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A Remark on the Regularity of Minima of Certain non Quadratic Functionals (**) 

SUMMARY. — We prove a global regularity result for minimizers of a class of integral functionals whose integrand \( f(x, u, \xi) \) grows polynomially like \(|\xi|^p\), \(1 < p < 2\).

Un’osservazione sulla regolarità dei minimi di alcuni funzionali non quadrattici

SUNTO. — Si prova un risultato di regolarità globale per minimi di una classe di integrali variationali il cui integrando \( f(x, u, \xi) \) ha una crescita polinomiale del tipo \(|\xi|^p\) con \(1 < p < 2\).

INTRODUCTION

Let \( f \) be a function defined on \( \Omega \times \mathbb{R} \times \mathbb{R}^n \) and set

\[
I(u, \Omega) = \int_{\Omega} f(x, u, Du) \, dx ,
\]

where \( \Omega \) is a bounded open set of \( \mathbb{R}^n \). We recall that a minimizer for \((0.1)\) is a function \( u \) such that

\[
I(u, \text{spt } \varphi) \leq I(u + \varphi, \text{spt } \varphi) ,
\]

for all \( \varphi \in C^1_0(\mathbb{R}^n) \).

In this paper we shall prove global regularity for minimizers of functionals of the type \((0.1)\) when the integrand \( f(x, u, \xi) \) grows polynomially like \(|\xi|^p\), \(1 < p < 2\).

More precisely we assume that for a suitable \( \mu \geq 0 \) the function \( f: \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \), verifies the following conditions:

\[
(H_1) \quad c_1 (\mu^2 + |\xi|^2)^{p/2} \leq f(x, u, \xi) \leq c (\mu^2 + |\xi|^2)^{p/2} ;
\]

\[
(H_2) \quad |f_0 (x, u, \xi)| \leq c (\mu^2 + |\xi|^2)^{p-2n/2} ;
\]


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\((H_1)\) \[ f(x, u, \xi) x, \eta \geq (\mu^2 + |\xi|^2)^{p-2/2} |\eta|^2; \]
\((H_2)\) \[ |f(x, u, \xi) - f(x, u, \eta)| \leq c(\mu^2 + |\xi|^2)^{p-2/2} (\mu^2 + |\xi|^2)^{2/2} (\mu^2 + |\xi|^2)^{p-2} - \kappa |\xi - \eta|^2; \]
for \(1 < p < 2, x \in (0, 2 - p)\) and \(\mu, c_1, c, \kappa\) independent of \((x, u)\);
\((H_3)\) \[ |f(x, u, \xi) - f(y, v, \xi)| \leq (\mu^2 + |\xi|^2)^{p/2} \omega(|u|, |x - y| + |u - v|); \]
where \(\omega(x, t) = k(t) \min \{\xi \gamma, L\} \) for some \(L > 0\) and \(\gamma \in (0, 1)\), and \(k\) is increasing.

Then we have:

**Theorem 1:** Let \(u \in W^{1,p}(\Omega)\) be a minimizer of (0.1) with \(f\) satisfying \((H_1)\) ... \((H_3)\). Then \(Du\) is locally \(\lambda\)-Holder continuous for some \(\lambda > 0\).

Moreover, if \(f = f(\xi)\) satisfies only \((H_1)\) ... \((H_3)\), \(u \in W^{2,2}_{0k}(\Omega)\).

In [4] M. Giaquinta and G. Modica proved a result of partial regularity for vector valued minima of the functional (0.1) in the case \(p \geq 2\). In the same paper they also showed that it is possible to deduce from this general result, global regularity in the scalar case.

On the other hand in a paper by E. Acerbi and N. Fusco [1], partial regularity is proved for vector valued minimizers of (0.1) in the case \(1 < p < 2\).

In this paper we show that, as M. Giaquinta and G. Modica did in [4] when \(p \geq 2\), it is still possible to deduce from the result of E. Acerbi and N. Fusco [1] global regularity in the scalar case when \(1 < p < 2\).

The proof of Theorem 1 is straightforward from the result of E. Acerbi and N. Fusco when \(f(x, u, \xi)\) satisfies \((H_1)\) ... \((H_3)\) with, \(\mu \neq 0\). Actually the case \(\mu = 0\) is the truly interesting one and require be treated an approximation argument based on Lemma 2.6.

The main tool of the proof of \(C^{1, \psi}\) regularity is given by the two integral estimates (1.4) and (1.5) for minimizers of functionals not depending on \((x, u)\), proved in [1]. Deriving (1.5) is the only part of the proof in which we use assumption \((H_1)\).

