma of monotonicity of C. Miranda (see Lemma 3.1 in [M],) and from (5.5) we deduce the following result:

$$\|u\|_{W^2(\Omega_i(x))} \leq c_7 (|Lu + \lambda \beta u|_{2, \Omega_i(x)} + |u|_{2, \Omega_i(x)} + \|u\|_{W^2(\Omega_i(x))} + |u|_{2, \tilde{\Omega}_i(x)}) \quad \forall \lambda \in [\lambda_1, +\infty[.$$

By using inequality (5.6) with $\lambda = 0$ and Lemma 1.1 we have $u \in W^2(\Omega)$. It follows that $\beta u \in L^2(\Omega)$ and, then, by (5.5) and Lemma 1.1 we deduce (5.3).

**Corollary 5.1:** If $i_1, i_2$ and the condition $\text{ess inf} \beta > 0$ hold, then there exist $\lambda_0$, $\lambda \in \mathbb{R}_+$ such that:

$$\|u\|_{W^2(\tilde{\Omega})} \leq c \|Lu + \lambda \beta u\|_{2, \tilde{\Omega}} \quad \forall u \in W^2(\Omega) \cap W^0_0(\tilde{\Omega}), \quad \forall \lambda \in [\lambda_0, +\infty[.$$
Proof: The result can be deduced from Theorem 5.1 using arguments similar to those used by G. Viola [V] in order to get Theorem 6 from Proposition 5.

In the following corollary we suppose that there exists \( \lambda_1 \in R \) satisfying
\[
\text{ess inf}_\Omega (\lambda_1 \beta + a) > 0
\]

Corollary 5.2: If \( (i_1), (i_2) \), (5.8) and the condition
\[
a_{\alpha} \cdot (a_\alpha) \in M^0_{\text{loc}}(\Omega),
\]
hold, then there exists \( c \in R^+ \) such that
\[
\| u \|_{W^2(\Omega)} \leq c \| Lu + \lambda \beta u \|_{2, \Omega} \quad \forall u \in W^2(\Omega) \cap W^1_0(\Omega), \quad \forall \lambda \in [\lambda_1, + \infty[.
\]

Proof: Using Theorem 4.2 in [TTV], the result is a consequence of Theorem 5.1.

Now we consider the condition
\[
a = a^* + a^* \quad a^* \in M^0_0(\Omega), \quad \text{ess inf}_\Omega a^* > 0.
\]

Corollary 5.3: If \( (i_1), (i_2) \), (5.9), (5.11) hold and if \( \lambda_2 \) is a real number satisfying the condition
\[
\text{ess inf}_\Omega (\lambda_2 \beta + a^*) > 0,
\]
then there exist \( c \in R^+ \) and a bounded open subset \( \Omega_0 \) in \( \Omega \) such that:
\[
\| u \|_{W^2(\Omega)} \leq c (|Lu + \lambda \beta u|_{2, \Omega} + |u|_{2, \Omega_0})
\]
\[
\forall u \in W^2(\Omega) \cap W^1_0(\Omega), \quad \forall \lambda \in [\lambda_2, + \infty[.
\]

Proof: From corollary 5.2 it follows that
\[
\| u \|_{W^2(\Omega)} \leq c_1 \left( \sum_{\ell=1}^N a_{\ell} u_{\ell} + \sum_{i=1}^M a_i u_i + a^* u + \lambda \beta u \right)_{2, \Omega} \leq c_1 (|Lu + \lambda \beta u|_{2, \Omega} + |a^* u|_{2, \Omega}),
\]
\[
c_1 \in R^+, \quad \forall u \in W^2(\Omega) \cap W^1_0(\Omega), \quad \forall \lambda \in [\lambda_2, + \infty[.
\]

From the last relation and Lemma 3.4 we deduce (5.12).

Corollary 5.4: If the hypotheses of Corollary 5.3 are satisfied and if \( \beta^{-1} \in L^\infty_\text{loc}(\Omega) \), then there exist \( c, \lambda_0 \in R^+ \) such that
\[
\| u \|_{W^2(\Omega)} \leq c |Lu + \lambda \beta u|_{2, \Omega} \quad \forall u \in W^2(\Omega) \cap W^1_0(\Omega), \quad \forall \lambda \in [\lambda_0, + \infty[.
\]
Proof: The result can be deduced from Corollary 5.3 in the same way. Corollary 5.1 follows from Theorem 5.1.

