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A Problem of Seakeeping (**)

SUMMARY. — The paper deals with the unsteady motion of a rigid body floating in a bounded basin, under the action of gravity, of liquid pressure and of given external forces. Existence and uniqueness is proved for the linearized model, assuming the radiation condition on the lateral surface of the basin.

Un problema di seakeeping

SUNTO. — Si considera il moto non stazionario di un corpo rigido galleggiante in un bacino limitato, sotto l'azione della gravità, della pressione del liquido e di forze esterne. Per il modello linearizzato si dimostrano esistenza ed unicità, assegnando la condizione di radiazione sulla superficie laterale del bacino.

1. - Introduction and statement of the problem

This paper deals with the combined motion of a system consisting of a fluid and of a body partially immersed in it. The fluid is assumed to be inviscid, incompressible (with constant density ρ) and its motion to be irrotational. We suppose the body is rigid and that it describes a forced motion under the action of gravity, of liquid pressure and of given external forces.

The motion of the body and the fluid is assumed to be near equilibrium and we use the same linear model as in [1] and [6].

The system may be described as follows. The position of the body (referred to a cartesian coordinate system (x_1, x_2, x_3) with the x_2-axis positive upward) is specified by the vector $\xi := (\xi_T, \xi_R)$ where $\xi_T(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$ is the center of mass position vector, while $\xi_R(t) = (\xi_4(t), \xi_5(t), \xi_6(t))$ is the vector whose components are the angles of rotation of the center of mass about axes through it.

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The state of the fluid is described by a function $\phi(x_1, x_2, x_3, t)$, the velocity potential, which is harmonic in a domain Ω. In absence of external forces, except for gravity g (the atmospheric pressure is assumed to vanish) the fluid occupies at rest the bounded domain Ω, lying in the lower half-space $x_2 < 0$, and the body itself is at rest; the boundary of Ω consists of the free-calm surface Γ_F, the fixed bottom surface Γ_B, the surface Γ_I of the immersed body in the rest position and a far-away «artificial» lateral surface Γ_L. At the initial time $t = 0$ some external forces act on the body and the system evolves consequently.

We assume that the resultant of the external forces is a first order quantity, like $\varepsilon F(t)$, where ε is a suitable small parameter, and that an expansion in powers of ε is permitted for the functions describing the system ϕ, ξ_T, ξ_R.

Than at the first order, if we denote the first deviation of these quantities also with ϕ, ξ_T, ξ_R, and assuming $\rho = g = 1$, the linearized model is:

(1.1a) $$\Delta\phi = 0 \quad \text{in } \Omega\,,$$

(1.1b) $$M_1\begin{pmatrix}\ddot{\xi}_T\\ \ddot{\xi}_R\end{pmatrix} = \begin{pmatrix}F\\ G\end{pmatrix} - \int_{\Gamma_I} \frac{\partial\phi}{\partial t}\begin{pmatrix}v\\ p\end{pmatrix} - M_2\begin{pmatrix}\xi_T\\ \xi_R\end{pmatrix},$$

where v is the unit normal to Γ_I pointing into the body, G and p the moments of F and v, respectively, about the center of mass of the body. Furthermore M_1 is the simmetric positive-define matrix of body-inertia coefficients, and M_2 is a non-negative matrix involving the various moments of the displaced volume and waterplane in equilibrium position.

Owing to linearization, the boundary conditions are assigned on the surfaces of equilibrium; they are the following:

(1.1c) $$\frac{\partial\phi}{\partial v} = \langle\dot{\xi}, q\rangle \quad \text{on } \Gamma_I\,, \qquad \text{where } q = \begin{pmatrix}v\\ p\end{pmatrix},$$

(1.1d) $$\frac{\partial\phi}{\partial x_2} = -\frac{\partial^2\phi}{\partial t^2}\,, \qquad \text{on } \Gamma_F\,,$$

(1.1e) $$\frac{\partial\phi}{\partial v} = -C_0\phi_t\,, \qquad \text{on } \Gamma_L, \quad C_0 > 0\,,$$

(1.1f) $$\frac{\partial\phi}{\partial v} = 0\,, \qquad \text{on } \Gamma_B\,.$$

Recall that [6] (1.1c), (1.1f) express the kinematic condition of continuity of the normal velocity of the fluid across the sides of Ω, while (1.1d) is the classical Fourier condition; the radiation condition (1.1e) means that, on the far-away surface Γ_L, ϕ has the character of a plane progressive wave of velocity $1/C_0$, proceeding outward, so that the flow of energy through Γ_L is negative. This condition on

Γ_L simulates the usual radiation condition imposed at infinity (for a discussion of the boundary conditions see, e.g. [3]).

The initial data are:

(1.1g) $\quad \phi = 0, \phi_t = 0, \quad$ on $\Gamma_F \quad$ for $t = 0$,

(1.1h) $\quad \phi = 0, \quad$ on $\Gamma_L \quad$ for $t = 0$,

(1.1i) $\quad \xi_T(0) = \xi_R(0) = 0, \quad \dot{\xi}_T(0) = \dot{\xi}_R(0) = 0$.

We observe at once that the uniqueness of the solution of (1.1) is a consequence of the following energy argument: the total energy of the system is

$$E(t) = \frac{1}{2}\int_{\Omega} |\nabla\phi|^2 + \frac{1}{2}\int_{\Gamma_F} \phi_t^2 + \frac{1}{2}M_1\dot{\xi}\cdot\dot{\xi} + M_2\xi\cdot\xi .$$

Owing to (1.1) the rate of energy is:

$$\frac{dE}{dt} = F\cdot\dot{\xi}_T + G\cdot\dot{\xi}_R - C_0\int_{\Gamma_L}\phi_t^2$$

while $E(0) = 0$, by (1.1a), (1.1c), (1.1f), (1.1g), (1.1h), (1.1i).

