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GIOCONDA MOSCARIELLO (*)

A Remark on the Regularity of Minima of Certain non Quadratic Functionals (**)

SUMMARY. — We prove a global regularity result for minimizers of a class of integral functionals whose integrand $f(x, u, \xi)$ grows polinomially like $|\xi|^p$, $1 < p < 2$.

Un'osservazione sulla regolarità dei minimi di alcuni funzionali non quadratici

SUNTO. — Si prova un risultato di regolarità globale per minimi di una classe di integrali variazionali il cui integrando $f(x, u, \xi)$ ha una crescita polinomiale del tipo $|\xi|^p$ con $1 < p < 2$.

INTRODUCTION

Let f be a function defined on $\Omega \times R \times R^n$ and set

$$I(u, \Omega) = \int_\Omega f(x, u, Du)\, dx, \tag{0.1}$$

where Ω is a bounded open set of R^n. We recall that a minimizer for (0.1) is a function u such that

$$I(u, \operatorname{spt} \varphi) \leq I(u + \varphi, \operatorname{spt} \varphi),$$

for all $\varphi \in C_0^1(R^n)$.

In this paper we shall prove global regularity for minimizers of functionals of the type (0.1) when the integrand $f(x, u, \xi)$ grows polinomially like $|\xi|^p$, $1 < p < 2$.

More precisely we assume that for a suitable $\mu \geq 0$ the function $f: \Omega \times R \times R^n \to R$, verifies the following conditions:

(H$_1$) $\quad c_1(\mu^2 + |\xi|^2)^{p/2} \leq f(x, u, \xi) \leq c(\mu^2 + |\xi|^2)^{p/2}$;

(H$_2$) $\quad |f_{\xi\xi}(x, u, \xi)| \leq c(\mu^2 + |\xi|^2)^{(p-2)/2}$;

(*) Indirizzo dell'autore: Dipartimento di Matematica ed Applicazioni «R. Caccioppoli», Complesso Monte S. Angelo, Edificio T, Via Cintia, 80126 Napoli.

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(H$_3$) $\quad (f_{\xi\xi}(x, u, \xi)\eta, \eta) \geq (\mu^2 + |\xi|^2)^{(p-2)/2} |\eta|^2$;

(H$_4$) $\quad |f_{\xi\xi}(x, u, \xi) - f_{\xi\xi}(x, u, \eta)| \leq$

$$\leq c(\mu^2 + |\xi|^2)^{(p-2)/2} (\mu^2 + |\eta|^2)^{(p-2)/2} (\mu^2 + |\xi|^2 + |\eta|^2)^{(2-p-\alpha)/2} |\xi - \eta|^\alpha;$$

for $1 < p < 2$, $\alpha \in (0, 2-p)$ and μ, c_1, c, α independent of (x, u);

(H$_5$) $\quad |f(x, u, \xi) - f(y, v, \xi)| \leq (\mu^2 + |\xi|^2)^{p/2} \omega(|u|, |x-y| + |u-v|)$;

where $\omega(s, t) = k(s) \min\{t^\gamma, L\}$ for some $L > 0$ and $\gamma \in (0, 1]$, and k is increasing.

Then we have:

THEOREM 1: *Let $u \in W^{1,p}(\Omega)$ be a minimizer of* (0.1) *with f satisfying* (H$_1$) ... (H$_5$). *Then Du is locally λ-Holder continuous for some $\lambda > 0$.*

Moreover, if $f = f(\xi)$ satisfies only (H$_1$) ... (H$_4$), *$u \in W^{2,2}_{loc}(\Omega)$.*

In [4] M. Giaquinta and G. Modica proved a result of partial regularity for vector valued minima of the functional (0.1) in the case $p \geq 2$. In the same paper they also showed that it is possible to deduce from this general result, global regularity in the scalar case.