6. Existence theorems

Theorem 6.1: Using either the hypotheses \(i_1\), \(i_2\) and the condition \(\text{ess inf } \beta > 0\) or the hypotheses \(i_1\), \(i_2\), (5.9), (5.11) and the condition \(\beta^{-1} \in L_{\infty}^2(\Omega)\), the problem

\[
(6.1) \\
\begin{align*}
    u & \in W^2(\Omega) \cap W_0^1(\Omega), \\
    Lu + \lambda \beta u &= f, \quad f \in L^2(\Omega)
\end{align*}
\]

is uniquely solvable for \(\lambda\) large enough.

Proof: We denote by \(A\) the operator \(-\Delta\) if \(i_1\), \(i_2\) and \(\text{ess inf } \beta > 0\) hold, the operator \(-\Delta + a^\ast\) in the other case.

It's known (see, e.g., Theorem 5.4 in [TT1]) that, for any \(\lambda \in \mathbb{R}_+\), the problem

\[
(6.1) \\
\begin{align*}
    u & \in W^2(\Omega) \cap W_0^1(\Omega), \\
    Au + \lambda \beta u &= f, \quad f \in L^2(\Omega)
\end{align*}
\]

is uniquely solvable.

For any \(\tau \in [0, 1]\) we set

\[
L_\tau = (1 - \tau)A + \tau(L).
\]

Using the Corollary 5.1 when \(i_1\), \(i_2\) and the condition \(\text{ess inf } \beta > 0\) are verified, the corollary 5.4 in the other case, we have that there exist \(c_0, \lambda_0 \in \mathbb{R}_+\) such that

\[
\|u\|_{W^2(\Omega)} \leq c_0 \|L_\tau u + \lambda \beta u\|_{L^2(\Omega)}
\]

\[
\forall u \in W^2(\Omega) \cap W_0^1(\Omega), \quad \forall \lambda \in [\lambda_0, +\infty], \quad \forall \tau \in [0, 1].
\]

Therefore observing that:

\[
L_\tau u + \lambda \beta u = (1 - \tau)(Au + \lambda \beta u) + \tau(Lu + \lambda \beta u),
\]

we obtain the result by means of the method of continuity.

Theorem 6.2: If \(i_1\), \(i_2\), (5.9) and (5.11) hold, then the problem

\[
(6.2) \\
\begin{align*}
    u & \in W^2(\Omega) \cap W_0^1(\Omega), \\
    Lu &= f, \quad f \in L^2(\Omega)
\end{align*}
\]

is a zero index problem.

Furthermore if the condition

\[
(6.3) \quad \text{ess inf } a > 0,
\]

holds, then the problem (6.2) is uniquely solvable.
Proof: We consider the function

$$\beta : x \in \Omega \to (1 + |x|^2)^{-1}.$$  

We observe that $\beta$ so defined verifies (5.1) and the condition $\beta^{-1} \in L^\infty_{sc} (\bar{\Omega})$. It follows, from Theorem 6.1, that the problem (6.1) is uniquely solvable for $\lambda$ large enough.

On the other hand, since $\beta \in M^d_0 (\Omega)$, from Lemma 3.4 we have that the operator

$$u \in W^2 (\Omega) \mapsto \beta u \in L^2 (\Omega)$$

is compact.

From known results we deduce that (6.2) is a zero index problem. The uniqueness of the solution follows from Corollary 5.2. \(\blacksquare\)

7. Appendix

In this section we derive an apriori local estimate (see (7.1)) which implies (5.3).

Using notations given in section 4, let us fix an open set $U$ in $R^*$ such that $U \subset \Omega$ or $U \cap \partial \Omega \neq \emptyset$ and $U \subset U_i$ for some $i \in I$.

If $U \cap \partial \Omega \neq \emptyset$, let us fix a bounded domain $\bar{\Omega}^*$ of class $C^2$ in $R^*$, an open set $V$ in $R^*$ and a diffeomorphism $g : \bar{\Omega} \to \bar{\Omega}$ of class $C^2$ such that:

$$g(U \cap \Omega) = V \cap \Omega^*, \quad g(U \cap \partial \Omega) = V \cap \partial \Omega^*.$$

Let us set

$$L_0 u = - \sum_{i,j = 1}^N a_{ij} u_{x_i x_j}.$$  

Theorem 7.1: If the conditions $i_1$, $i_2$) are satisfied and $\lambda_1$ is a real number, then there exists a constant $c \in R_+$ such that

$$|v\omega|_{2,\Omega} \leq c (|L v + \lambda \partial v|_{2,\Omega} + |v|_{2,\Omega})$$

for any $\lambda \in [\lambda_1, + \infty[$ and for any function $v : \Omega \to R$ with the conditions

$$v \in W^2 (\Omega) \cap W^1_0 (\Omega), \quad \text{supp} \ v \subset U.$$

We give the following lemmas before the proof of Theorem 7.1.