If $F = G = 0$, we have $dE/dt \leq 0\ \forall t$, hence $E(t) \leq 0\ \forall t$; than the conclusion (see [6]) is that the system stays at rest for all times. The solution of (1.1) is than unique. In [8] existence was obtained for the linearized model, assuming that the bounded region Ω in which the body is immersed has «fixed» phisical lateral walls (a bounded basin). In the present paper we obtain analogous results, assuming, as we have seen, on the «geometrical» lateral surface Γ_L the condition of radiation.

2. - The transformed problem

The existence of the solution of (1.1) will be obtained, as in [8] using the method of the Laplace transform.

We apply this transformation L with respect to t to problem (1.1) and we denote with $\bar{\phi}(x;s) := L\phi(x;t)$, $\bar{\xi}(s) := L\xi(t)$ and $\bar{f}(s) := Lf(t)$ where $f = (F, G)$.

If we seek for a solution $\bar{\phi}$ of the form

(2.1) $\quad \bar{\phi}(x;s) = s\,\bar{\xi}(s)\cdot u(x;s)$,

than ϕ and ξ satisfy (1.1) if the new unknown $u(x;s)$ solves the following self-contained boundary value problem:

$$(2.2)\quad \begin{cases} \Delta u = 0 & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial \nu} = -\chi(x;s)\,u & \text{on } \Gamma := \Gamma_F \cup \Gamma_L\,, \quad \text{where } \chi(x;s) := \begin{cases} s^2 & \text{on } \Gamma_F\,, \\ s & \text{on } \Gamma_L\,, \end{cases} \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_B\,, \\ \dfrac{\partial u}{\partial \nu} = q & \text{on } \Gamma_I \end{cases}$$

and $\bar{\xi}$ solves the system of algebraic equations

$$(2.3)\qquad (M_1 + s^{-2} M_2)\,\bar{\xi} = s^{-2}\tilde{f} - \int_{\Gamma_I} (u\cdot\bar{\xi})\,q$$

where u is the solution of (2.2). The problem is thus decoupled since system (2.2) is independent on $\bar{\xi}$.

The problem (2.2) is a vector system of six scalar equations of the same kind; in the following we will consider only one of these, denoting again with q the scalar assigned real function and with u the scalar complex unknown function.

An existence-uniqueness theorem for the variational solution of the problem (2.2) may be proved easily, like in [8], if we consider on $H^1(\Omega)$ the sesquilinear form:

$$a_s(u,v) := \int_\Omega \nabla u\,\nabla\bar{v} + \int_\Gamma \chi(x;s)\,u\bar{v}\,.$$

It is continuous and coercive relatively to $L^2(\Omega)$, *i.e.* $\forall s \in \mathbb{C}$ fixed, $\operatorname{Re} s > 0$, $\exists K = K(s) > 0$ such that

$$\operatorname{Re} a_s(u,u) + K(s)\,\|u\|^2_{L^2(\Omega)} \geq \frac{1}{2}\,\|u\|^2_{H^1(\Omega)}\,.$$

This enables us to assert the following statement:

«If $q \in L^2(\Gamma_I)$, there exists only one $u \in H^1(\Omega)$, holomorphic in s, $\operatorname{Re} s > 0$ such that:

$$(2.4)\qquad a_s(u,v) = \int_{\Gamma_I} q\bar{v} \qquad \forall v \in H^1(\Omega)\,.$$

Moreover there exists a constant $C(s) > 0$ such that

$$(2.5)\qquad \|u\|_{H^1(\Omega)} \leq C(s)\cdot\|q\|_{L^2(\Gamma_I)}\text{»}.$$

In order to apply the inverse Laplace transform, we are now interested to evaluate the asymptotic behaviour of the constant $C(s)$, in order to know the behaviour of the original unknowns as functions of the time variable.

To reach this goal we adapt to our case a technique used in [4] and [8]: we decompose the solution u of (2.4) as the sum (up to a suitable constant) of u_1 and u_2, where these functions are «weak» solutions of two auxiliary Neumann problems.

More precisely we consider the linear space $\tilde{H}^1(\Omega)$ defined by:

$$\tilde{H}^1(\Omega) := \left\{ u = u(x;s) : u(\cdot;s) \in H^1(\Omega), \int_\Omega \chi(x,s)\, u(x)\, dx = 0\ \forall s \in \mathbb{C}, \operatorname{Re} s > 0 \right\}.$$

It is a Hilbert space with the norm $\|u\|_{\tilde{H}^1(\Omega)} = \left\{ \int_\Omega |\nabla u|^2 \right\}^{1/2}$; moreover there is a constant independent on s, $\operatorname{Re} s > 0$ (always denoted with $\tilde{c}$), such that, $\forall u \in \tilde{H}^1(\Omega)$:

$$\frac{1}{\tilde{c}} \|u\|_{\tilde{H}^1(\Omega)} \le \|u\|_{H^1(\Omega)} \le \tilde{c}\, \|u\|_{\tilde{H}^1(\Omega)}. \tag{2.6}$$

Furthermore we define:

$$\tilde{H}^{1/2}(\Gamma) := \{ u \in L^2(\Gamma) : \exists w \in \tilde{H}^1(\Omega)\ \ w|_\Gamma = u \}$$

with the norm $\|u\|_{\tilde{H}^{1/2}(\Gamma)} := \inf_{w|_\Gamma = u} \|w\|_{H^1(\Omega)}$.