On the other hand in a paper by E. Acerbi and N. Fusco [1], partial regularity is proved for vector valued minimizers of (0.1) in the case $1 < p < 2$.

In this paper we show that, as M. Giaquinta and G. Modica did in [4] when $p \geq 2$, it is still possible to deduce from the result of E. Acerbi and N. Fusco [1] global regularity in the scalar case when $1 < p < 2$.

The proof of Theorem 1 is straightforward from the result of E. Acerbi and N. Fusco when $f(x, u, \xi)$ satisfies (H$_1$) ... (H$_5$) with, $\mu \neq 0$. Actually the case $\mu = 0$ is the truly interesting one and require be treated an approximation argument based on Lemma 2.6.

The main tool of the proof of $C^{1,\alpha}$ regularity is given by the two integral estimates (1.4) and (1.5) for minimizers of functionals not depending on (x, u), proved in [1]. Deriving (1.5) is the only part of the proof in which we use assumption (H$_4$).

By different and more involved methods, which can only work in the scalar case, J. Manfredi in [6] proved a similar result when $p > 1$. However we remark that here we get also an existence result for second derivatives, result not contained in [6].

1. - PRELIMINARY RESULTS

In the following the letter c will denote any constant and $B_R(x_0)$ the ball centred at $x_0 \in R^n$ of radius R.

If $u \in L^p(\Omega)$, for any $B_R(x_0) \subset \Omega$, we set

$$u_{x_0, R} = \frac{1}{\operatorname{mes} B_R} \int_{B_R(x_0)} u(x)\, dx = \fint_{B_R(x_0)} u(x)\, dx\,.$$

If no confusion is possible we omit the indication of the centre of the ball, $B_R = B_R(x_0)$.

LEMMA 1.1: *For every $\gamma \in (-1/2, 0)$ and $\mu \geqslant 0$ we have:*

$$(2\gamma + 1)|\xi - \eta| \leqslant \frac{|(\mu^2 + |\xi|^2)^\gamma \xi - (\mu^2 + |\eta|^2)^\gamma \eta|}{(\mu^2 + |\xi|^2 + |\eta|^2)^\gamma} \leqslant \frac{c(n)}{2\gamma + 1} |\xi - \eta| ,$$

for every $\xi, \eta \in R^n$.

PROOF: see [1].

Let $u \in W^{1,p}_{loc}$ be a minimizer of (0.1), with $\mu \geqslant 0$, $1 < p < 2$.
We set

$$H(\xi) = (\mu^2 + |\xi|^2)^{p/2},$$

$$V(\xi) = (\mu^2 + |\xi|^2)^{(p-2)/4} \xi,$$

$$\Phi(x_0, R) = \fint_{B_R(x_0)} |V(Du) - (V(Du))_{x_0, R}|^2 dx .$$

Let us recall some results proved in [1], that we'll use in the following

PROPOSITION 1.2: *If f satisfies (H_1), there are two constants $c > 0$ and $q > 1$, both independent of μ, such that:*

$$\left(\fint_{B_{R/2}} H^q(Du)\, dx \right)^{1/q} \leqslant c \fint_{B_R} H(Du)\, dx ,$$

for every $B_R \subset\subset \Omega$.

From now let $\mu > 0$, and $f: \Omega \to R$ independent of (x, u).

PROPOSITION 1.3: (See [1]) *Let be a function of class C^2 satisfying (H_1), (H_3).*
Then $u \in W^{2,p}_{loc}$, $V(Du) \in W^{1,2}_{loc}$. Moreover

$$\int_{B_{R/2}} |D(V(Du))|^2 dx \leqslant \frac{c}{R^2} \int_{B_R} H(Du)\, dx , \tag{1.1}$$

$$\int_{B_{R/2}} (\mu^2 + |Du|^2)^{(p-2)/2} |D^2 u|^2 dx \leqslant \frac{c}{R^2} \int_{B_R} H(Du)\, dx , \tag{1.2}$$

$$\int_{B_{R/2}} |D^2 u|^p \, dx \leq \frac{c}{R^p} \int_{B_R} H(Du) \, dx, \tag{1.3}$$

for a suitable c independent of μ.