Lemma 7.1: Any function $v$ satisfying (7.2) is a $W^2$-limit of a sequence of functions $v_n$, such that

$$v_n \in C^2 (\bar{\Omega}), \quad v_n |_{\partial \Omega} = 0, \quad \text{supp} \ v_n \subset K,$$

where $K$ is a compact set in $R^*$ contained in $\bar{\Omega} \cap U$ and independent of $n$. 
PROOF: Let \( f \in W^2(\Omega^*) \cap W_0^1(\Omega^*) \) such that 
\[ f(z) = v(g^{-1}(z)) \text{ if } z \in V \cap \Omega^*, \]
\[ f(z) = 0 \text{ if } z \in \Omega^* \setminus V. \]
Let \( (f_n)_{n \in \mathbb{N}} \) be a sequence of functions in \( C^2(\bar{\Omega}^*) \) convergent to \( f \) in \( W^2(\Omega^*) \).

We know (see, e.g., Theorem 6.14 of D. Gilbarg - N. S. Trudinger \([G{T}]\)) that there exists a function \( \omega_n \in C^2(\Omega^*) \) satisfying the relations
\[
\Delta \omega_n = 0, \quad \omega_n |_{\partial \Omega^*} = f_n |_{\partial \Omega^*}.
\]

Therefore, if \( \varphi_n = f_n - \omega_n \), we have
\[
\varphi_n - f \in W^2(\Omega^*) \cap W_0^1(\Omega^*), \quad \Delta(\varphi_n - f) = \Delta(f_n - f)
\]
and, from Theorem 9.17 in \([G{T}]\), it follows
\[
\| \varphi_n - f \|_{W^2(\Omega^*)} \leq c |\Delta(f_n - f)|_{2, \Omega^*}, \quad c \in \mathbb{R}^+, \quad \forall n \in \mathbb{N}.
\]

Let \( \psi \in C_0^\infty(\mathbb{R}^*) \) be a function such that \( \text{supp} \psi \subset U \) and \( \psi |_{\text{supp} \psi} = 1 \). We have the assertion if we define \( \nu_n \) as follows:
\[
\nu_n(x) = \psi \varphi_n(g(x)) \quad \text{if} \quad x \in \Omega \cap U, \quad \nu_n(x) = 0 \quad \text{if} \quad x \in \Omega \setminus U.
\]

**Lemma 7.2:** If \( \nu : \Omega \to \mathbb{R} \) is a function which belongs to \( C^2(\bar{\Omega}) \) with \( \text{supp} \nu \subset \bar{\Omega} \cap U \), then there exist a compact set \( K \subset \bar{\Omega} \cap U \) and a sequence of functions \( \nu_n \in C_0^\infty(\Omega) \), with \( \text{supp} \nu_n \subset K \), convergent to \( \nu \) in \( C^2(\bar{\Omega}) \).

**Proof:** It is known (see, e.g., Theorem 6.37 in \([G{T}]\)) that there exists a function \( \omega \in C_0^\infty(\mathbb{R}^*) \) such that \( \omega |_\Omega = \nu \). Let \( \psi \in C_0^\infty(\mathbb{R}^*) \) be a function verifying the conditions:
\[
\psi |_{\text{supp} \psi} = 1, \quad \text{supp} \psi \subset U
\]
and let \( (f_n)_{n \in \mathbb{N}} \) be a sequence of mollifiers.

If we set
\[
\nu_n = \psi(f_n * \nu)
\]
the lemma follows.

**Proof of Theorem 7.1:** By Lemma 7.1 it is enough to prove (7.1) when
\[
(7.4) \quad \nu \in C^2(\bar{\Omega}), \quad \nu |_{\partial \Omega} = 0, \quad \text{supp} \nu \in \bar{\Omega} \cap U.
\]

We denote by \( X_1, \ldots, X_n \) the direction cosines of the exterior normal to \( \Omega \).