Via the Lax-Milgram theorem, it's easy to prove the following lemmas:

Lemma 2.1: If $q \in L^2(\Gamma_1)$, there is a unique $u_1 \in \tilde{H}^1(\Omega)$ such that

$$\int_\Omega \nabla u_1 \nabla \bar{w} = \int_{\Gamma_1} q \bar{w} \qquad \forall w \in \tilde{H}^1(\Omega) \tag{2.7}$$

and there is $C_1 > 0$ independent on s, $\operatorname{Re} s > 0$, such that

$$\|u_1\|_{H^1(\Omega)} \le C_1 \|q\|_{L^2(\Gamma_1)}. \tag{2.8}$$

Lemma 2.2: If $k \in H^{-1/2}(\Gamma)$ (the antidual space of $H^{1/2}(\Gamma)$) there is a unique $u_2 \in \tilde{H}^1(\Omega)$ such that

$$\int_\Omega \nabla u_2 \nabla \bar{w} = \underset{H^{-1/2}(\Gamma)}{\langle} k, w \underset{H^{1/2}(\Gamma)}{\rangle} \qquad \forall w \in \tilde{H}^1(\Omega) \tag{2.9}$$

and there is $C_2 > 0$ independent of s, $\operatorname{Re} s > 0$, such that

$$\|u_2\|_{H^1(\Omega)} \le C_2 \|k\|_{H^{-1/2}(\Gamma)}. \tag{2.10}$$

Then we introduce the following linear operators:

$$N : L^2(\Gamma_1) \to \tilde{H}^{1/2}(\Gamma) \qquad Nq := u_1|_\Gamma,$$

$$P : H^{-1/2}(\Gamma) \to \tilde{H}^{1/2}(\Gamma) \qquad Pk := u_2|_\Gamma,$$

where u_1 and u_2 are the solutions respectively of (2.7) and (2.9).

It follows, as a consequence of Lemmas 2.1, 2.2, that the norms of P and N are bounded by a constant independent on s, $\operatorname{Re} s > 0$.

PROPOSITION 2.3: Let $q \in L^2(\Gamma_1)$; if $k \in H^{-1/2}(\Gamma)$ is solution of the equation

$$k + \chi P k = -\chi \left\{ Nq + \frac{\int q}{\int \chi} \right\} \quad \text{in } H^{-1/2}(\Gamma) \tag{2.11}$$

than $u := u_1 + u_2 + \int q / \int \chi$, where u_1 and u_2 are the solutions respectively of (2.7) and (2.9), is the solution of (2.4).

As a consequence $k \equiv (\partial u/(\partial \nu)|_{\Gamma}$, and $\langle k, 1 \rangle = -\int_{\Gamma_1} q$.

PROOF: Any $v \in H^1(\Omega)$ may be decomposed as $v = w + \beta(s)$, where $w \in \hat{H}^1(\Omega)$ and $\beta(s) = \left(1 / \int \chi\right) \int \chi v$. Therefore, taking (2.7) and (2.9) into account, we get:

$$a_s\left(u_1 + u_2 + \frac{\int q}{\int \chi}\right) = \int_{\Gamma_1} q\bar{w} + \langle k, w \rangle + \int_{\Gamma} \chi \left(u_1 + u_2 + \frac{\int q}{\int \chi}\right)\bar{v} =$$

$$= \int_{\Gamma_1} q\bar{v} - \bar{\beta}(s) \int_{\Gamma_1} q + \langle k, v \rangle - \bar{\beta}(s)\langle k, 1 \rangle + \int_{\Gamma} \chi \left(Nq + Pk + \frac{\int q}{\int \chi}\right)\bar{v}.$$

If k is a solution of (2.11) than $\langle k, 1 \rangle = -\int_{\Gamma_1} q$; than the assertion follows. ∎

3. - THE INTEGRAL EQUATION

In order to solve the «integral» equation (2.11), it is usefull to introduce the operator $Q: H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ so defined:

$$Qv := Pv + |\Gamma|^{-1}\langle v, 1 \rangle.$$

At soon, we have:

$$\int_{\Gamma} \chi Q v = |\Gamma|^{-1}\langle v, 1 \rangle \int_{\Gamma} \chi. \tag{3.1}$$

PROPOSITION 3.1: Q is an isomorphism from $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$ and the norms of Q and Q^{-1} are bounded by a constant independent on s, $\operatorname{Re} s > 0$. Moreover there is a constant $\gamma > 0$ such that $\forall b \in H^{1/2}(\Gamma)$:

$$\underset{H^{-1/2}(\Gamma)}{\langle} Q^{-1}b, b \underset{H^{1/2}(\Gamma)}{\rangle} \geq \gamma \|b\|^2_{H^{1/2}(\Gamma)}. \tag{3.2}$$

PROOF: The linearity and boundedness of Q follow easily from the analogous properties of P. The iniectivity follows from (3.1).