LEMMA 1.4: (See [1]) *Let f be a function of class C^2 satisfying $(H_1) \dots (H_3)$ Then*

$$H(Du) \in W^{1,s}_{loc}, \quad \text{where } s = \frac{2n}{2n-p} > 1,$$

$$f_{\xi_\alpha}(Du) \in W^{1,2}_{loc}, \quad D(f_{\xi_\alpha}(Du)) = f_{\xi_\alpha \xi_\beta}(Du) \, D(D_\beta u).$$

From now on, $(H_1) \dots (H_4)$ are always assumed for $f = f(\xi)$.

PROPOSITION 1.5: *Let $u \in W^{1,p}(\Omega)$ be a minimizer of (0.1). Then the function $H(Du)$ is locally bounded in Ω. Moreover there are two constants c and $\sigma > 1$ both independent of $\mu > 0$, s.t.*

$$\int_{B_\rho(x_0)} H(Du) \, dx \leq c \left(\frac{\rho}{R} \right)^\sigma \int_{B_R} H(Du) \, dx, \tag{1.4}$$

$$\Phi(x_0, \rho) \leq c \left(\frac{\rho}{R} \right)^\sigma \Phi(x_0, R), \tag{1.5}$$

for every $B_R \subset\subset \Omega$ and $\rho < R$.

PROOF: The proof follows the same lines of the proof of Proposition 2.11 of [1]. For completeness we sketch it.

From Proposition 1.2 and Lemma 1.3 we get

$$V(Du), \, H^{1/2}(Du) \in W^{1,2}_{loc}(\Omega),$$

$$f_{\xi_\alpha}(Du) \in W^{1,2}_{loc} \quad \text{and} \quad D(f_{\xi_\alpha}(Du)) = f_{\xi_\alpha \xi_\beta}(Du) \, D(D_\beta u).$$

We now set

$$A_{\alpha\beta}(x) = (\mu^2 + |Du|^2)^{(2-p)/2} f_{\xi_\alpha \xi_\beta}(Du).$$

We remark that if $(H_1) \dots (H_3)$ hold, then A is a uniformly elliptic matrix with bounded coefficients, and the ratio of the greatest to the least eigenvalue are bounded independent of μ. Then by the same arguments used in [1] (Proposition 2.6), we get

$$\int A_{\alpha\beta} D_\alpha (H(Du)) \, D_\beta \varphi \, dx \leq -c \int |DV(Du)|^2 \varphi \, dx$$

for all $\varphi \in C^1_0(\Omega)$ with $\varphi \geq 0$.

Now the proof continues exactly as in [1] (Proposition 2.7, 2.8).

The relation (1.4) follows from Proposition 1.2.

To extend this result to the case $\mu = 0$ we will approximate the function f. We remark that the approximation used in the case $p \geq 2$ doesn't work in this case.

Let N be a positive integer and $\eta(t)$ a function such that

$$\eta(t) = \begin{cases} 0 & 0 \leq t \leq 1, \\ 1 & t \geq N, \end{cases}$$

$$0 \leq \eta(t) \leq 1, \quad \eta(t) \in C^\infty(R),$$

$$|\eta'| \leq c/(N-1), \quad |\eta''| \leq c/(N-1)^2, \quad |\eta'''| \leq c/(N-1)^3.$$

LEMMA 1.6: *If f satisfy* $(H_1) \dots (H_4)$ *with* $\mu = 0$, we set for $\varepsilon \in (0, 1)$

$$f_\varepsilon(\xi) = \eta\left(\frac{|\xi|^2}{\varepsilon^2}\right) f(\xi) + \left(1 - \eta\left(\frac{|\xi|^2}{\varepsilon^2}\right)\right)c(|\xi|^2 + \varepsilon^2)^{p/2}.$$

Then for a suitable N, the function $f_\varepsilon(\xi)$ *verifies* $(H_1) \dots (H_4)$ *with* $\mu = \varepsilon$ *and the constants* c, c_1, α *independent of* ε *and* N.