We start proving that, for any \( \nu \) satisfying (7.4), it holds
\[
(7.5) \quad \nu^2 |\nu_n|_{2, \Omega}^2 \leq |L_0 \nu|^2_{2, \Omega} + I_1(\nu) + I_2(\nu),
\]
where

\[ I_1 (v) = \sum_{i,k,t,s} \int_{\Omega} (a_{\theta} a_{\theta} - a_{\theta} a_{\theta}) X_{k} v_{n} v_{n-t} \, d\sigma, \]

\[ I_2 (v) = -\sum_{i,k,t,s} \int_{\Omega} (a_{\theta} a_{\theta} - a_{\theta} a_{\theta}) \eta v_{n} v_{n-t} \, d\sigma. \]

We remark that, by Lemma 7.2, it is enough to prove (7.5) when

\[ v \in \mathcal{W}(\bar{\Omega}), \quad \text{supp } v \subset \bar{\Omega} \cap U. \]

Therefore the inequality (7.5) is proved integrating on \( \Omega \) both sides of the following well-known inequality (see, e.g., C. Miranda [M1])

\[ v^2 v_{\|,t}^2 \leq (L_0 v)^2 + \sum_{i,k,t,s} (a_{\theta} a_{\theta} - a_{\theta} a_{\theta}) \frac{\partial}{\partial x_t} (v_{n} v_{n-t}) \]

and, then, integrating by parts.

We will use well known methods (see, e.g., C. Miranda [M2], O. A. Ladyženskaja - N. N. Urал'ceva [LU], A. A. Sharovskii [S], G. Viola [VI]) to prove that, if \( v \) verifies (7.4), for any \( \epsilon \in R_+ \), there exists \( c_1 (\epsilon) \) such that

\[ |I_1 (v)| \leq \epsilon |v_{\|,t}|_{2,\Omega}^2 + c_1 (\epsilon) |v|_{2,\Omega}^2, \]

where \( c_1 (\epsilon) \) and the other constants in this proof are independent of \( v \) and \( \lambda \).

Moreover, if we set \( g = \sum_{i,j} (a_{\theta})_{x_j} \), it is easy to see that

\[ |I_2 (v)| \leq \epsilon |v_{\|,t}|_{2,\Omega}^2 + c_2 (\epsilon) |g v_{n-t}|_{2,\Omega}^2. \]

From previous relations and from (4.7) we can deduce the estimate

\[ |v_{\|,t}|_{2,\Omega} \leq c_3 \left( |L_0 v|_{2,\Omega} + |v|_{2,\Omega} \right), \]

for any function \( v \) satisfying (7.2).

Now we prove that (7.8) implies (7.1) by means of known techniques (see, e.g., M. Chicco [C]).

We have for \( \lambda \geq 0 \):

\[ \int_{\Omega} (L_0 v + \lambda v)^2 \, dx = \int_{\Omega} (L_0 v)^2 \, dx + \lambda^2 \int_{\Omega} v^2 \, dx + \]

\[ + 2 \lambda \sum_{i,j=1}^{n} \int_{\Omega} \left( \beta \frac{\partial v_{n-t}}{\partial x_t} v_{n-t} + (\beta (a_{\theta})_{x_j} a_{\theta} \beta v_{n-t}) v_{n-t} \right) \, dx \geq \]

\[ \geq \int_{\Omega} (L_0 v)^2 \, dx + \lambda^2 \int_{\Omega} v^2 \, dx + 2 \lambda \int_{\Omega} v^2 \, dx - 2 \lambda \int_{\Omega} |\beta v| |v| v_{n-t} \, dx, \]
where
\[ \eta = \sum_{i,j=1}^n (a_{ij}x_i + b_i y) . \]

Since \( \eta \in \hat{M}^{1,n-1} (\Omega) \), by Lemma 3.3 we have
\[
(7.10) \quad \int_{\Omega} |\beta \eta| \gamma \, dx \leq \frac{\lambda}{2} \int_{\Omega} \beta^2 \gamma^2 \, dx + \frac{1}{2 \lambda} \int_{\Omega} |\eta \gamma| \, dx \leq \\
\leq \frac{\lambda}{2} \int_{\Omega} \beta^2 \gamma^2 \, dx + \frac{1}{2 \lambda} \left( \epsilon \int_{\Omega} \eta^2 \, dx + c_\delta (\epsilon) \int_{\Omega} \gamma^2 \, dx \right).
\]

From (7.8), (7.9), (7.10) we deduce the bound
\[ |\nu_{\infty}|_{2,\Omega} \leq c_5 (|L_0 v + \lambda \beta v|_{2,\Omega} + |v|_{2,\Omega}) . \]

From last relation and from (4.7), (4.8) the inequality (7.1) follows for \( \lambda \geq 0 \).

Starting from (7.1) with \( \lambda = 0 \), we deduce (7.1) for \( \lambda \in [\lambda_1, 0] \) by means of the following inequality which is a consequence of the hypothesis \( \beta \in \hat{M}^1 (\Omega) \) and of the Lemma 3.4:
\[ |\lambda \beta v|_{2,\Omega} \leq |\lambda_1| (\epsilon \int_{\Omega} |\nu_{\infty}|^2 + c_6 (\epsilon) |v|_{2,\Omega}) . \]

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