Let us assume momentarely that Q is onto; we get, from (2.9) with $w = u_2$:

$$\|u_2\|^2_{\dot{H}^1(\Omega)} = {}_{H^{-1/2}(\Gamma)}\langle k, u_2 \rangle_{H^{1/2}(\Gamma)} = \langle k, Qk - |\Gamma|^{-1}\langle k, 1\rangle\rangle = \langle k, Qk\rangle - \\ - \| |\Gamma|^{-1}\langle k, 1\rangle\|^2_{H^{1/2}(\Gamma)} .$$

Therefore

$$\|Qk - |\Gamma|^{-1}\langle k, 1\rangle\|^2_{H^{1/2}(\Gamma)} \leq \tilde{c}^2 \|u_2\|^2_{\dot{H}^1(\Omega)} \leq \tilde{c}^2 \{\langle k, Qk\rangle - \| |\Gamma|^{-1}\langle k, 1\rangle\|^2_{H^{1/2}(\Gamma)}\},$$

hence

$$\gamma \|Qk\|^2_{H^{1/2}(\Gamma)} \leq {}_{H^{-1/2}(\Gamma)}\langle k, Qk \rangle_{H^{1/2}(\Gamma)} ,$$

with γ independent on s. Than (3.2) follows from surjectivity of Q.

It remains to show that Q is onto. Let $b \in H^{1/2}(\Gamma)$; if $b' := b - Q\left(\int \chi b / \int \chi\right)$, we get $b' \in \dot{H}^{1/2}(\Gamma)$, owing to (3.1). Let $z \in \dot{H}^1(\Omega)$ the variational solution of the mixed problem: $\Delta z = 0$ in Ω, $z = b'$ on Γ, $\partial z/\partial \nu = 0$ on $\partial\Omega - \Gamma$. We claim that z solves the equation

$$\int_\Omega \nabla z \nabla \bar{w} = {}_{H^{-1/2}(\Gamma)}\langle k', w \rangle_{H^{1/2}(\Gamma)} \qquad \forall w \in H^1(\Omega), \tag{3.2}$$

for suitable choice of $k' \in H^{-1/2}(\Gamma)$. Indeed, owing to the Green's formula for the mixed problems, we have

$$\int_\Omega \nabla z \nabla \bar{v} = 0 \quad \forall v \in V := \{u \in H^1(\Omega) : u|_\Gamma = 0\}. \tag{3.3}$$

Furthermore:

$$\int_\Omega \nabla z \nabla \bar{w} = {}_{H^{-1/2}(\partial\Omega)}\langle \partial z/\partial \nu, w \rangle_{H^{1/2}(\partial\Omega)} =: l(w) \quad \forall w \in H^1(\Omega).$$

Now we prove that the right-hand side $l(w)$ defines a continuous antilinear functional on $H^{1/2}(\Gamma)$. Indeed, taking into account (3.3), $l(v) = 0$ $\forall v \in V$; therefore

$$|l(w)| = |l(w + v)| \leq C\|w + v\|_{H^1(\Omega)} \quad \forall w \in H^1(\Omega), \quad \forall v \in V,$$

that is

$$|l(w)| \leq C' \inf_{v \in V} \|w + v\|_{H^1(\Omega)} \quad \forall w \in H^1(\Omega).$$

But the space $H^{1/2}(\Gamma)$ is isomorphic to the quotient space $H^1(\Omega)/_V$, hence

$|l(w)| \leq C' \|w\|_{H^{1/2}(\Gamma)}$, $\forall w \in H^1(\Omega)$. So we get (3.2), for a suitable $k' \in H^{-1/2}(\Gamma)$, and $\langle k', 1 \rangle = 0$. By definition $Qk' = z|_\Gamma = b' = b - Q\left(\int \chi b / \int \chi \right)$. Finally let $k := k' + \int \chi b / \int \chi$; it's easy to verify that $Qk = b$. From the construction of Q^{-1}, we see that its norm is bounded by a constant independent of s, $\operatorname{Re} s > 0$. ■

Now we consider a linear functional equation which is equivalent to (2.11).

Proposition 3.2: Let $q \in L^2(\Gamma_1)$; if $k \in H^{-1/2}(\Gamma)$ is solution of (2.11), than $k^* := k/\chi + \int q / \int \chi \in H^{1/2}(\Gamma)$ and satisfies the following equation, in $H^{-1/2}(\Gamma)$:

$$Q^{-1} k^* + \chi k^* = q^*, \tag{3.4}$$

where

$$q^* = -Q^{-1} N q + \int q \cdot \left\{ \frac{\chi}{\int \chi} - \frac{\bar{\chi}}{\int \bar{\chi}} \right\}. \tag{3.5}$$

Also viceversa is true.

Proof: Let $k \in H^{-1/2}(\Gamma)$ be solution of (2.11); because of $\chi P k \in L^2(\Gamma)$ and $\chi \left\{ Nq + \int q / \int \chi \right\} \in L^2(\Gamma)$, than $k \in L^2(\Gamma)$ indeed; moreover $\int_\Gamma k = -\int_{\Gamma_1} q$. So $k^* \in L^2(\Gamma)$; substituting k^* in (2.11), we obtain $k^* + Q\left\{ \chi k^* - \chi\left(\int q / \int \chi \right) \right\} = -Nq - |\Gamma|^{-1} \int q$, which implies $k^* \in H^{1/2}(\Gamma)$. Applying the operator Q^{-1}, we get (3.4), taking into account that $Q^{-1}(1) = \left(\bar{\chi} / \int \bar{\chi} \right) |\Gamma|$. As $\langle Q^{-1} N q, 1 \rangle = 0$ owing to (3.1), also $\langle q^*, 1 \rangle = 0$. Hence we get, by (3.1)

$$\int \chi k^* = 0, \quad \text{which implies} \tag{3.6}$$

$$\langle Q^{-1} k^*, 1 \rangle = 0. \tag{3.7}$$

In order to prove the vice-versa, let $k^* \in H^{1/2}(\Gamma)$ be solution of (3.4), (3.5). If $k := \chi k^* - \chi\left(\int q / \int \chi \right)$, than $k \in L^2(\Gamma)$, and by vertue of (3.6), $\int k = -\int_{\Gamma_1} q$. Appling Q to (3.4), we obtain easily (2.11), taking into account that $Q^{-1}(1) = \left(\bar{\chi} / \int \bar{\chi} \right) |\Gamma|$. ■