PROOF: The property (H_1) of f_ε is immediately verified. The properties $(H_2) \dots (H_4)$ require many, but easy calculations.

PROPOSITION 1.7: *The result of Proposition* 1.5 *holds also in the case* $\mu = 0$.

PROOF: For every $\varepsilon \in (0, 1)$, let u_ε be the (only) minimum point of

$$\int_B f_\varepsilon(Dv)\,dx,$$

in the space $u + W_0^{1,p}(B)$, $B \subset\subset \Omega$.

By (H_1) and the relations (1.1), (1.3) it's easy to verify that, at least for a subsequence:

$$u_\varepsilon \to u \quad \text{weakly in } W^{2,p}_{loc}(B) \text{ and weakly in } u + W_0^{1,p}(B),$$

$$(\varepsilon^2 + |Du_\varepsilon|^2)^{(p-2)/4} Du_\varepsilon \to |Du|^{(p-2)/2} Du \quad \text{weakly in } W^{1,2}_{loc}(B).$$

Then the result follows by letting $\varepsilon \to 0$ in Proposition 1.5.

2. - PROOF OF THEOREM 3

The following theorem is proved in [3].

THEOREM 2.1: *Let* $u \in W^{1,p}(\Omega)$ *be a minimizer of the functional* (0.1) *with* $f = f(x, u, \xi)$ *verifying* (H_1).

Then u *is locally Hölder-continuous with some exponent* $\alpha > 0$. *Moreover for any*

$x_0 \in \Omega$ *and* $R < \operatorname{dist}(x_0, \partial\Omega)$ *we have*:

$$\int_{B_R} (\mu^2 + |Du|^2)^{p/2} dx \leq c R^{n-p+p\sigma}.$$

Throughout this section we shall assume that the function $f = f(x, u, \xi)$ satisfies $(H_1) \dots (H_5)$.

In order to prove the Theorem 1 we compare a minimizer u of (0.1) with the solution of a problem independent of (x, u).

PROPOSITION 2.2: *Let* $u \in W^{1,p}(\Omega)$ *be a minimizer of* (0.1). *Then* $u \in C^{0,\sigma}(\Omega)$ *for any* $0 < \sigma < 1$. *Moreover for any* $0 < \sigma < 1$ *there are constants* R_0, c *such that for any* $x_0 \in \Omega$ *and* ρ, R, $0 < \rho < R < \inf(R_0, \operatorname{dist}(x_0, \partial\Omega))$ *we have*

$$(2.1) \qquad \fint_{B_\rho} H(Du)\, dx \leq c \left(\frac{\rho}{R}\right)^{-p+p\sigma} \fint_{B_R} H(Du)\, dx.$$

PROOF: Let $x_0 \in \Omega$, $R < \operatorname{dist}(x_0, \partial\Omega)$, $u_0 = u_{x_0, R/2}$ and $f^0(\xi) = f(x_0, u_0, \xi)$.

Let $v \in W^{1,p}(B_{R/2}(x_0))$ be the minimum point of

$$\int_{B_{R/2}} f^0(Dv)\, dx,$$

in the space $u + W_0^{1,p}(B_{R/2})$.