By different and more involved methods, which can only work in the scalar case, J. Manfredi in [6] proved a similar result when \(p > 1\). However we remark that here we get also an existence result for second derivatives, result not contained in [6].

1. - Preliminary results

In the following the letter \(c\) will denote any constant and \(B_R(x_0)\) the ball centred at \(x_0 \in \mathbb{R}^n\) of radius \(R\).

If \(u \in L^p(\Omega)\), for any \(B_R(x_0) \subset \Omega\), we set
\[
u_{x_0, R} = \frac{1}{\text{mes} B_R} \int_{B_R(x_0)} u(x) \, dx = \int_{B_R(x_0)} u(x) \, dx.
\]
If no confusion is possible we omit the indication of the centre of the ball, $B_R = B_R(x_0)$.

**Lemma 1.1:** For every $\gamma \in (-1/2, 0)$ and $\mu \geq 0$ we have:

$$
(2\gamma + 1)||\xi - \eta|| \leq \frac{\left|\left(\mu^2 + |\xi|^2\right)^{\gamma} \xi - \left(\mu^2 + |\eta|^2\right)^{\gamma} \eta\right|}{(\mu^2 + |\xi|^2 + |\eta|^2)^{\gamma}} \leq \frac{c(n)}{2\gamma + 1} ||\xi - \eta||,
$$

for every $\xi, \eta \in \mathbb{R}^n$.

**Proof:** see [1].

Let $u \in W^{1,p}_{\text{ex}}$ be a minimizer of (0.1), with $\mu \geq 0, 1 < p < 2$. We set

$$
H(\xi) = (\mu^2 + |\xi|^2)^{\rho/2},
$$
$$
V(\xi) = (\mu^2 + |\xi|^2)^{\rho/4} \xi,
$$
$$
\Phi(x, R) = \int_{B_R(x_0)} |V(Du) - (V(Du))_{x_0,R}|^2 dx.
$$

Let us recall some results proved in [1], that we'll use in the following.

**Proposition 1.2:** If $f$ satisfies (H$_1$), there are two constants $c > 0$ and $q > 1$, both independent of $\mu$, such that:

$$
\left(\int_{B_R(x_0)} H^q(Du) dx\right)^{1/q} \leq c \int_{B_R(x_0)} H(Du) dx,
$$

for every $B_R \subset \subset \Omega$.

From now let $\mu > 0$, and $f: \Omega \to \mathbb{R}$ independent of $(x, u)$.

**Proposition 1.3:** (See [1]) Let be a function of class $C^2$ satisfying (H$_1$), (H$_2$).

Then $u \in W^{2,p}_{\text{ex}}, V(Du) \in W^{1,2}_{\text{ex}}$. Moreover

$$
\int_{B_R(x_0)} |D(V(Du))|^2 dx \leq \frac{c}{R^2} \int_{B_R(x_0)} H(Du) dx,
$$

(1.1)

$$
\int_{B_R(x_0)} (\mu^2 + |Du|^2)^{\rho/2} |D^2 u|^2 dx \leq \frac{c}{R^2} \int_{B_R(x_0)} H(Du) dx,
$$

(1.2)
for a suitable \( c \) independent of \( \mu \).

**Lemma 1.4:** (See [1]) Let \( f \) be a function of class \( C^2 \) satisfying (H1) ... (H4). Then

\[
H(Du) \in W^{1,\frac{s}{s+1}}_{\text{loc}}, \quad \text{where } s = \frac{2\rho}{2n - p} > 1,
\]

\[
f_{\mu}(Du) \in W^{1,\frac{s}{s+2}}_{\text{loc}}, \quad D(f_{\mu}(Du)) = f_{\mu}(Du)D(D\mu).
\]

From now on, (H1) ... (H4) are always assumed for \( f = f(\xi) \).

**Proposition 1.5:** Let \( u \in W^{1,p}_0(\Omega) \) be a minimizer of (0.1). Then the function \( H(Du) \) is locally bounded in \( \Omega \). Moreover there are two constants \( c \) and \( \sigma > 1 \) both independent of \( \mu > 0 \), s.t.

\[
\int_{B(x_0, \rho)} H(Du) \, dx \leq c \left( \frac{\rho}{R} \right)^{\sigma} \int_{B(x_0)} H(Du) \, dx,
\]

\[
\phi(x_0, \rho) \leq c \left( \frac{\rho}{R} \right)^{\sigma} \phi(x_0, R),
\]

for every \( B \subset \subset \Omega \) and \( \rho < R \).