Now we are able to state the following result:

THEOREM 3.3: Let $q \in L^2(\Gamma_I)$; than equation (3.4) admits for every $s \in \mathbb{C}$ fixed, $\operatorname{Re} s > 0$, a unique solution $k^* = k^*(s) \in H^{1/2}(\Gamma)$. Moreover there exist constants $C > 0$ and $\alpha_0 > 0$ such that, for all complex numbers s, $\operatorname{Re} s > \alpha_0$, the following estimates hold:

(3.8) $$\|k^*\|_{H^{1/2}(\Gamma)} \leq C\|q\|_{L^2(\Gamma_I)},$$

(3.9) $$\|k^*\|_{H^{-1/2}(\Gamma_I)} \leq C|s|^{-2}\|q\|_{L_2(\Gamma_I)},$$

(3.10) $$\|k^*\|_{H^{-1/2}(\Gamma_L)} \leq C|s|^{-1}\|q\|_{L_2(\Gamma_I)},$$

(3.11) $$\|k^*\|_{L^2(\Gamma_I)} \leq C|s|^{-1}\|q\|_{L_2(\Gamma_I)},$$

(3.12) $$\|k^*\|_{L^2(\Gamma_L)} \leq C|s|^{-1/2}\|q\|_{L_2(\Gamma_I)}.$$

As a consequence, we have the analogous result concerning equation (2.11).

COROLLARY 3.4: Let $q \in L^2(\Gamma_I)$; than equation (2.11) admits for every $s \in \mathbb{C}$ fixed, $\operatorname{Re} s > 0$, a unique solution $k = k(s) \in H^{-1/2}(\Gamma)$. Moreover there exist constants $C > 0$, and $\beta_0 > 0$ such that, for all complex numbers s, $\operatorname{Re} s > \beta_0$, the following estimates hold

(3.13) $$\|k\|_{H^{-1/2}(\Gamma)} \leq C\|q\|_{L^2(\Gamma_I)},$$

(3.14) $$\|k\|_{H^{1/2}(\Gamma_I)} \leq C|s|^2\|q\|_{L^2(\Gamma_I)}, \quad \|k\|_{H^{1/2}(\Gamma_L)} \leq C|s|\,\|q\|_{L^2(\Gamma_I)},$$

(3.15) $$\|k\|_{L^2(\Gamma_I)} \leq C|s|\,\|q\|_{L^2(\Gamma_I)}, \quad \|k\|_{L^2(\Gamma_L)} \leq C\sqrt{|s|}\,\|q\|_{L^2(\Gamma_I)}.$$

PROOF:

Existence and uniqueness. We recall that $\langle Q^{-1}Nq, 1\rangle = 0$. Than, by vertue of known regularity results for the solutions of elliptic boundary value problems in Lipschitz domains, the operator $Q^{-1}N$ is a bounded map of $L^2(\Gamma_I)$ into $L^2(\Gamma)$.

Now we introduce the following continuous sesquilinear form, in $H^{1/2}(\Gamma)$:

$$b_s(u,v) := {}_{H^{-1/2}(\Gamma)}\langle Q^{-1}u, v\rangle_{H^{1/2}(\Gamma)} + \int_\Gamma \chi u\bar{v}.$$

By vertue of (3.2), $b_s(u,v)$ is $H^{1/2}(\Gamma)$-coercive, relatively to $L^2(\Gamma)$, *i.e.* there is $M(s) > 0$ such that

$$\operatorname{Re} b_s(u,u) + M(s)\|u\|^2_{L^2(\Gamma)} \geq C\|u\|^2_{H^{1/2}(\Gamma)}$$

where $C > 0$ is independent on s, $\operatorname{Re} s > 0$. Than the existence and uniqueness of the solution of (3.4) follows from the Fredholm alternative by using the fact that $q^* \in L^2(\Gamma)$, that $H^{1/2}(\Gamma)$ is completely immersed in $L^2(\Gamma)$, and that $b_s(u,u) = 0$ only if $u = 0$ (see [2]).

Estimates. We adapt to our case the method of partition of unity. Let B_F, B_L, B_ε be a covering of Γ such that $B_F \subset \Gamma_F$, $B_L \subset \Gamma_L$, $B_\varepsilon \supset (\bar{\Gamma}_F \cap \bar{\Gamma}_L)$ and $|B_\varepsilon| \to 0$ as $\varepsilon \to 0$.

Let $\xi_F, \xi_L, \xi_\varepsilon \in C^\infty(\bar{\Gamma})$ such that $\operatorname{supp} \xi_i \subset \bar{\Gamma}_i$, $0 \leqslant \xi_i \leqslant 1$, $i = L, F, \varepsilon$ and $\sum \xi_i = 1$. Let now $k^* \in H^{1/2}(\Gamma)$ be the solution of (3.4); than $k^* = \sum k^* \xi_i$ and $k^* \xi_i \in H^{1/2}(\Gamma)$. Moreover $\|k^* \xi_\varepsilon\|_{H^{1/2}(\Gamma)} \to 0$ when $\varepsilon \to 0$; as a consequence

$$\|k^* - k^* \xi_F\|_{H^{1/2}(\Gamma_F)} \to 0 \; \varepsilon \to 0; \qquad \|k^* - k^* \xi_L\|_{H^{1/2}(\Gamma_L)} \to 0 \; \varepsilon \to 0. \tag{3.16}$$

Let us multiply the equation (3.4) by ξ_F, and than represent it in the form

$$Q^{-1}(k^* \xi_F) + s^2(k^* \xi_F) = [Q^{-1}(k^* \xi_F) - (Q^{-1} k^*) \xi_F] + q^* \xi_F. \tag{3.17}$$

Now we state a lemma, whose proof will be led at the end of the proof of the theorem.