From (1.4) we get

$$(2.2) \qquad \int_{B_\rho} H(Du)\, dx \leq c \left(\frac{\rho}{R}\right)^n \int_{B_{R/2}} H(Du)\, dx + c \int_{B_{R/2}} |D(u-v)|^p dx.$$

On the other hand by Lemma 1.2 and Hölder inequality

$$(2.3) \qquad \int_{B_{R/2}} |D(u-v)|^p dx \leq c \int_{B_{R/2}} [|V(Du) - V(Dv)|(\mu^2 + |Du|^2 + |Dv|^2)^{(2-p)/4}]^p dx \leq$$

$$\leq c \left(\int_{B_{R/2}} |V(Du) - V(Dv)|^2 \right)^{p/2} \left(\int_{B_{R/2}} (\mu^2 + |Du|^2 + |Dv|^2)^{p/2} \right)^{(2-p)/2} \leq$$

$$\leq c \left(\int_{B_{R/2}} [f^0(Du) - f^0(Dv)]\, dx \right)^{p/2} \left(\int_{B_{R/2}} (\mu^2 + |Du|^2 + |Dv|^2)^{p/2} \right)^{(2-p)/2},$$

where the last inequality is obtained by (H_1). Now we observe that (see [1], Lemma 3.3)

$$(2.4)\qquad \fint_{B_{R/2}} [f^0 D(u) - f^0 D(v)]\,dx \le ck(|u_0|) \fint_{B_R} H(Du)\,dx \left(R^p \fint_{B_R} (1 + |Du|^p)\,dx \right)^{(q-1)/q},$$

where q is the exponent in Proposition 1.2.

So by (2.2), (2.3), (2.4) we get:

$$\int_{B_\rho} H(Du)\,dx \le c\left[\left(\frac{\rho}{R}\right)^n + k(|u_0|)^{\beta/2}\left(R^p + R^p \fint_{B_R} H(Du)\,dx\right)^{\beta/2}\right]\int_{B_R} H(Du)\,dx,$$

where $\beta = (q-1)/q$.

The result follows from Theorem 2.1 as in [3].

We are now ready to prove the Theorem 1.

Proof of Theorem 1: Let v be the function defined in Proposition 2.1. Then by relation (1.5) we get:

$$\int_{B_\rho} |V(Du) - V(Du)_{x_0,\rho}|^2\,dx \le c\left(\frac{\rho}{R}\right)^{n+2\sigma} \int_{B_{R/2}} |V(Du) - V(Dv)_{x_0,R/2}|^2\,dx +$$

$$+c\int_{B_{R/2}} |V(Du) - V(Dv)|^2\,dx.$$

By (H_1):

$$\int_{B_{R/2}} |V(Du) - V(Dv)|^2\,dx \le c\int_{B_{R/2}} |f^0(Du) - f^0(Dv)|\,dx.$$

Then we get result as in Theorem 6.4 of [4].

To prove $u \in W^{2,2}_{loc}(\Omega)$ it is enough to remark that (1.2) still holds if $\mu = 0$ and moreover Du is locally bounded.

REFERENCES

[1] E. Acerbi - N. Fusco, *Regularity for minimizers of non-quadratic functionals: the case* $1 < p < 2$, J. Math. Anal. Appl., 140 (1989), 115-135.

[2] E. Di Benedetto, $C^{1+\alpha}$ *local regularity of weak solution of degenerate elliptic equations*, Nonlinear Anal., 7 (1983), 827-850.

[3] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Princeton Univ. Press, Princeton (1983).

[4] M. Giaquinta - G. Modica, *Remarks on the regularity of the minimizers of certain degenerate functionals*, Manuscripta Math., 57 (1986), 55-99.

[5] D. Gilbarg - N. S. Trudinger, *Elliptic Differential Equations of Second Order*, 2nd ed., Springer, Berlin (1984).

[6] J. S. Manfredi, *Regularity for minima of functionals with p-growth*, J. Diff. Eq., 76 (1988), 203-212.
[7] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Eq., 51 (1984), 126-150.
[8] K. Uhlenbeck, *Regularity for a class of nonlinear elliptic systems*, Acta Math., 138 (1977), 219-240.

Direttore responsabile: Prof. A. Ballio - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5 - Bologna