**Proof:** The proof follows the same lines of the proof of Proposition 2.11 of [1]. For completeness we sketch it.

From Proposition 1.2 and Lemma 1.3 we get

\[
V(Du), H^{1/2}(Du) \in W^{1,2}_{\text{loc}}(\Omega),
\]

\[
f_{\mu}(Du) \in W^{1,2}_{\text{loc}} \quad \text{and} \quad D(f_{\mu}(Du)) = f_{\mu}(Du)D(D\mu).
\]

We now set

\[
A_{ud}(x) = (\mu^2 + |Du|^2)^{1/2 - \sigma/2}f_{\mu}(Du).
\]

We remark that if (H1) ... (H4) hold, then \( A \) is a uniformly elliptic matrix with bounded coefficients, and the ratio of the greatest to the least eigenvalue are bounded independent of \( \mu \). Then by the same arguments used in [1] (Proposition 2.6), we get

\[
\int A_{ud} D\phi (H(Du)) D\phi \, dx \leq -c \int |DV(Du)|^2 \phi \, dx
\]

for all \( \phi \in C^0_0(\Omega) \) with \( \phi \geq 0 \).

Now the proof continues exactly as in [1] (Proposition 2.7, 2.8). The relation (1.4) follows from Proposition 1.2.
To extend this result to the case $\mu = 0$ we will approximate the function $f$. We remark that the approximation used in the case $p \geq 2$ doesn't work in this case.

Let $N$ be a positive integer and $\eta(t)$ a function such that

$$
\eta(t) = \begin{cases} 
0 & 0 \leq t \leq 1, \\
1 & t \geq N,
\end{cases}
$$

$0 \leq \eta(t) \leq 1$, $\eta(t) \in C^\infty(\mathbb{R})$, $|\eta'| \leq c/(N-1)$, $|\eta''| \leq c/(N-1)^2$, $|\eta'''| \leq c/(N-1)^3$.

**Lemma 1.6:** If $f$ satisfy $(H_1) \ldots (H_4)$ with $\mu = 0$, we set for $\varepsilon \in (0, 1)$

$$
f_\varepsilon(\xi) = \eta\left(\frac{|\xi|^2}{\varepsilon^2}\right)f(\xi) + \left(1 - \eta\left(\frac{|\xi|^2}{\varepsilon^2}\right)\right)(|\xi|^2 + \varepsilon^2)^{p/2}.
$$

Then for a suitable $N$, the function $f_\varepsilon(\xi)$ verifies $(H_1) \ldots (H_4)$ with $\mu = \varepsilon$ and the constants $c, c_1, \alpha$ independent of $\varepsilon$ and $N$.

**Proof:** The property $(H_1)$ of $f_\varepsilon$ is immediately verified. The properties $(H_2) \ldots (H_4)$ require many, but easy calculations.

**Proposition 1.7:** The result of Proposition 1.5 holds also in the case $\mu = 0$.

**Proof:** For every $\varepsilon \in (0, 1)$, let $u_\varepsilon$ be the (only) minimum point of

$$
\int_B f_\varepsilon(Du) \, dx,
$$
in the space $u + W_0^{1,p}(B)$, $B \subset \Omega$.

By $(H_1)$ and the relations (1.1), (1.3) it's easy to verify that, at least for a subsequence:

$$
\begin{align*}
u_\varepsilon & \rightharpoonup u \quad \text{weakly in } W^{2,p}_{\text{loc}}(B) \text{ and weakly in } u + W_0^{1,p}(B), \\
(\varepsilon^2 + |Du|^2)^{p/2} - 2^{1/p} Du_\varepsilon & \rightharpoonup |Du|^{p-2/2} Du \quad \text{weakly in } W_0^{1,2}(B).
\end{align*}
$$

Then the result follows by letting $\varepsilon \to 0$ in Proposition 1.5.

2. - Proof of Theorem 3

The following theorem is proved in [3].

**Theorem 2.1:** Let $u \in W^{1,p}(\Omega)$ be a minimizer of the functional (0.1) with $f = f(x, u, \xi)$ verifying $(H_1)$.

Then $u$ is locally Hölder-continuous with some exponent $\alpha > 0$. Moreover for any
Let \( x_0 \in \Omega \) and \( R < \text{dist} (x_0, \partial \Omega) \) we have:

\[
\int_{B_R} \left( \mu^2 + |D\nu|^2 \right) dx \leq c R^{-p - p_0}
\]

Throughout this section we shall assume that the function \( f = f(x, u, \xi) \) satisfies \((H_1), \ldots, (H_3)\).