LEMMA 3.5: Let $k^* \in H^{1/2}(\Gamma)$ be the solution of (3.4). Than there exist constants $C > 0$ and $\alpha_1 > 0$ such that, $\forall s \in \mathbb{C}$, $\operatorname{Re} s > \alpha_1$ and $\forall \sigma$, $0 < \sigma \leqslant 1/2$

$$\|Q^{-1}(k^* \xi_F) - (Q^{-1} k^*) \xi_F\|_{L^2(\Gamma)} \leqslant C \|k^*\|_{H^\sigma(\Gamma)}.$$

An analogous estimate is valid for $k^* \xi_L$.

Owing to the positivity of the operator Q^{-1}, we may also state the following result, which is a direct application of a classical theorem, concerning positive isomorphisms between a Hilbert space and its antidual (see [2]).

«If $g \in L^2(\Gamma)$ the equation $(s^2 I + Q^{-1}) u = g$ has a unique solution $u \in H^{1/2}(\Gamma)$ $\forall s \in \mathbb{C}$, $\operatorname{Re} s > 0$; moreover the norm of the operator $(s^2 I + Q^{-1})^{-1} : L^2(\Gamma) \to H^{1/2}(\Gamma)$ is bounded independently on s, *i.e.* there exist positive constants C and α_2 such that $\forall s \in \mathbb{C}$, $\operatorname{Re} s > \alpha_2$:

$$\|u\|_{H^{1/2}(\Gamma)} \leqslant C \|g\|_{L^2(\Gamma)}$$

and also

$$\|u\|_{H^\vartheta(\Gamma)} \leqslant \frac{C}{|s|^{1-2\vartheta}} \|g\|_{L^2(\Gamma)} \qquad \forall \vartheta, \; -\frac{1}{2} \leqslant \vartheta \leqslant \frac{1}{2}». \tag{3.18}$$

Let us now go back to equation (3.17); applying the lemma and estimate (3.18) with $0 < \vartheta \leqslant 1/2$, and $0 < \sigma < 1/2$ we have $\forall s \in \mathbb{C}$, $\operatorname{Re} s > \alpha_3 > 0$:

$$\|k^* \xi_F\|_{H^\vartheta(\Gamma_F)} \leqslant \|k^* \xi_F\|_{H^\vartheta(\Gamma)} \leqslant \frac{C}{|s|^{1-2\vartheta}} (\|k^*\|_{H^\sigma(\Gamma)} + \|q^*\|_{L^2(\Gamma)}).$$

Letting $\varepsilon \to 0$, owing to (3.16) we get

$$\|k^*\|_{H^\vartheta(\Gamma_F)} \leqslant \frac{C}{|s|^{1-2\vartheta}} (\|k^*\|_{H^\sigma(\Gamma)} + \|q^*\|_{L^2(\Gamma)}). \tag{3.19}$$

Analogously, multipling (3.4) by ξ_L, and taking s instead of s^2 in (3.17), we have $\forall s \in \mathrm{C}$, $\mathrm{Re}\, s > \alpha_4 > 0$, and again $0 < \delta \leq 1/2$, $0 < \sigma < 1/2$:

$$(3.20) \qquad \|k^*\|_{H^\delta(\Gamma_L)} \leq \frac{C}{|s|^{1/2-\delta}} (\|k^*\|_{H^\sigma(\Gamma)} + \|q^*\|_{L^2(\Gamma)}).$$

Let us take now $\delta = \sigma$ in (3.19), (3.20) with $0 < \sigma < 1/2$; than, if $\mathrm{Re}\, s > \beta_1(\sigma) > 0$

$$(3.21) \qquad \|k^*\|_{H^\sigma(\Gamma_V)} \leq \frac{C}{|s|^{1-2\sigma}} \{\|k^*\|_{H^\sigma(\Gamma_L)} + \|q^*\|_{L^2(\Gamma)}\},$$

$$(3.22) \qquad \|k^*\|_{H^\sigma(\Gamma_L)} \leq \frac{C}{|s|^{1/2-\sigma}} \{\|k^*\|_{H^\sigma(\Gamma_V)} + \|q^*\|_{L^2(\Gamma)}\}.$$

Replacing (3.21) in (3.22) and vice-versa, we obtain, if $\mathrm{Re}\, s > \beta_2(\sigma) > 0$ and $0 < \sigma < 1/2$:

$$(3.23) \qquad \|k^*\|_{H^\sigma(\Gamma_V)} \leq \frac{C}{|s|^{1-2\sigma}} \|q^*\|_{L^2(\Gamma)},$$

$$(3.24) \qquad \|k^*\|_{H^\sigma(\Gamma_L)} \leq \frac{C}{|s|^{1/2-\sigma}} \|q^*\|_{L^2(\Gamma)}.$$

Now we take in (3.19) and (3.20) $\delta = 1/2$ and $\sigma = 1/4$; if $s \in \mathrm{C}$, $\mathrm{Re}\, s > \max(\alpha_3, \alpha_4)$:

$$(3.25) \qquad \|k^*\|_{H^{1/2}(\Gamma_V)} \leq C(\|k^*\|_{H^{1/4}(\Gamma)} + \|q^*\|_{L^2(\Gamma)}),$$

$$(3.26) \qquad \|k^*\|_{H^{1/2}(\Gamma_L)} \leq C(\|k^*\|_{H^{1/4}(\Gamma)} + \|q^*\|_{L^2(\Gamma)}).$$

By vertue of (3.23), (3.24), finally we have, if $\mathrm{Re}\, s > \alpha_0 > 0$

$$(3.27) \qquad \|k^*\|_{H^{1/2}(\Gamma)} \leq C\|q^*\|_{L^2(\Gamma)} \leq C\|q\|_{L^2(\Gamma_V)}, \quad \text{that is (3.8)}.$$

Returning back to equation (3.4) and taking into account of (3.27)

$$|s|^2 \|k^*\|_{H^{-1/2}(\Gamma_V)} \leq \|Q^{-1}k^*\|_{H^{-1/2}(\Gamma_V)} + \|q^*\|_{L^2(\Gamma_V)} \leq$$
$$\leq C\{\|k^*\|_{H^{1/2}(\Gamma)} + \|q^*\|_{L^2(\Gamma)}\} \leq C\|q\|_{L^2(\Gamma)},$$

that is (3.9). Analougsly we obtain (3.10). Interpolating between (3.27) and (3.9) and also between (3.27) and (3.10), we have (3.11) and (3.12). ■

Proof of Lemma 3.5: Let $k^* \in H^{1/2}(\Gamma)$ be the solution of (3.4). Owing to $q^* \in L^2(\Gamma)$, also $Q^{-1}k^* \in L^2(\Gamma)$ and, by (3.7), $\int_\Gamma Q^{-1}k^* = 0$. Than the Neumann problem: $\Delta u = 0$ in Ω, $\partial u/\partial\nu = Q^{-1}k^*$ on Γ, $\partial u/\partial\nu = 0$ on $\partial\Omega - \Gamma$, admits a variational solution, unique up to constants; let u be the solution such that $u \in H^1(\Omega)$. Actually $u \in H^{3/2}(\Omega)$ (see [5]) and $u/_\Gamma = k^*$; moreover

$$\|u\|_{H^{1/2+\sigma}(\Omega)} \leqslant C\|k^*\|_{H^\sigma(\Gamma)} \qquad \forall\sigma, \quad 0 < \sigma \leqslant \frac{1}{2}.$$

Now we consider the following mixed problem: $\Delta v = 0$ in Ω, $v = \xi_F$ on Γ, $\partial v/\partial\nu = 0$ on $\partial\Omega - \Gamma$. Noticing that, owing to the definition of ξ_F, the compatibility conditions on the data are satisfied, the solution $v \in C^1(\bar{\Omega})$ (see [7]).

Now we consider $g := u \cdot v \in H^1(\Omega)$; it satisfies the variational mixed problem: $\Delta g = 2\nabla u \cdot \nabla v$ in Ω, $g = \xi_F \cdot k^*$ on Γ, $\partial g/\partial\nu = 0$ on $\partial\Omega - \Gamma$. We notice that

$$\left.\frac{\partial g}{\partial\nu}\right|_\Gamma = (Q^{-1}k^*)\xi_F + k^* \cdot \left.\frac{\partial v}{\partial\nu}\right|_\Gamma \in L^2(\Gamma).$$

Than we decompose $g = g_1 + g_2$, where g_1 satisfies the mixed variational problem: $\Delta g_1 = 2\nabla u \cdot \nabla v$ in Ω, $g_1 = 0$ on Γ, $\partial g_1/\partial\nu = 0$ on $\partial\Omega - \Gamma$, while $g_2 := g - g_1$ satisfies the mixed problem $\Delta g_2 = 0$ in Ω, $g_2 = \xi_F \cdot k^*$ on Γ, $\partial g_2/\partial\nu = 0$ on $\partial\Omega - \Gamma$.

Now we notice that:

$$Q\left(\left.\frac{\partial g_2}{\partial\nu}\right|_\Gamma\right) = g_2|_\Gamma - \frac{\int \chi g_2}{\int \chi},$$

so that

$$\left.\frac{\partial g_2}{\partial\nu}\right|_\Gamma = Q^{-1}(k^* \cdot \xi_F) - \frac{1}{\int\chi} Q^{-1}(1) \int_\Gamma \chi k^* \xi_F.$$

Finally we have

$$\left.\frac{\partial g}{\partial\nu}\right|_\Gamma = (Q^{-1}k^*)\xi_F + k^* \left.\frac{\partial v}{\partial\nu}\right|_\Gamma = Q^{-1}(k^*\xi_F) - \frac{1}{\int\chi} Q^{-1}(1) \int_\Gamma \chi k^* \xi_F + \left.\frac{\partial g_1}{\partial\nu}\right|_\Gamma,$$

so that

$$Q^{-1}(k^*\xi_F) - (Q^{-1}k^*)\xi_F = k^* \cdot \left.\frac{\partial v}{\partial\nu}\right|_\Gamma + \frac{1}{\int\chi} Q^{-1}(1) \cdot \int_\Gamma \chi k^* \xi_F - \left.\frac{\partial g_1}{\partial\nu}\right|_\Gamma.$$

The $L^2(\Gamma)$-norm of the first two terms on the right hand side may be easily bounded from above by the L^2-norm of k^*, if Re s sufficiently large. Moreover

$$\left\|\frac{\partial g_1}{\partial\nu}\right\|_{L^2(\Gamma)} \leqslant C\|g_1\|_{H^{3/2-\sigma}(\Omega)} \leqslant C\|\nabla u\|_{H^{-1/2+\sigma}(\Omega)} \leqslant C\|u\|_{H^{1/2+\sigma}(\Omega)} \leqslant C\|k^*\|_{H^\sigma(\Gamma)},$$

$$\forall\sigma, \quad 0 < \sigma < 1/2.$$

The assertion is proved. ■

Now we go back to estimate the constant $C(s)$ in (2.5).