In order to prove the Theorem 1 we compare a minimizer \( u \) of (0.1) with the solution of a problem independent of \((x, u)\).

**Proposition 2.2:** Let \( u \in W^{1,p} (\Omega) \) be a minimizer of (0.1). Then \( u \in C^{0,\gamma} (\Omega) \) for any \( 0 < \gamma < 1 \). Moreover for any \( 0 < \gamma < 1 \) there are constants \( R_0, \epsilon \) such that for any \( x_0 \in \Omega \) and \( \rho, R, 0 < \rho < R < \inf (R_0, \text{dist} (x_0, \partial \Omega)) \) we have

\[
\int_{B_R} H(Du) dx \leq c \left( \frac{\rho}{R} \right)^{-p - p_0} \int_{B_\rho} H(Du) dx.
\]

**Proof:** Let \( x_0 \in \Omega \), \( R < \text{dist} (x_0, \partial \Omega) \), \( u_0 = u_{x_0, R/2} \) and \( f^0 (\xi) = f(x_0, u_0, \xi) \).

Let \( v \in W^{1,p} (B_{R/2} (x_0)) \) be the minimum point of

\[
\int_{B_{R/2}} f^0 (Dv) dx,
\]

in the space \( u + W_0^{1,p} (B_{R/2}) \).

From (1.4) we get

\[
\int_{B_R} H(Du) dx \leq c \left( \frac{\rho}{R} \right)^{p} \int_{B_\rho} H(Du) dx + c \int_{B_{R/2}} |D(u - v)|^p dx.
\]

On the other hand by Lemma 1.2 and Hölder inequality

\[
\int_{B_{R/2}} |D(u - v)|^p dx \leq c \int_{B_{R/2}} \left( |V(Du) - V(Dv)| \left( |\mu|^2 + |D\nu|^2 \right)^{p/2} \right)^{p/2} dx
\]

\[
\leq c \left( \int_{B_{R/2}} \left( |V(Du) - V(Dv)| \right)^{p/2} \left( \int_{B_{R/2}} (|\mu|^2 + |D\nu|^2)^{p/2} \right)^{p/2} \right)^{2-p/2}
\]

\[
\leq c \left( \int_{B_{R/2}} \left( |f^0 (Du) - f^0 (Dv)| \right) dx \right)^{p/2} \left( \int_{B_{R/2}} (|\mu|^2 + |D\nu|^2)^{p/2} \right)^{2-p/2},
\]

where the last inequality is obtained by \((H_1)\). Now we observe that (see [1], Lem-
(2.4) \[ \int_{B_R} [f^p(Du) - f^p(Dv)] \, dx \leq c \left( k(|u_0|) \right) \int_{B_R} H(Du) \, dx \left( R^p \int_{B_{R/2}} (1 + |Du|^p) \, dx \right)^{\frac{q-1}{q}}, \]

where \( q \) is the exponent in Proposition 1.2.

So by (2.2), (2.3), (2.4) we get:

\[ \int_{B_R} H(Du) \, dx \leq c \left[ \left( \frac{2}{R} \right)^p + k(|u_0|)^{p/2} \left( R^p + R^p \int_{B_{R/2}} H(Du) \, dx \right)^{\beta/2} \right] \int_{B_R} H(Du) \, dx, \]

where \( \beta = (q - 1)/q \).

The result follows from Theorem 2.1 as in [3].

We are now ready to prove the Theorem 1.

**Proof of Theorem 1:** Let \( \nu \) be the function defined in Proposition 2.1. Then by relation (1.5) we get:

\[ \int_{B_R} |V(Du) - V(Dv)|^2 \, dx \leq c \left( \frac{2}{R} \right)^{2p} \int_{B_{R/2}} |V(Du) - V(Dv)| \, dx + \]

\[ + c \int_{B_{R/2}} |V(Du) - V(Dv)| \, dx. \]

By (H1):

\[ \int_{B_{R/2}} |V(Du) - V(Dv)| \, dx \leq c \int_{B_{R/2}} |f^q(Du) - f^q(Dv)| \, dx. \]

Then we get result as in Theorem 6.4 of [4].

To prove \( \mu \in W^{1,q}_{\text{loc}}(\Omega) \) it is enough to remark that (1.2) still holds if \( \mu = 0 \) and moreover \( Du \) is locally bounded.

**REFERENCES**