THEOREM 3.6: The unique solution $u = u(s)$ to problem (2.2) is a holomorphic function of s in the half-plane $\operatorname{Re} s > \gamma_0 > 0$, valued in $H^1(\Omega)$ and its norm is bounded independently on s:

$$\|u\|_{H^1(\Omega)} \leqslant C\|q\|_{L^2(\Gamma_F)}. \tag{3.28}$$

Moreover, its trace on Γ_F and Γ_L satisfies the estimates:

$$\|u\|_{H^{1/2}(\Gamma)} \leqslant C\|q\|_{L^2(\Gamma_F)}, \tag{3.29}$$

$$\|u\|_{H^{-1/2}(\Gamma_F)} \leqslant C|s|^{-2}\|q\|_{L^2(\Gamma_F)}; \quad \|u\|_{H^{-1/2}(\Gamma_L)} \leqslant C|s|^{-1}\|q\|_{L^2(\Gamma_F)}, \tag{3.30}$$

$$\|u\|_{L^2(\Gamma_F)} \leqslant C|s|^{-1}\|q\|_{L^2(\Gamma_F)}; \quad \|u\|_{L^2(\Gamma_L)} \leqslant C|s|^{-1/2}\|q\|_{L^2(\Gamma_F)}. \tag{3.31}$$

PROOF: We know (same notations of Proposition 2.3) that $u = u_1 + u_2 + \int q/\int \chi$; hence (3.28) (3.29) follow easily from (2.8), (2.10), (3.13), if $\operatorname{Re} s > \gamma_0 > 0$. On the other hand $u|_\Gamma = u_1|_\Gamma + u_2|_\Gamma + \int q/\int \chi = Nq + Pk + \int q/\int \chi = k/\chi$; taking into account of (3.13), (3.14), (3.15), we have the desired estimates. ∎

Now we return back to the algebraic system (2.3); proceeding in a quite similar way as in [8] and using (3.28) and (3.31), we obtain the following:

THEOREM 3.7: Assume that $\|M_1\| \geqslant K_1$ with K_1 sufficiently large, and let $\bar{f}(s)$ be a holomorphic function of s. There exists $\sigma_0 > 0$ such that, if $\operatorname{Re} s > \sigma_0$, the system of algebraic equations (2.3) admits a unique solution $\bar{\xi} = \bar{\xi}(s)$ which is a holomorphic function of s and satisfies

$$|\bar{\xi}(s)| \leqslant C|s|^{-2}|\bar{f}(s)|$$

where C does not depend on s.

4. - CONCLUSIONS

If now we combine Theorems 3.6 and 3.7 and take into account (2.1), we can apply the inversion formula for the Laplace transform. In order to have a solution of original problem (1.1) with the desired regularity with respect to time we need to request a suitable behaviour of $\bar{f}(s)$ as $s \to \infty$.

We may state the following:

Theorem 4.1: Suppose that

i) $q \in [L^2(\Gamma_I)]^6$;

ii) $\|M_1\| \geqslant K_1$ with K_1 sufficiently large;

iii) $f(t)$ is a Laplace-transformable function having support in $[0, +\infty)$ and whose transform $\tilde{f}(s)$ satisfies

$$|\tilde{f}(s)| \leqslant C|s|^{-1} \qquad \forall s \in \mathbb{C}, \quad \mathrm{Re}\, s > \sigma_1 \quad \text{for some } \sigma_1 > 0 .$$

Than there is a unique pair

$$(\xi, \phi) \in C^1([0, +\infty)) \times C^0([0, +\infty); H^1(\Omega))$$

such that:

$$\phi|_{\Gamma} \in C^0([0, +\infty); H^{1/2}(\Gamma)),$$

$$\phi|_{\Gamma_F} \in C^1([0, +\infty); L^2(\Gamma_F)) \cap C^2([0, +\infty); H^{-1/2}(\Gamma_F)),$$

$$\phi|_{\Gamma_L} \in C^{0,1/2}([0, +\infty); L^2(\Gamma_L)) \cap C^1([0, +\infty); H^{-1/2}(\Gamma_L)),$$

and such that (1.1) hold.

BIBLIOGRAPHY

[1] J. T. Beale, *Eigenfunction expansion for objects floating in an open sea*, Comm. Pure Appl. Math., 30 (1977), 283-313.

[2] R. Dautray - J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Springer-Verlag.

[3] D. Euvrard - G. P. Fernandez - A. Jami - M. Lenoir, *Méthodes numériques en hydrodynamique navale*, cours de D.E.A. commun. ENSTA, Paris VI, (1980-81).

[4] R. M. Garipov, *On the linear theory of gravity waves: the theorem of existence and uniqueness*, Arch. Rational Mech. Anal., 24 (1967).

[5] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman Publishing (1985).

[6] F. John, *On the motion of floating bodies* - I, Comm. Pure Appl. Math., 2 (1949), 13-57.

[7] A. Lorenzi, *A mixed problem for the Laplace equation in a right angle with an oblique derivative given on a side of the angle*, Ann. Mat. Pura Appl., 157 (1974), 259-306.

[8] C. Pagani - M. Verri - A. Zaretti, *The problem of seakeeping*, Appl. Anal. (1993).

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